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EMBEDDABILITY OF GENERALISED WREATH PRODUCTS

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Abstract

Given two finitely generated groups that coarsely embed into a Hilbert space, it is known that their wreath product also embeds coarsely into a Hilbert space. We introduce a wreath product construction for general metric spaces X, Y, Z and derive a condition, called the (δ-polynomial) path lifting property, such that coarse embeddability of X, Y and Z implies coarse embeddability of $X \wr_Z Y$. We also give bounds on the compression of $X \wr_Z Y$ in terms of δ and the compressions of X, Y and Z.

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1. Introduction

Ever since the recently discovered relations between the Novikov conjecture and coarse embeddability [15], this latter property has been the focal point of much research. Concretely, for a finitely generated group with a word metric relative to a finite generating subset, coarse embeddability into a Hilbert space implies the Novikov conjecture. This result was suggested by Gromov in [10, Problems (4) and (5)] and proved in [15]. Later the same result was proved for embeddings into uniformly convex Banach spaces, providing one of the motivations for studying embeddings into l^p-spaces for $p \neq 2$ [13].

Definition 1.1 (see [11]). Fix $p \geq 1$. A metric space (X, d) is *coarsely embeddable into an L^p-space* if there exist a measure space (Ω, μ), nondecreasing functions $\rho_-, \rho_+ : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \rho_-(t) = +\infty$ and a map $f : X \to L^p(\Omega, \mu)$ such that

$$\rho_-(d(x, x')) \leq \|f(x) - f(x')\|_p \leq \rho_+(d(x, x')) \quad \forall x, x' \in X.$$

The map f is called a *coarse embedding* of X into $L^p(\Omega, \mu)$ and the map ρ_- is called a *compression function* for f. A metric space (X, d_X) is *coarsely embeddable into a Hilbert space* if there exists such a map whose codomain is a Hilbert space. We usually shorten this property to saying that X is CE.

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In 2004, Guentner and Kaminker introduced a numerical invariant that can be used to quantify 'how well' a metric space (X, d) embeds coarsely into a Hilbert space [12]. This links coarse embeddability to the well-studied notion of quasi-isometric embeddability [8].

Definition 1.2. Fix $p \geq 1$. Given a metric space (X, d) and a measure space (Ω, μ), the *L^p-compression* $R(f)$ of a coarse embedding $f : X \to L^p(\Omega, \mu)$ is defined as the supremum of $r \in [0, 1]$ such that

$$\exists C, D > 0 \quad \forall x, x' \in X : \frac{1}{C} d(x, x')^r - D \leq \|f(x) - f(x')\| \leq C d(x, x') + D.$$

If such r does not exist, then we set $R(f) = 0$. The *L^p-compression* $\alpha_p(X)$ of X is defined as the supremum of $R(f)$ taken over all coarse embeddings of X into all possible L^p-spaces.

In the setting of groups, compression is related to interesting group-theoretic properties. For example, it is known that finitely generated groups with nonequivariant compression $>\frac{1}{2}$ satisfy property A (which is equivalent to exactness of the reduced C^*-algebra) [12]. The converse is not true [1]. Other interesting facts occur in the amenable case. If G is an amenable group, then, given a coarse embedding $f : G \to \mathcal{H}$ with compression $R(f)$, one can always find an affine isometric action of the group on a Hilbert space such that the associated 1-cocycle also has compression $R(f)$ [8]. This is related to properties such as Kazhdan's property (T) and the Haagerup property.

Definition 1.3. Let G be a group and (Ω, μ) a measure space. Fix $p \geq 1$. A map $f : G \to L^p(\Omega, \mu)$ is called *G-equivariant* if there is an affine isometric action α of G on $L^p(\Omega, \mu)$ such that for all $g, h \in G : f(gh) = \alpha(g)(f(h))$. A compactly generated, locally compact, second countable group G equipped with the word length metric relative to a compact generating subset is said to satisfy the *Haagerup property* if it admits an equivariant coarse embedding into a Hilbert space.

A lot of effort has gone into studying the behaviour of coarse embeddability and the Haagerup property under group constructions [3, 6]. Li gave a proof that the wreath product of two countable groups with the Haagerup property is again Haagerup [14]. Although using similar ideas, his proof is more concise than that of [7], where the authors proved a more general statement. Instead of looking only at standard wreath products $G \wr H$, de Cornulier *et al.* considered permutational wreath products $G \wr_X H := G^{(X)} \rtimes H$, where X is a countable H-set and H acts on $G^{(X)}$ by shifting indices. They conjectured that the Haagerup property for G and H would imply the Haagerup property for any permutational wreath product $G \wr_X H$, but only proved it in the case where $X = H/L$ with L *co-Haagerup* in H. Here a subgroup $L < H$ is called *co-Haagerup* if there exists a proper G-invariant conditionally negative definite kernel on H/L. It was shown that the above-mentioned conjecture is false and so the choice of X is restricted [4].

The nonequivariant analogue of the Haagerup property is *coarse embeddability*. By the work of Dadarlat and Guentner [5], it follows that $G \wr H$ is coarsely embeddable

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if G is coarsely embeddable and H has property A. It is known that property A implies coarse embeddability into a Hilbert space, but the converse is unknown in the case of finitely generated groups. Li [14] and de Cornulier *et al.* [7] showed that coarse embeddability into a Hilbert space is preserved under wreath products, without referring to property A. Even stronger, for $p \in [1, 2]$, Li proved that the L^p-compression of a wreath product $G \wr H$ is strictly positive whenever G and H have strictly positive L^p-compression [14]. Although coarse embeddability and compression are defined for arbitrary metric spaces, the behaviour of compression and coarse embeddability had not been studied for any type of permutational wreath products.

