ON THE *PROBLÈME DES MÉNAGES*

MAX WYMAN AND LEO MOSER

Introduction. The classical *problème des ménages* asks for the number of ways of seating at a circular table n married couples, husbands and wives alternating, so that no husband is next to his own wife.

An outline of the history of the problem to 1946 was given by Kaplansky and Riordan **(11)**. They also presented a bibliography, which is augmented and brought up to date in the bibliography of the present paper.

The first explicit solution of the problem is due to Touchard **(23)** and the simplest derivation of Touchard's formula is due to Kaplansky **(9)**. In the present paper a new explicit solution to the problem is obtained, via an exponential generating function for certain numbers closely related to the ménage numbers and introduced by Cayley **(4)**. Although the new explicit expression is quite complicated, it does lead to some new and deep results concerning the ménage numbers. In particular, it is shown that the usual asymptotic formula for these numbers can actually be used to compute the numbers exactly.

Several other new explicit expressions for the ménage numbers are obtained and one of these suggests a strong conjecture concerning Latin rectangles for which some evidence is presented.

The most extensive published tables of the ménage numbers are those given by Lucas **(13)**. These go up to $n = 25$. In the present paper we present tables which give the numbers up to $n = 65$. These were computed by F. L. Miksa, using a recursion formula of Cayley **(4)**, and checked by means of congruences due to Riordan **(20)**.

1. A Generating Function. Rather than deal directly with the ménage numbers M_n many authors introduce the number U_n defined by

(1.1) $$M_n = 2\,(n!)\,U_n.$$

Further, Cayley **(4)** introduced an auxiliary sequence q_n defined by

(1.2) $$U_n = q_n - q_{n-2},$$

and showed that the q_n satisfy the recurrence relation

(1.3) $$q_n = n\,q_{n-1} + q_{n-2} + (-1)^{n-1}(n-2).$$

If we introduce the generating function $F(t)$ by

(1.4) $$F(t) = \sum_{n=0}^{\infty} q_n \frac{t^n}{n!},$$

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then it is easily shown that $F(t)$ is the solution of

$$(1.5)\qquad (1-t)\ddot{F} - 2\dot{F} - F = t\,e^{-t},\quad F(0) = \dot{F}(0) = 0,$$

where the "dot" means differentiation with respect to t.

The substitution

$$(1.6)\qquad F = (1-t)^{\frac{1}{2}}y,\ x = 2(1-t)^{\frac{1}{2}}$$

makes (1.5) take the form

$$(1.7)\qquad y'' + x^{-1}y' - (1+x^{-2})y = \tfrac{1}{2}x(1-\tfrac{1}{4}x^2)\,e^{(x^2/4-1)},\quad y(2) = y'(2) = 0,$$

where the prime denotes differentiation with respect to x. The homogeneous equation is well known and the complementary function can be expressed in terms of the modified Bessel functions as

$$(1.8)\qquad A\,I_1(x) + B\,K_1(x),$$

where A, B are constants.

In order to determine a particular integral $P(x)$ of (1.7), we assume a series solution of the form

$$(1.9)\qquad P(x) = \sum_{n=0}^{\infty} a_n x^{n+3}.$$

Substituting into (1.7) we immediately are led to

$$(1.10)\qquad a_0 = e^{-1}/16,\ a_{2n+1} = 0,\quad 4a_{2n}(n+1)(n+2) - a_{2n-2} = e^{-1}(1-n)/2^{2n+1}\,n!$$

This recurrence relation is easily solved and our particular solution can be put into the form

$$(1.11)\qquad P(x) = e^{-1}\left[I_1(x) - \tfrac{1}{2}x\,e^{x^2/4} + 2\sum_{n=1}^{\infty} b_n\,(\tfrac{1}{2}x)^{2n+1}\right],$$

where

$$b_n = \left(\sum_{s=1}^{n} s!\right)\Big/ n!(n+1)!\,.$$

Replacing $s!$ by

$$\int_0^\infty e^{-z}\,z^s dz,$$

we find

$$(1.12)\qquad P(x) = e^{-1}\left[I_1(x) - \tfrac{1}{2}x\,e^{x^2/4} + 2\int_0^\infty F(x,z)dz\right],$$

where $F(x,z) = z\,e^{-z}\,(I_1(x) - z^{\frac{1}{2}}\,I_1\,(x\,z^{-\frac{1}{2}}))/(1-z)$.