In this paper, we define a general permutational wreath product $X \wr_Z^C Y$ of arbitrary metric spaces X, Y, Z, where $C \in \mathbb{R}^+$, and we investigate under which conditions the coarse embeddability of X, Y, Z implies coarse embeddability of $X \wr_Z Y$. This leads to the definition of the (δ-polynomial) path lifting property. Precisely, we obtain the following result. Our proof uses similar ideas as in [7] and [14].

Theorem 1.4 (see Theorem 4.6). *Let X, Y, Z be metric spaces and $p : Y \to Z$ be a C-dense bornologous map with the coarse path lifting property. Assume that Y is uniformly discrete and that Z has C-bounded geometry. If X, Y, Z are coarsely embeddable into a Hilbert space, then so is $X \wr_Z^C Y$.*

We also give bounds on the Hilbert space compression of this wreath product in terms of δ and the Hilbert space compression of X, Y and Z. The bounds we obtain coincide with the bounds given in [14] when applied to standard wreath products.

2. Preliminaries

Given two finitely generated groups G and H, the wreath product, written as $G \wr H$, is the set of pairs $(\mathbf{f}, h)$, where $h \in H$ and $\mathbf{f} : H \to G$ is a finitely supported function (that is, $\mathbf{f}(h) = e_G$ for all but finitely many $h \in H$) together with a group operation

$$(\mathbf{f}, h) \cdot (\mathbf{g}, h') = (\mathbf{f} \cdot (h\mathbf{g}), hh'),$$

where $(h\mathbf{g})(z) = \mathbf{g}(h^{-1}z)$ for all $z \in G$. One can think of $G \wr H$ as being the semi-direct product $\bigoplus_H G \rtimes H$, where H acts on $\bigoplus_H G$ by permuting the indices. If finite sets S and T generate G and H, respectively, then $G \wr H$ is generated by the finite set

$$\{(\mathbf{e}, t) : t \in T\} \cup \{(\delta_s, e_H) : s \in S\},$$

where $\mathbf{e}(h) = e_G$ for all $h \in H$ and

$$\delta_s(h) = \begin{cases} s & \text{if } h = e_H, \\ e_G & \text{otherwise.} \end{cases}$$

The word metric on $G \wr H$ coming from this generating set can be thought of as follows. Given two elements $(\mathbf{f}, x)$ and $(\mathbf{g}, y)$, take the shortest path in the Cayley

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graph $\mathrm{Cay}(H,T)$ going from x to y that passes through the points in $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) = \{h_1,\dots,h_n\}$. At each point $h_i \in \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$, travel from $\mathbf{f}(h_i)$ to $\mathbf{g}(h_i)$ in G. Explicitly, for $(\mathbf{f},x),(\mathbf{g},y) \in \bigoplus_{g\in H} G \rtimes H$ and $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) = \{h_1,\dots,h_n\}$, define

$$p_{(x,y)}(\mathbf{f},\mathbf{g}) = \inf_{\sigma\in S_n}\Big(d_H(x,h_{\sigma(1)}) + \sum_{i=1}^{n} d_H(h_{\sigma(i)},h_{\sigma(i+1)}) + d_H(h_{\sigma(n)},y)\Big),$$

where the infimum is taken over all permutations in S_n. The number $p_{(x,y)}(\mathbf{f},\mathbf{g})$ corresponds to the shortest path between x and y in H going through each element in $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$. Hence, the distance between $(\mathbf{f},x)$ and $(\mathbf{g},y)$ is

$$d_{G\wr H}((\mathbf{f},x),(\mathbf{g},y)) = p_{(x,y)}(\mathbf{f},\mathbf{g}) + \sum_{h\in H} d_G(\mathbf{f}(h),\mathbf{g}(h)).$$

Suppose that G,H are groups and H acts transitively on a set X. Fix a base point $x_0 \in X$ and define the *permutational wreath product* to be the group $G \wr_X H := \bigoplus_X G \rtimes H$, where

$$\bigoplus_X G = \{\mathbf{f} : X \to G : \mathbf{f}(x) = e_G \text{ for all but finitely many } x \in X\}$$

and H acts on $\bigoplus_X G$ by permuting the indices. If S and T generate G and H, respectively, then $G \wr_X H$ is generated by

$$\{(\mathbf{e},t) : t \in T\} \cup \{(\delta_s, e_H) : s \in S\},$$

where $\mathbf{e}(x) = e_G$ for all $x \in X$ and

$$\delta_s(x) = \begin{cases} s & \text{if } x = x_0,\\ e_G & \text{otherwise.}\end{cases}$$

The metric on $G \wr_X H$ from the generating set can be thought of as follows. Given two elements $(\mathbf{f},x)$ and $(\mathbf{g},y)$, take the shortest path going from x to y in $\mathrm{Cay}(H,T)$ that passes through points $\{h_1,\dots,h_n\}$ such that $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) = \{h_1x_0,\dots,h_nx_0\}$. At each element $h_i \in \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$, travel from $\mathbf{f}(h_ix_0)$ to $\mathbf{g}(h_ix_0)$ in G. In general, the shortest path is not necessarily unique.

Explicitly, for $(\mathbf{f},x),(\mathbf{g},y) \in \bigoplus_{x\in X} G \rtimes H$, let $I = \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$ and let $n = |\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})|$. Define $\mathcal{P}_I$ to be the set

$$\mathcal{P}_I := \{(h_1,\dots,h_n) \subset H^n : \{h_1x_0,\dots,h_nx_0\} = I\}.$$

In particular, if $(h_1,\dots,h_n) \in \mathcal{P}_I$, then any permutation of $(h_1,\dots,h_n)$ is also in $\mathcal{P}_I$. Hence, the length of the shortest path between x and y in H passing though the points that project onto $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$ is precisely

$$\rho_{(x,y)}(\mathbf{f},\mathbf{g}) := \inf_{(h_1,\dots,h_n)\in\mathcal{P}_I}\Big(d(x,h_1) + \sum_{i=1}^{n-1} d(h_i,h_{i+1}) + d(h_n,y)\Big).$$

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Hence, the distance between $(\mathbf{f}, x)$ and $(\mathbf{g}, y)$ is

$$d_{G\wr_X H}((\mathbf{f}, x), (\mathbf{g}, y)) = \rho_{(x,y)}(\mathbf{f}, \mathbf{g}) + \sum_{z\in X} d_G(\mathbf{f}(z), \mathbf{g}(z)).$$

One can ask whether we can generalise this construction. Suppose that Y and Z are metric spaces and $p : Y \to Z$ is a C-dense map, that is, $B_Z(p(Y), C) = Z$. Given two points $y, y' \in Y$ and a finite sequence of points $I = \{z_1, \dots, z_n\}$ in Z, we define $\mathcal{P}_I$ to be the set

$$\mathcal{P}_I := \{(y_1, \dots, y_n) \subset Y^n : \exists\sigma \in S_n \text{ such that } \forall i, p(y_i) \in B(z_{\sigma(i)}, C)\}.$$

In particular, if $(y_1, \dots, y_n) \in \mathcal{P}_I$, then any permutation of $(y_1, \dots, y_n)$ also lies in $\mathcal{P}_I$. We now define the *length of the path from y to y' going through I* by

$$\mathrm{path}_I(y, y') = \inf_{(y_1,\dots,y_n)\in\mathcal{P}_I} \Big(d_Y(y, y_1) + \sum_{i=1}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y_n, y')\Big).$$

Let X be another metric space and fix a base point $x_0 \in X$. Define $\bigoplus_Z X$ to be the set

$$\bigoplus_Z X = \{\mathbf{f} : Z \to X : \mathbf{f}(z) = x_0 \text{ for all but finitely many } z \in Z\}.$$

For $\mathbf{f}, \mathbf{g} \in \bigoplus_Z X$, define $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) = (\mathrm{Supp}(\mathbf{f}) \cup \mathrm{Supp}(\mathbf{g}))\backslash\{z \in Z : \mathbf{f}(z) = \mathbf{g}(z)\}$. Let $(\mathbf{f}, y), (\mathbf{g}, y') \in \bigoplus_Z X \times Y$ and let $I = \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$. Define a metric on the set $\bigoplus_Z X \times Y$ by

$$d((\mathbf{f}, y), (\mathbf{g}, y')) = \mathrm{path}_I(y, y') + \sum_{z\in Z} d_X(\mathbf{f}(z), \mathbf{g}(z)).$$

We obtain a metric space $(\bigoplus_Z X \times Y, d)$, which we denote by $X \wr_Z^C Y$. When there is no risk for confusion, we will omit C from this notation. When X and Y are graphs, the metric wreath product $X \wr_Y Y$ coincides with the wreath product of graphs. See [9, Definition 2.1].

3. Measured walls

Let X be a set and 2^X the power set of X. We endow 2^X with the product topology. For $x \in X$, denote $\mathcal{A}_x = \{A \subset X : x \in A\}$. This is a clopen subset in 2^X. For two elements $x, y \in X$, we say that a set $A \subset X$ *cuts* x and y, denoted $A \vdash \{x, y\}$, if $x \in A$ and $y \in A^c$ or $x \in A^c$ and $y \in A$. Likewise, we say that A cuts another set Y if neither $Y \subset A$ nor $Y \subset A^c$.

Definition 3.1. A *measured walls structure* on a set X is a Borel measure μ on 2^X such that, for every $x, y \in X$,

$$d_\mu(x, y) := \mu(\{A \in 2^X : A \vdash \{x, y\}\}) < \infty.$$

Since $\{A \in 2^X : A \vdash \{x, y\}\} = \mathcal{A}_x \vartriangle \mathcal{A}_y$, the set is measurable. It follows that d_μ is well defined and is a pseudometric on X, called the *wall metric associated to μ*.

If $f : X \to Y$ is a map between sets and (Y, μ) is a measured walls structure, then we can push forward the measure μ by the inverse image map $f^{-1} : 2^Y \to 2^X$ and obtain a measured walls structure $(X, f^*\mu)$, where, for $A \subset 2^X$, $f^*\mu(A) = \mu(\{f(B) \mid B \in A, B = f^{-1}(f(B))\})$. It follows that $d_{f^*\mu}(x, x') = d_\mu(f(x), f(x'))$.

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Given a family of spaces X_i with measured walls space structures μ_i and the natural projection maps $p_i : \bigoplus X_j \to X_i$, the measure $\mu = \sum_I p_i^* \mu_i$ defines a measured walls space structure on $\bigoplus_i X_i$. The associated wall metric is $d_\mu((x_i),(y_i)) = \sum_i d_{\mu_i}(x_i, y_i)$.