If we introduce the principal value of the integral at $z = 1$ we can rearrange the terms so that

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$$P(x) = e^{-1}\left[L\, I_1(x) - \tfrac{1}{2}x\, e^{x^2/4} + 2\int_0^\infty G(x, z)dz\right], \tag{1.13}$$

where

$$L = 2\int_0^\infty \frac{e^{-z}}{1-z}dz - 1, \quad G(x, z) = \frac{z^{\frac{1}{2}}e^{-z}I_1(xz^{\frac{1}{2}})}{z-1}. \tag{1.14}$$

Thus the general solution of (1.7) must be of the form

$$y = A\, I_1(x) + B\, K_1(x) + P(x), \tag{1.15}$$

where the constants A, B must be chosen to satisfy $y(2) = y'(2) = 0$.

The analysis so far is straight-forward and it seems likely that it has been carried thus far before. The major difficulty is in the evaluation of the constants A and B. In view of the complexity of the functions involved it is, indeed, remarkable that these constants can be evaluated in a tractable form. The evaluation of the constants is given in the next section.

2. Evaluation of the constants. If $f_1(x)$, $f_2(x)$ denote two functions of x we introduce the usual Wronskian notation $W(f_1, f_2)$ by

$$W(f_1, f_2) = f_1 f_2' - f_2 f_1'. \tag{2.1}$$

In order to satisfy the boundary conditions $y(2) = y'(2) = 0$ we have

$$\begin{aligned} A\, I_1(2) + B\, K_1(2) + P(2) &= 0 \\ A\, I_1'(2) + B\, K_1'(2) + P'(2) &= 0. \end{aligned} \tag{2.2}$$

Since it is well known that $W(I_1(2), K_1(2)) = -\frac{1}{2}$ we have

$$A = 2\, W(P(2), K_1(2)), \quad B = 2\, W(I_1(2), P(2)). \tag{2.3}$$

We evaluate these Wronskians, by the usual procedure, from the differential equations satisfied by $P(x)$ and $I_1(x)$. These differential equations are

$$x\, P'' + P' - (x + x^{-1})\, P = \tfrac{1}{2}x^2(1 - \tfrac{1}{4}x^2) \exp(\tfrac{1}{4}x^2 - 1), \tag{2.4}$$

$$x\, I_1'' + I_1' - (x + x^{-1})\, I_1 = 0. \tag{2.5}$$

We multiply (2.4) by I_1 and (2.5) by P. By subtraction of the resulting equations and integration from $x = 0$ to $x = 2$ we obtain

$$2\, W(I_1(2), P(2)) = \tfrac{1}{2}e^{-1}\int_0^2 x^2(1 - \tfrac{1}{4}x^2)\, e^{x^2/4}\, I_1(x)\, dx. \tag{2.6}$$

Hence

$$B = \tfrac{1}{2}e^{-1}\int_0^2 x^2(1 - \tfrac{1}{4}x^2)\, e^{x^2/4}\, I_1(x)\, dx, \tag{2.7}$$

and similarly

$$A = -\tfrac{1}{2}\, e^{-1}\int_0^2 x^2(1 - \tfrac{1}{4}x^2)e^{x^2/4} K_1(x)\, dx. \tag{2.8}$$

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In order to evaluate (2.7) we write (2.5) in the form

(2.9) $$I_1'' + (x^{-1}I_1)' - I_1 = 0.$$

Multiplying (2.9) by $\exp(x^2/4)$ and integrating from 0 to 2 we can show, by integrating by parts, that

(2.10) $$\int_0^2 e^{x^2/4}\,(\tfrac{1}{4}x^2 - 1)\,I_1(x)\,dx = 1 - e\,I_1'(2) + \tfrac{1}{2}\,e\,I_1(2).$$

Similarly by multiplying the differential equation by $x^2\exp(x^2/4)$ and repeating the process we find

(2.11) $$\int_0^2 e^{x^2/4}(x^2 + \tfrac{1}{4}x^4)\,I_1(x)dx = 6\,e\,I_1(2) - 4\,e\,I_1'(2).$$