Definition 3.2. Let X be a set. A function $k : X \times X \to \mathbb{R}_+$ is a *conditionally negative definite kernel* if $k(x,x) = 0$ and $k(x,y) = k(y,x)$ for all $x, y \in X$ and for every pair of finite sequences $x_1, \ldots, x_n \in X$, $\lambda_1, \ldots, \lambda_n$ real numbers such that $\sum_{i=1}^n \lambda_i = 0$,

$$\sum_{i,j} \lambda_i \lambda_j k(x_i, x_j) \leq 0.$$

Proposition 3.3 ([2, Proposition 6.16], see also [7, Proposition 2.6]). *Let X be a set and $k : X \times X \to \mathbb{R}_+$. The following are equivalent:*

(1) *there exists $f : X \to L^1(X)$ such that $k(x,y) = \|f(x) - f(y)\|_1$ for all $x, y \in X$;*
(2) *for every $p \geq 1$, there exists $f : X \to L^p(X)$ such that $(k(x,y))^{1/p} = \|f(x) - f(y)\|_p$ for all $x, y \in X$;*
(3) *$k = d_\mu$ for some measured walls structure (X, μ).*

In order to prove our main result, we make use of a method of lifting measured walls structures. First we require some technical definitions. Let W, X be sets and $\mathcal{A} = 2^{(X)}$, the set of finite subsets of X.

Definition 3.4. An $\mathcal{A}$-gauge on W is a function $\phi : W \times W \to \mathcal{A}$ such that

$$\phi(w, w') = \phi(w', w) \quad \forall w, w' \in W,$$
$$\phi(w, w'') \subset \phi(w, w') \cup \phi(w', w'') \quad \forall w, w', w'' \in W.$$

If W is a group, then ϕ is called *left invariant* if $\phi(ww', ww'') = \phi(w', w'')$ for all $w, w', w'' \in W$.

Definition 3.5. Let G be a group and X a G-set. A measured walls structure (X, μ) is *uniform* if for all $x, y \in X$ the map $g \mapsto d_\mu(gx, gy)$ is bounded on G.

Theorem 3.6 [7, Theorem 4.2]. *Let X, W be sets, $\mathcal{A} = 2^{(X)}$. Let ϕ be an $\mathcal{A}$-gauge on W and assume that $\phi(w,w) = \emptyset$ for all $w \in W$. Let (X, μ) be a measured walls structure. Then there is a naturally defined measure $\widetilde{\mu}$ on $2^{W \times X}$ such that $(W \times X, \widetilde{\mu})$ is a measured walls structure with corresponding pseudometric*

$$d_{\widetilde{\mu}}(w_1 x_1, w_2 x_2) = \mu(\{A \in \mathcal{A} : A \vdash \phi(w_1, w_2) \cup \{x_1, x_2\}\}).$$

A consequence of this theorem is that if X, Y, Z are metric spaces, where X has a fixed point $x_0 \in X$, then $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$ is an $\mathcal{A}$-gauge on $\bigoplus_Z X$, where $\mathcal{A} = 2^{(Z)}$. Hence, if Z has a measured walls structure, there exists a lifted measured walls structure on $\bigoplus_Z X \times Z$.

4. Coarse embeddings of wreath products

Definition 4.1. A metric space (X, d) is *uniformly discrete* if there exists $\delta > 0$ such that for all $x \in X$, $B(x, \delta) = \{x\}$. We say that a metric space has C-bounded geometry

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for some $C > 0$ if there exists a constant $N(C) > 0$ such that $|B(x, C)| \le N(C)$ for all $x \in X$. A metric space has *bounded geometry* if it has C-bounded geometry for every $C > 0$.

Example 4.2. Note that C-bounded geometry for some C does not in general imply bounded geometry. As an easy example, one can consider an infinite metric space equipped with the discrete metric, that is, $d(x, y) = 1$ for every $x, y \in X$ distinct.

Definition 4.3. Let Y and Z be metric spaces. A C-dense map $p : Y \to Z$ has the *coarse path lifting property* if there exists a nondecreasing function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for any $z, z' \in Z$ and $y \in Y$ with $d_Y(p(y), z) \le C$, there exists a $y' \in Y$ with $d(p(y'), z') \le C$ and $d(y, y') \le \theta(d(z, z'))$.

Definition 4.4. A map between metric spaces $f : Y \to Z$ is *bornologous* if for every $R > 0$ there is $S_R > 0$ such that if $d_Y(y, y') \le R$, then $d_Z(f(y), f(y')) \le S_R$ for all $y, y' \in Y$.

Example 4.5. The path lifting property occurs naturally in the setting of groups. Let $Y = H$ be a group and let $N \triangleleft H$. The most natural way of defining a distance function on $Z := H/N$ is by setting $d(hN, h'N)$ to be the infimum of $d(hn, h'n')$ over all $n, n' \in N$. The projection map $p : H \to H/N$ is a bornologous map and one checks easily that it satisfies the coarse path lifting property. Actually, one only needs the fact that N is 'almost normal' in H, that is, that for every finite subset F of H, there exists a finite subset $F' \subset H$ with $NF \subset F'N$.

Another example can be obtained by taking Z to be the set of right N-cosets of H, where N is any (not necessarily normal) subgroup of H. In this case, the projection map $p : H \to N\backslash H, g \mapsto Ng$ is a bornologous map that has the coarse path lifting property.

Theorem 4.6. *Let X, Y, Z be metric spaces and $p : Y \to Z$ be a C-dense bornologous map with the coarse path lifting property. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function satisfying the properties in Definition 4.3. Assume that Y is uniformly discrete and that Z has C-bounded geometry. If X, Y, Z are coarsely embeddable into an L^1-space, then so is $X \wr_Z^C Y$.*

Remark 4.7. Note that, by Proposition 3.3, the conclusion of the theorem also implies L^p-embeddability of $X \wr_Z Y$ for any $p \ge 1$. On the other hand, it is known that L^p embeds isometrically into L^1 for $1 \le p \le 2$. Hence, in the formulation of Theorem 4.6, we can just as well replace L^1-embeddability by L^p-embeddability for $1 \le p \le 2$.