Multiplying (2.10) by eight and subtracting (2.11) we obtain

(2.12) $$\int_0^2 e^{x^2/4}(x^2 - \tfrac{1}{4}x^4)\,I_1(x)dx = 8 - 4e\,I_1'(2) - 2e\,I_1(2) + 8\int_0^2 e^{x^2/4}I_1(x)dx$$

From the known recurrence relations of the modified Bessel functions we have

(2.13) $$2\,I_1'\,(2) + I_1(2) = 2\,I_0(2).$$

Hence

(2.14) $$\int_0^2 e^{x^2/4}\,(x^2 - \tfrac{1}{4}x^4)\,I_1(x)\,dx = 8 - 4\,e\,I_0(2) + 8\int_0^2 e^{x^2/4}I_1(x)\,dx.$$

Let us now consider the integral

$$J = \int_0^2 e^{x^2/4}\,I_1(x)\,dx.$$

The substitution $x = 2u^{\frac{1}{2}}$ transforms J into

(2.15) $$\begin{aligned} J &= \int_0^1 e^u\,I_1(2\,u^{\frac{1}{2}})\,u^{-\frac{1}{2}}du \\ &= \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}\int_0^1 e^u u^n du \\ &= \sum_{n=0}^{\infty} \frac{(1 - n + n(n-1)\dots(-1)^n n!)e + (-1)^{n+1}n!}{n!(n+1)!} \\ &= e[I_1(2) - I_2(2) + I_3(2)\dots] + e^{-1} - 1 \\ &= e\sum_{n=1}^{\infty}(-1)^{n+1}I_n(2) + e^{-1} - 1. \end{aligned}$$

However, from the generating function for $I_n(x)$ we can prove that

(2.16) $$e^{-2} = I_0(2) + 2\sum_{n=1}^{\infty}(-1)^n\,I_n(2).$$

Thus

(2.17) $$J = \tfrac{1}{2}e^{-1} + \tfrac{1}{2}eI_0(2) - 1$$

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and, from (2.14),

$$\int_0^2 e^{x^2/4}\,(x^2 - \tfrac{1}{4}x^4)\,I_1(x)\,dx = 4e^{-1}. \tag{2.18}$$

Finally from (2.7), (2.18) we have that the constant B is given by

$$B = 2e^{-2}. \tag{2.19}$$

The evaluation of the constant A can also be carried out with the help of the integral representation.

$$2\,K_1\,(2\,u^{\frac{1}{2}})\,u^{-\frac{1}{2}} = \int_0^\infty \exp(-\,uz - z^{-1})dz. \tag{2.20}$$

The final result is that

$$A = e^{-1} + 2e^{-1}\int_0^\infty e^{-z}/(z-1)dz. \tag{2.21}$$

These results imply that the desired solution of (1.7) is

$$y = 2e^{-2}K_1(x) - \tfrac{1}{2}e^{-1}xe^{\frac{1}{4}x^4} - 2e^{-1}\int_0^\infty \frac{z^{\frac{1}{2}}e^{-z}I_1(x(z)^{\frac{1}{2}})dz}{1-z} \tag{2.22}$$

and that the generating function $F(t)$, for q_n is given by

$$F(t) = 2e^{-2}(1-t)^{-\frac{1}{2}}K_1(2(1-t)^{\frac{1}{2}}) - e^{-t} - 2e^{-1}\int_0^\infty H(z,t)dz \tag{2.23}$$

where

$$H(z,t) = z^{\frac{1}{2}}\,e^{-z}\,I_1(2(z-zt)^{\frac{1}{2}})/(1-z)(1-t)^{\frac{1}{2}}.$$

The modified Bessel functions satisfy the well known differentiation formulae

$$\left(\frac{d}{zdz}\right)^m z^{-\alpha}I_\alpha(z) = z^{-\alpha-m}I_{\alpha+m}(z), \tag{2.24}$$

$$\left(\frac{d}{zdz}\right)^m z^{-\alpha}\,K_\alpha(z) = (-1)^m z^{-\alpha-m}K_{\alpha+m}(z). \tag{2.25}$$

Hence

$$q_n = F^{(n)}(0) = 2e^{-2}K_{n+1}(2) + (-1)^{n+1} + 2(-1)^{n+1}e^{-1}\int_0^\infty M_{n+1}(z)dz, \tag{2.26}$$

where

$$M_{n+1}(z) = z^{\frac{1}{2}(n+1)}\,e^{-z}I_{n+1}\,(2z^{\frac{1}{2}})/(1-z).$$

Since the ménage numbers U_n are given by $U_n = q_n - q_{n-2}$ we find that

$$U_n = 2e^{-2}nK_n(2) + 2(-1)^n + 2n(-1)^n e^{-1}\int_0^\infty M_n(z)dz. \tag{2.27}$$