Proof of Theorem 4.6. By Proposition 3.3, there exist measured walls structures (X, σ), (Y, ν), (Z, μ) and functions $\rho_X, \rho_Y, \rho_Z, \eta_X, \eta_Y, \eta_Z : \mathbb{R}^+ \to \mathbb{R}^+$, increasing to infinity, such that

$$\rho_X(d_X(x_1, x_2)) \le d_\sigma(x_1, x_2) \le \eta_X(d_X(x_1, x_2)) \quad \forall x_1, x_2 \in X, \tag{4.1}$$

$$\rho_Y(d_Y(y_1, y_2)) \le d_\nu(y_1, y_2) \le \eta_Y(d_Y(y_1, y_2)) \quad \forall y_1, y_2 \in Y, \tag{4.2}$$

$$\rho_Z(d_Z(z_1, z_2)) \le d_\mu(z_1, z_2) \le \eta_Z(d_Z(z_1, z_2)) \quad \forall z_1, z_2 \in Z. \tag{4.3}$$

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By Theorem 3.6, there exists a measured walls structure $\widetilde{\mu}$ on $\bigoplus_Z X \times Z$, where, for $(\mathbf{f}, z), (\mathbf{g}, z') \in \bigoplus_Z X \times Z$,

$$d_{\widetilde{\mu}}((\mathbf{f}, z), (\mathbf{g}, z')) = \mu(\{A : A \vdash \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) \cup \{z, z'\}\}).$$

We have a projection map $p : \bigoplus_Z X \times Y \to \bigoplus_Z X \times Z$, where $(\mathbf{f}, y) \mapsto (\mathbf{f}, p(y))$. Using this, we can pull back a measured walls structure on $\bigoplus_Z X \times Y$, where

$$d_{p\widetilde{\mu}}((\mathbf{f}, y), (\mathbf{g}, y')) = d_{\widetilde{\mu}}((\mathbf{f}, p(y)), (\mathbf{g}, p(y'))).$$

We define three other walls structures, $\widetilde{\sigma}, \widetilde{\nu}$ and $\widetilde{\omega}$, on $X \wr_Z Y$, where

$$d_{\widetilde{\sigma}}((\mathbf{f}, y), (\mathbf{g}, y')) = \sum_{z \in Z} d_\sigma(\mathbf{f}(z), \mathbf{g}(z)),$$
$$d_{\widetilde{\nu}}((\mathbf{f}, y), (\mathbf{g}, y')) = d_\nu(y, y'),$$
$$d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y')) = |\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})|.$$

It is clear from our comments in Section 3 on pushing forward and summing up walls space structures that $\widetilde{\sigma}$ and $\widetilde{\nu}$ are indeed walls space structures. To see that the latter is associated to a measured walls space structure, note that $d_{\widetilde{\omega}}$ is associated as in Proposition 3.3 to the map $\Lambda : \bigoplus_Z X \times Y \to L^1(X \times Z), (\mathbf{f}, y) \mapsto \Lambda(\mathbf{f}, y)$, where

$$\Lambda(\mathbf{f}, y) : (x, z) \mapsto \begin{cases} \frac{1}{2} & \text{if } f(z) = x, \\ 0 & \text{if } f(z) \neq x. \end{cases}$$

We now aim to show that we can coarsely embed $X \wr_Z Y$ into an L^1-space. Define $\lambda = p\widetilde{\mu} + \widetilde{\sigma} + \widetilde{\nu} + \widetilde{\omega}$ to be a measured walls space structure on $X \wr_Z Y$. By Proposition 3.3, it suffices to show that for every $R > 0$, if $d_\lambda((\mathbf{f}, y), (\mathbf{g}, y')) \le R$, then $d_{X \wr_Z Y}((\mathbf{f}, y), (\mathbf{g}, y')) \le C_1(R)$ and, if $d_{X \wr_Z Y}((\mathbf{f}, y), (\mathbf{g}, y')) \le R$, then $d_\lambda((\mathbf{f}, y), (\mathbf{g}, y')) \le C_2(R)$, where C_1, C_2 are constants depending only on R.

Fix $R > 0$ and suppose that $d_\lambda((\mathbf{f}, y), (\mathbf{g}, y')) \le R$. In particular,

$$d_{\widetilde{\mu}}((\mathbf{f}, p(y)), (\mathbf{g}, p(y'))) \le R, \tag{4.4}$$
$$\sum_{z \in Z} d_\sigma((\mathbf{f}(z), \mathbf{g}(z)) \le R, \tag{4.5}$$
$$d_\nu(y, y') \le R, \tag{4.6}$$
$$|\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})| \le R. \tag{4.7}$$

Define $p(y) = z_0$ and write $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) = \{z_1, z_2, \ldots, z_n\}$ for some $n \le R$. By (4.4), it follows that $\mu(A : A \vdash \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) \cup \{p(y), p(y')\}) \le R$. In particular, $d_\mu(z_i, z_j) \le R$ for all $z_i, z_j \in \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) \cup \{p(y), p(y')\}$. By (4.3), this implies that $d_Z(z_i, z_j) \le \rho_Z^{-1}(R)$ for all z_i, z_j. Starting from $y_0 = y$, by the path lifting property, we can find y_1 such that $d_Z(p(y_1), z_1) \le C$ and $d_Y(y, y_1) \le \theta(\rho_Z^{-1}(R))$. We can then find y_2 with $d_Z(p(y_2), z_2) \le C$ and $d_Y(y_1, y_2) \le \theta(\rho_Z^{-1}(R))$. Continuing inductively and by the triangle inequality,

$$\sum_{i=0}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y_n, y_0) \le 2 \sum_{i=0}^{n-1} \theta(\rho_Z^{-1}(R)) \le 2R\theta(\rho_Z^{-1}(R)).$$

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Using (4.6) and denoting $y_0 = y$,

$$\mathrm{path}_I(y, y') \leq \sum_{i=0}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y_n, y_0) + d_Y(y, y') \leq 2R\theta(\rho_Z^{-1}(R)) + \rho_Y^{-1}(R). \quad (4.8)$$

Now we can deduce that

$$\begin{aligned} &\mathrm{path}_I(y, y') + \sum_{z\in Z} d_X((\mathbf{f}(z), \mathbf{g}(z)) \\ &\quad \leq 2R\theta(\rho_Z^{-1}(R)) + \rho_Y^{-1}(R) + \sum_{z\in \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})} \rho_X^{-1}(R) \quad \text{by (4.1), (4.5) and (4.8)} \\ &\quad \leq 2R\theta(\rho_Z^{-1}(R)) + \rho_Y^{-1}(R) + R\rho_X^{-1}(R) \quad \text{by (4.7).} \end{aligned}$$

It suffices to set $C_1(R) = 2R\theta(\rho_Z^{-1}(R)) + \rho_Y^{-1}(R) + R\rho_X^{-1}(R)$.

Now suppose conversely that $d_{X\wr_Z Y}((\mathbf{f}, y), (\mathbf{g}, y')) \leq R$. In particular,

$$\mathrm{path}_I(y, y') \leq R, \quad (4.9)$$

$$\sum_{z\in Z} d_X(f(z), g(z)) \leq R. \quad (4.10)$$

Let $(y_1, \ldots, y_n) \in \mathcal{P}_I$ be such that

$$d_Y(y, y_1) + \sum_{i=1}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y_n, y') \leq R + 1. \quad (4.11)$$

As Y is uniformly discrete, we have $\delta_Y := \inf(d(a, b) \mid a, b \in Y) > 0$. This implies that, although some of the y_i may be equal, the number of distinct y_i is bounded by $(R + 1)/\delta_Y$. Any point in the support of $\mathbf{f}^{-1}\mathbf{g}$ lies, by definition, in a C-neighbourhood of some $p(y_i)$. As such neighbourhoods contain at most $N(C)$ elements, we can conclude that

$$n = |\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})| \leq E(R) := N(C)\frac{R + 1}{\delta_Y}. \quad (4.12)$$

From (4.11) and the triangle inequality, it follows that

$$d_Y(a, b) \leq R + 1 \quad \forall a, b \in \{y, y', y_1, \ldots, y_n\}.$$

As p is bornologous, there exists $S = S(R + 1)$ such that for all $z, z' \in \{p(y), p(y'), p(y_1), \ldots, p(y_n)\}$, we have $d_Z(z, z') \leq S$. By definition of $(y_1, \ldots, y_n)$ and the triangle inequality, it follows that $d_Z(z, z') \leq S + 2C$ for every $z, z' \in \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) \cup \{p(y), p(y')\}$. By (4.3), it follows that

$$d_\mu(z, z') \leq \eta_Z(S + 2C) \quad \forall z, z' \in \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) \cup \{p(y), p(y')\}. \quad (4.13)$$

Let us enumerate $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) \sqcup \{p(y), p(y')\} = \{p(y) = z_0, z_1, \ldots, z_{n+1} = p(y')\}$. Note that, if A cuts $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) \cup \{p(y), p(y')\}$, then A must cut $\{z_i, z_{i+1}\}$ for some $i \in \{0, 1, \ldots, m - 1\}$. Hence, $d_{p\widetilde{\mu}}((\mathbf{f}, y), (\mathbf{g}, y')) \leq \sum_{i=0}^{n} d_\mu(z_i, z_{i+1})$.

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It now follows by (4.2), (4.10), (4.13) and (4.12) that

$$\begin{aligned} d_\lambda((\mathbf{f}, y), (\mathbf{g}, y')) &= (d_{\widetilde{\nu}} + d_{p\widetilde{\mu}} + d_{\widetilde{\sigma}} + d_{\widetilde{\omega}})((\mathbf{f}, y), (\mathbf{g}, y')) \\ &\leq d_\nu(y, y') + \sum_{i=0}^{n} d_\mu(z_i, z_{i+1}) + \sum_{z\in Z} d_\sigma(\mathbf{f}(z), \mathbf{g}(z)) + d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y')) \\ &\leq \eta_Y(R) + \sum_{i=0}^{n} \eta_Z(S + 2C) + \sum_{z\in \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})} \eta_X(R) + E(R) \\ &\leq \eta_Y(R) + E(R)\eta_Z(S + 2C) + E(R)\eta_X(R) + E(R). \end{aligned}$$

Hence, it suffices to set $C_2(R) := \eta_Y(R) + E(R)(\eta_Z(S + 2C) + \eta_X(R) + 1)$. This shows by Proposition 3.3 that $X \wr_Z Y$ embeds coarsely into an L^p-space. □

REMARK 4.8. The only time we used the condition that Y is uniformly discrete was to show that (4.9) and (4.10) imply that $|\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})|$ is bounded by some function of R. One checks easily that this condition can be replaced by the condition that Y has bounded geometry. Alternatively, it would also be sufficient to require nothing on Y and Z but to ask that X is a uniformly discrete metric space.