If we replace $K_n(2)$, $I_n(2\,z^{\frac{1}{2}})$ by their known series expansions we can obtain an explicit series expression for U_n in terms of n. This expression is very complicated. However (2.27) is a useful expression in that one can derive many of

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the known results directly without resorting to the series expression. For example, it is readily shown from (2.27) that

$$\sum_{n=2}^{\infty} U_n I_n(2t) = e^{-2t}/(1-t) - I_0(2t) + I_1(2t). \tag{2.28}$$

Hence, by redefining U_0, U_1, to be 1 and -1 respectively we obtain Touchard's result **(24)**:

$$\sum_{n=0}^{\infty} U_n I_n(2t) = e^{-2t}/(1-t). \tag{2.29}$$

In the next section we shall use (2.27) to derive some new results for the ménage numbers.

3. New results. It has been shown **(11)** that an asymptotic expansion for U_n is given by

$$U_n \sim e^{-2}\, n!\left[1 - \frac{1}{(n-1)} + \frac{1}{2!(n-1)(n-2)} \cdots\right]. \tag{3.1}$$

By means of (2.27) we shall prove a much deeper result.

To prove this result we write (2.27) in the form

$$U_n = 2e^{-2}n\, K_n(2) + J_n, \tag{3.2}$$

where

$$J_n = 2(-1)^n\left\{1 + n\, e^{-1}\int_0^{\infty} \frac{z^{n/2}e^{-z}I_n(2z^{\frac{1}{2}})}{1-z}\, dz\right\}. \tag{3.3}$$

In (3.3) we replace the first term of the bracket by means of

$$1 = e^{-1}\sum_{m=0}^{\infty} 1/m! \tag{3.4}$$

and $I_n(2\, z^{\frac{1}{2}})$ by its series expression

$$I_n(2z^{\frac{1}{2}}) = z^{\frac{1}{2}n}\sum_{m=0}^{\infty}\frac{z^m}{m!(m+n)!}. \tag{3.5}$$

Hence J_n takes the form

$$J_n = 2(-1)^n e^{-1}\left[\sum_{m=0}^{\infty}\left\{(1/m!) + n\int_0^{\infty}\frac{e^{-z}}{1-z}\sum_{m=0}^{\infty}\frac{z^{m+n}}{m!(m+n)!}\, dz\right\}\right]. \tag{3.6}$$

This can be put in the form

$$J_n = 2(-1)^n e^{-1}\left\{Cn\, I_n(2) + \sum_{m=0}^{\infty}\frac{b_{mn}}{m!(m+n)!}\right\}, \tag{3.7}$$

where

$$\begin{aligned} C &= \int_0^{\infty}\frac{e^{-z}}{1-z}\, dz, \\ b_{mn} &= (m+n)! - n\{(m+n-1)! + (m+n-2)! + \ldots + 1\} \\ &= (m+n-1)!m - n\{(m+n-2)! + (m+n-3)! + \ldots + 1\}. \end{aligned} \tag{3.8}$$

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It is trivial to show

$$|C| < 4e^{-1}, \tag{3.9}$$

and

$$|nI_n(2)| \leqslant e/(n-1)!. \tag{3.10}$$

Hence

$$|CnI_n(2)| \leqslant 4/(n-1)!. \tag{3.11}$$

Let us consider the series term of (3.7) and write

$$\begin{aligned} H_n &= \sum_{m=0}^{\infty} \frac{b_{mn}}{m!(m+n)!} \\ &= \frac{n! - n\{(n-1)! + \ldots + 1)\}}{n!} \\ &\qquad + \frac{(n+1)! - n(n! + (n-1)! + \ldots + 1)}{(n+1)!} \\ &\quad + \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m+n)!} \\ &= -\frac{(n-2)! + (n-3)! + \ldots + 1}{(n-1)!}\left(1 + \frac{1}{n+1}\right) \\ &\quad + \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m+n)!}. \end{aligned} \tag{3.12}$$

If $n \geqslant 7$ it is easily shown that

$$\frac{(n-2)! + (n-3)! + \ldots + 1}{(n-1)!}\left(1 + \frac{1}{n+1}\right) \leqslant \frac{2}{n+1} \tag{3.13}$$

and

$$\left| \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m+n)!} \right| \leqslant \frac{2(e-1)}{n+1}. \tag{3.14}$$

Hence for $n \geqslant 7$,

$$|H_n| \leqslant \frac{2e}{n+1}. \tag{3.15}$$

Actually (3.15) is a very crude inequality. It is, however, sufficient for our purposes.