5. The compression of $X \wr_Z Y$ in terms of X, Y, Z

We can modify the previous proof to give information on the L^1-compression of $X \wr_Z Y$ in terms of the growth behaviour of θ and the L^1-compression of X, Y and Z.

DEFINITION 5.1. Let Y and Z be metric spaces and let $p : Y \to Z$ be a C-dense map with the coarse path lifting property, that is, there exists a nondecreasing function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $z, z' \in Z$ and $y \in Y$ with $d_Y(p(y), z) \leq C$, there exists a $y' \in Y$ with $d(p(y'), z') \leq C$ and $d(y, y') \leq \theta(d(z, z'))$. If $\delta > 0$ is such that $\theta(r) \lesssim r^\delta + 1$ for every $r \in \mathbb{R}^+$, then we say that p has the *δ-polynomial path lifting property*. We say that p has the polynomial path lifting property if it has the *δ-polynomial path lifting property* for some $\delta > 0$.

THEOREM 5.2. *Let X, Y, Z be metric spaces and $p : Y \to Z$ be a C-dense bornologous map with the coarse path lifting property, where Y is uniformly discrete and Z has C-bounded geometry. Let $\theta : \mathbb{R}^+ \to \mathbb{R}$ be a nondecreasing function satisfying the properties in Definition 4.3. Suppose that there are constants $a, b > 0$ such that $d_Z(p(y), p(y')) \leq a d_Y(y, y') + b$ for every $y, y' \in Y$. If p has the δ-polynomial path lifting property for some $\delta > 0$ and if X, Y, Z have L^1-compression equal to α, β, γ, respectively, then the L^1-compression of $X \wr_Z^C Y$ is bounded from below by* $\min(\alpha, \beta, \gamma/(\gamma + \delta))$.

REMARK 5.3. Our bound generalises the bound of [14, Theorem 1.1]. Note further that, as both X and Y can be considered as metric subspaces of $X \wr_Z Y$, one also has an upper bound, namely $\min(\alpha, \beta)$, for the compression of $X \wr_Z Y$. As in the previous remark, we can replace the assumption on Y by supposing that Y has bounded geometry or requiring nothing of Y and Z and assuming that X is uniformly discrete.

PROOF OF THEOREM 5.2. The starting point for this proof is the proof of Theorem 4.6 and we will often refer to inequalities stated there. For now, assume that α, β, γ are real numbers and that $f_1 : X \to L^1$, $f_2 : X \to L^1$ and $f_3 : Z \to L^1$ are large-scale Lipschitz functions into L^1-spaces such that

$$d_X(x,x')^\alpha \lesssim \|f_1(x) - f_1(x')\|_1,$$
$$d_Y(y,y')^\beta \lesssim \|f_2(y) - f_2(y')\|_1,$$
$$d_Z(z,z')^\gamma \lesssim \|f_3(z) - f_3(z')\|_1.$$

Here $\lesssim$ denotes inequality up to a multiplicative constant. The reason that we can take the lower bounds as above is that by taking the direct sum of f_i with the coarse embedding $\widetilde{f_i} : W \to l^2(W), w \mapsto \delta_w$, where $W = X, Y$ or Z, we can always assume that $\|f_1(w) - f_1(w')\|_1 \geq 1$ for distinct w, w'.

Let d_σ, d_ν and d_μ be the measured walls space structures associated to the functions f_1, f_2, f_3 by Proposition 3.3. Define the measured walls $d_{p\widetilde{\mu}}, d_{\widetilde{\mu}}, d_{\widetilde{\sigma}}, d_{\widetilde{\omega}}$ on $X \wr_Z Y$ as in Theorem 4.6. As a first step, we are going to show that the function associated to the measured wall $d_\lambda = d_{p\widetilde{\mu}} + d_{\widetilde{\mu}} + d_{\widetilde{\sigma}} + d_{\widetilde{\omega}}$ is Lipschitz. That is, there is a constant $\widetilde{C} \in \mathbb{R}$ such that for every $(\mathbf{f}, y), (\mathbf{g}, y') \in X \wr_Z Y$,

$$\begin{aligned} &d_{p\widetilde{\mu}}((\mathbf{f},y),(\mathbf{g},y')) + d_{\widetilde{\nu}}((\mathbf{f},y),(\mathbf{g},y')) + d_{\widetilde{\sigma}}((\mathbf{f},y),(\mathbf{g},y')) + d_{\widetilde{\omega}}(((\mathbf{f},y),(\mathbf{g},y')) \\ &\quad \leq \widetilde{C} d_{X\wr_Z Y}((\mathbf{f},y),(\mathbf{g},y')). \end{aligned}$$

By (4.12), it follows that $d_{\widetilde{\omega}}$ corresponds to a large-scale Lipschitz function if Y is uniformly discrete and Z has C-bounded geometry. Starting from (4.9) and (4.10), one can easily show the same fact using only uniform discreteness of X.