Combining these results we have from (3.7)

$$|J_n| \leqslant \frac{4}{n+1} + \frac{8}{e(n-1)!} \tag{3.16}$$

if $n \geqslant 7$.

Hence for $n \geqslant 8$ we have

$$|J_n| \leqslant 0.45. \tag{3.17}$$

Let us now return to (3.2) and examine the series expression for $K_n(2)$. This is given by

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$$K_n(2) = \tfrac{1}{2}\sum_{m=0}^{n-1}\frac{(-1)^m(n-m-1)!}{m!} + \tfrac{1}{2}(-1)^n\sum_{m=0}^{\infty}\frac{\Psi(n+m+1)+\Psi(m+1)}{m!(n+m)!}, \tag{3.18}$$

where

$$\Psi(k+1) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{k} - \gamma, \Psi(1) = -\gamma \tag{3.19}$$

and γ is Euler's constant.

It is easily shown that

$$\left|\sum_{m=0}^{\infty}\frac{\Psi(n+m+1)+\Psi(m+1)}{m!(n+m)!}\right| \leqslant \frac{e}{2(n-1)!}. \tag{3.20}$$

This implies

$$2n\, K_n(2) = n\sum_{m=0}^{n-1}\frac{(-1)^m(n-m-1)!}{m!} + R_n, \tag{3.21}$$

where the remainder satisfies $|R_n| \leqslant n\, e/(n-1)!$

Combining the results of (3.2), (3.17) and (3.21) we obtain

$$U_n = e^{-2}n\sum_{m=0}^{n-1}\frac{(-1)^m(n-m-1)!}{m!} + R_n' \tag{3.22}$$

where for $n \geqslant 8$ the remainder R_n' is definitely less than $\tfrac{1}{2}$.

Using the notation $\{x\}$ to denote the closest integer to x, we have shown that, for $n \geqslant 8$

$$U_n = \left\{e^{-2}n\sum_{m=0}^{n-1}\frac{(-1)^m(n-m-1)!}{m!}\right\}. \tag{3.23}$$

It is easy to verify that (3.23) remains valid for $0 \leqslant n \leqslant 7$. Hence we have proved the following theorem:

THEOREM. *For all values of n the ménage numbers U_n are given by* (3.23).

It is thus seen that the asymptotic expansion obtained in **(11)** is much more than an asymptotic expansion.

In concluding this section we might remark that about half of the terms in (3.23) are redundant in that their sum adds up to less than $\tfrac{1}{2}$. Further our analysis also implies that

$$U_n = \{2e^{-2}\, n\, K_n(2)\}. \tag{3.24}$$

We shall make use of (3.24) in the next section to make an interesting conjecture.

4. A Conjecture. The modified Bessel function $K_n(2)$ has the integral representation

$$K_n(2) = \tfrac{1}{2}\int_0^{\infty} t^{n-1}e^{-t-t^{-1}}dt. \tag{4.1}$$

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Hence (3.24) may be written

$$U_n = \left\{e^{-2}\, n \int_0^\infty t^{n-1}\, e^{-t-t^{-1}} dt\right\}. \tag{4.2}$$

The discovery of (4.2) led us to re-examine some of the known results in Latin rectangles. The simplest problem in this class is the so-called "problème des rencontres." This asks for the number of ways R_n of writing a second line of integers 1, 2, . . . n which is discordant with a first line of integers written in their normal order. It is well known that

$$R_n = \{e^{-1}\, n!\} = \left\{e^{-1} \int_0^\infty x^n\, e^{-x}\, dx\right\}. \tag{4.3}$$

Next in simplicity, in this class of problems, is the so-called reduced three line Latin rectangle problem. This asks for the number of ways P_n of having two lines of integers each of which is discordant with the first line of integers, written in normal order. For this case it was shown by Yamamoto **(26)** that

$$P_n \sim e^{-3}\, (n!)^2 \left[1 + \frac{H_1(-\frac{1}{2})}{n} + \frac{H_2(-\frac{1}{2})}{n(n-1)} + \dots\right], \tag{4.4}$$

where $H_n(x)$ is a Hermite polynomial.

We have been able to prove an equivalent formula, namely

$$P_n \sim e^{-3}\, (n!) \int_0^\infty x^n\, e^{-x-x^{-1}-x^{-2}}\, dx. \tag{4.5}$$

Finally Erdös and Kaplansky **(7)** have shown that the number P_n^k of reduced (n by $(k+1)$), Latin rectangles is given asymptotically by

$$P_n^k \sim e^{-\frac{1}{2}k(k-1)} (n!)^{k-1} \left[1 - \binom{k}{3} n^{-1} + \left(\tfrac{1}{2}\binom{k}{3}^2 + \tfrac{1}{2}\binom{k}{3}(k-5)\right) n^{-2} + \dots\right] \tag{4.6}$$

for $K \leqslant (\log n)^{3/2-\epsilon}$. The validity of the same formula was proved by Yamamoto **(26)** for $k < n^{1/3-\delta}$. The structure of the formula suggests an integral representation of the type

$$P_n^k \sim e^{-\frac{1}{2}k(k-1)} (n!)^{k-2} \int_0^\infty x^n \exp\left(-x - \binom{k}{3} x^{-1} + \tfrac{1}{2}\binom{k}{3}(k-5)\, x^{-2} + \dots\right) dx. \tag{4.7}$$

Formula (4.7) is, as we have seen, true for $k = 2,3$. If it were possible to prove an integral relation of this type then the asymptotic behavior of P_n^k could be determined for all values of k.

5. An exact expression for the ménage numbers. The usual explicit expression given for the ménage numbers U_n is

$$U_n = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!. \tag{5.1}$$

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In this section we shall derive a second expression from Touchard's generating function (2.9)

$$\sum_{n=0}^{\infty} U_n\, I_n(2t) = e^{-2t}/(1-t). \tag{5.2}$$

Touchard has remarked that (5.2) constitutes a Neumann expansion for the, function $e^{-2t}/(1-t)$ in terms of the modified Bessel functions $I_n(2t)$. However as far as we are aware, (5.2) has never been inverted to give an explicit expression for the U_n.

If we expand $e^{-2t}/(1-t)$ into a Maclaurin expansion of the form

$$\frac{e^{-2t}}{1-t} = \sum_{r=0}^{\infty} \frac{k_r t^r}{r!}. \tag{5.3}$$

then

$$k_r = \left[\frac{d^r}{dt^r}\frac{e^{-2t}}{1-t}\right]_{t=0} = r!\sum_{s=0}^{r}\frac{(-2)^s}{s!}. \tag{5.4}$$

Further from the well formulae for the coefficients of a Neumann expansion, (5.2) gives

$$U_n = \frac{2(i^n)}{\pi}\int_C \frac{e^{-2t}\,O_n(2it)}{1-t}\,dt, \tag{5.5}$$

where C is any closed contour, enclosing $t = 0$, such that $|t| < 1$. $O_n(z)$ are the so-called Neumann polynomials given explicitly by

$$O_n(z) = \tfrac{1}{4}\sum_{m=0}^{[\frac{1}{2}n]}\frac{n(n-m-1)!\,(\tfrac{1}{2}z)^{2m-n-1}}{m!}. \tag{5.6}$$

It follows immediately from (5.4), (5.5) and (5.6) that

$$U_n = \sum_{m=0}^{[\frac{1}{2}n]}\frac{(-1)^m n(n-m-1)!\,k_{n-2m}}{m!(n-2m)!} \tag{5.7}$$

If we use the umbral convention of replacing k_r by k^r we obtain the neat, mnemonic, formula

$$U_n = 2\,T_n(\tfrac{1}{2}k). \tag{5.8}$$

where $T_n(k)$ is the Chebyshev polynomial.