As d_ν and d_σ both correspond to large-scale Lipschitz functions, this implies that so does $d_{\widetilde{\nu}} + d_{\widetilde{\sigma}}$:

$$\begin{aligned} d_{\widetilde{\nu}}((\mathbf{f},y),(\mathbf{g},y')) + d_{\widetilde{\sigma}}((\mathbf{f},y),(\mathbf{g},y')) &= d_\nu(y,y') + \sum_{z\in Z} d_\sigma(\mathbf{f}(z),\mathbf{g}(z)) \\ &\lesssim d_Y(y,y') + 1 + \sum_{z\in Z} d_X(\mathbf{f}(z),\mathbf{g}(z)) + d_{\widetilde{\omega}}((\mathbf{f},y),(\mathbf{g},y')) \\ &\lesssim d_{X\wr_Z Y}((\mathbf{f},y),(\mathbf{g},y')) + 1. \end{aligned}$$

It thus remains to show that $d_{p\widetilde{\mu}}$ corresponds to a Lipschitz function. Denote $y_0 = y, y_{n+1} = y'$ and choose $(y_1, \ldots, y_n) \in \mathcal{P}_I$ such that

$$\text{path}_I(y,y') \leq \sum_{i=0}^{n} d_Y(y_i, y_{i+1}) \leq \text{path}_I(y,y') + 1.$$

Write $z_0 = p(y), z_{n+1} = p(y')$ and enumerate the elements of $\text{Supp}(\mathbf{f}^{-1}\mathbf{g})$ as $\{z_1, z_2, \ldots, z_n\}$, where each z_i lies in a C-ball around $p(y_i)$. As p is bornologous, we

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have that $d_Z(z_i, z_{i+1}) \leq 2C + ad(y_i, y_{i+1}) + b$ for each i. Hence,

$$
\begin{aligned}
d_{p\widetilde{\mu}}((\mathbf{f}, y), (\mathbf{g}, y')) &\leq \sum_{i=0}^{n} d_\mu(z_i, z_{i+1}) \lesssim \sum_{i=0}^{n} d_Z(z_i, z_{i+1}) + d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y')) \\
&\leq n(2C + b) + a \sum_{i=0}^{n} d_Y(y_i, y_{i+1}) + d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y')) \\
&= d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y'))(2C + b + 1) + a \sum_{i=0}^{n} d_Y(y_i, y_{i+1}) \\
&\leq d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y'))(2C + b + 1) + a + a \,\mathrm{path}_I(y, y') \\
&\lesssim d_{X \wr_Z Y}((\mathbf{f}, y), (\mathbf{g}, y')) + 1,
\end{aligned}
$$

where we used that $d_{\widetilde{\omega}}$ corresponds to a large-scale Lipschitz function. We conclude that d_λ is associated to a large-scale Lipschitz map of $X \wr_Z Y$ into an L^1-space.

As a second step, we calculate the compression of d_λ. Assume first that $d_\lambda((\mathbf{f}, y), (\mathbf{g}, y')) \leq R$ for some $R > 0$ such that (4.4), (4.5), (4.6) and (4.7) are valid. Enumerate the elements of $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$, say $z_1, z_2, \ldots, z_n$. Set $z_0 = p(y)$. Denote $y_0 = y$ and then use the path lifting property to take y_1 such that $d_Z(p(y_1), z_1) < C$ and $d(y_0, y_1) \leq ad(z_0, z_1)^\delta + b$. Next take y_2 such that $d_Z(p(y_2), z_2) < C$ and such that $d(y_1, y_2) \leq ad_Z(z_1, z_2)^\delta + b$ and so on. By definition,

$$
\mathrm{path}_I(y, y') \leq \left(\sum_{i=0}^{n-1} d_Y(y_i, y_{i+1}) \right) + d_Y(y_n, y').
$$

We now obtain

$$
\begin{aligned}
\mathrm{path}_I(y, y') &\leq \sum_{i=0}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y_n, y') \lesssim \sum_{i=0}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y, y') \\
&\lesssim \sum_{i=0}^{n-1} (d_Z(z_i, z_{i+1})^\delta + 1) + d_Y(y, y') \\
&\lesssim R + \sum_{i=0}^{n-1} d_Z(z_i, z_{i+1})^\delta + d_\nu(y, y')^{1/\beta} \\
&\leq R + \sum_{i=0}^{n-1} d_Z(z_i, z_{i+1})^\delta + R^{1/\beta} \\
&\lesssim R + \sum_{i=0}^{n-1} d_\mu(z_i, z_{i+1})^{\delta/\gamma} + R^{1/\beta} \\
&\lesssim R + RR^{\delta/\gamma} + R^{1/\beta},
\end{aligned}
$$

where the last inequality follows from the fact that

$$
d_\mu(z_i, z_{i+1}) \leq d_{p\widetilde{\mu}}((\mathbf{f}, y), (\mathbf{g}, y')) \leq R.
$$

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Consequently,

$$\begin{aligned} d_{X\wr_Z Y}((\mathbf{f}, y), (\mathbf{g}, y')) &= \mathrm{path}_I(y, y') + \sum_{z\in Z} d_X(f(z), g(z)) \\ &\lesssim R^{(\delta/\gamma)+1} + R^{1/\beta} + \sum_{z\in Z} d_\sigma(f(z), g(z))^{1/\alpha} \\ &\lesssim R^{(\delta+\gamma)/\gamma} + R^{1/\beta} + \Big(\sum_{z\in Z} d_\sigma(f(z), g(z))\Big)^{1/\alpha} \lesssim R^X, \end{aligned}$$

where $X = \max((\delta+\gamma)/\gamma, 1/\alpha, 1/\beta)$. Consequently, the compression of d_λ, and hence of $X \wr_Z Y$, is bounded from below by

$$\min\Big(\alpha, \beta, \frac{\gamma}{\delta+\gamma}\Big). \qquad \square$$

REMARK 5.4. At the end of [14, Section 2], the author showed that the L^p-compression $\alpha_p^*(X)$ of a metric space X is always greater than $\max(\frac{1}{2}, 1/p)\alpha_1^*(X)$. Moreover, L^p embeds isometrically into L^1 for any $p \in [1,2]$. So, for $p \in [1,2]$, we deduce that the positivity of the L^p-compression is preserved under generalised wreath products with the polynomial path lifting property.

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