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Table of Ménage Numbers, U_n

n										
0										1
1										−1
2										0
3										1
4										2
5										13
6										80
7										579
8										4738
9										43387
10									4	39792
11									48	90741
12									592	16642
13									7755	96313
14								1	09274	34464
15								16	48064	35783
16								264	93914	69058
17								4522	64356	01207
18								81705	64062	24416
19							15	57461	89109	94665
20							312	40021	86712	53762
21							6577	61864	45769	02053
22						1	45051	25042	12302	24304
23						33	43382	81820	37841	46955
24						803	99425	36462	30706	80706
25						20136	19745	87449	39236	99123
26					5	24412	12770	21518	36760	81296
27					141	80874	54121	35441	26917	90045
28					3976	29238	67612	00144	54828	24194
29				1	15464	79231	29989	49665	85597	50193
30				34	68204	08266	14983	47273	40955	31712
31				1076	37754	44394	44821	25463	33529	40175
32				34481	07559	89439	56929	18585	03293	19426
33			11	39021	31602	21345	03795	43638	02432	51567
34			387	63360	88757	64510	83282	09689	42454	55168
35			13579	25683	97610	83548	12838	24806	55155	91633
36		4	89263	68181	72674	64273	50357	97412	89388	39554
37		181	17111	44161	23578	95013	36816	90501	14249	74653
38		6889	66679	77874	33823	33907	79975	80757	02511	45232
39	2	68887	96926	13377	25044	79310	17322	96268	42696	37331
40	107	62771	05129	32852	47921	55467	77103	56797	10498	56642

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Table of Ménage Numbers, U_n.

n											
41		4415	56290	19891	48194	39830	83196	99970	42707	08660	48747
42	1	85566	65097	95828	03659	83212	57515	14716	68334	59763	96848
43	79	83996	94833	63418	59137	63816	96992	08396	12446	35031	15589
44	3514	90268	88496	81285	48747	33216	99334	22942	19228	03980	73090
45										1	58254
		17445	46717	35843	13657	70852	22706	45836	45728	90212	00713
46										72	83366
		69590	77881	51946	38308	62111	11982	87007	50904	26641	27392
47										3424	83522
		40098	53669	54471	72547	68779	40243	86115	44706	26361	30391
48									1	64468	09110
		06041	87840	60152	09219	74918	57252	67810	16409	45316	09090
49									80	62507	03682
		27218	60142	82965	59317	42716	23933	33754	57054	84141	72839
50									4032	96672	76936
		58890	36142	10938	08808	59685	47971	75030	78168	58457	34752
51								2	05765	21900	19435
		33778	81355	42153	61997	39439	63306	20885	84756	93390	60409
52								107	03985	67349	78651
		61744	28069	71363	85025	44933	81813	48096	08655	89107	05410
53								5675	25075	13866	33831
		27158	19299	47659	39404	93066	88177	80601	40960	60903	31861
54							3	06574	69734	91799	35488
		42199	94397	97715	89238	83812	11946	29345	13169	39005	76112
55							168	67497	76536	19957	88857
		92576	17576	87982	88650	64735	97608	01398	80030	29273	13563
56							9448	97804	17604	12841	09142
		94695	41458	72821	08832	08427	22881	97653	69427	61227	49570
57						5	38766	55699	35481	92625	84146
		19035	75909	00166	11666	74714	78157	33164	78496	20304	86819
58						312	58246	74716	90470	28455	64948
		52151	42090	06485	17961	70756	96110	13984	56829	80382	26128
59						18447	94228	72968	63947	37917	08535
		14181	37324	89815	08658	95856	70696	19067	11401	86377	75277
60					11	07199	89841	10584	13191	32048	20675
		94487	31311	19751	93291	00461	34299	17114	32556	45129	57442
61					675	58267	09933	77006	40277	16176	92091
		00422	09590	14676	72585	11993	77206	51822	90952	23565	78401
62					41897	56666	72062	88667	24148	39418	91007
		40473	50277	33700	66678	14090	58913	27371	86013	61045	83552
63				26	40244	43295	64975	42616	59667	86983	89232
		61742	01966	97316	33064	88618	71601	99260	68529	43996	56223
64				1690	18892	81029	16685	73828	37219	43143	46766
		12623	19720	95291	01110	79691	33663	80938	68220	78795	03874
65			1	09889	52094	38550	08753	98369	25269	57562	74720
		74686	09288	18130	98379	85654	60435	38029	33627	62308	89183

NOTE: $U_{45} = 15825417445$ . . . *etc.*

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University of Alberta

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