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Differentiability Properties of Optimal Value Functions

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Abstract. Differentiability properties of optimal value functions associated with perturbed optimization problems require strong assumptions. We consider such a set of assumptions which does not use compactness hypothesis but which involves a kind of coherence property. Moreover, a strict differentiability property is obtained by using techniques of Ekeland and Lebourg and a result of Preiss. Such a strengthening is required in order to obtain genericity results.

1 Introduction

Optimal value functions of optimization problems depending on parameters are of excruciating importance in analysis and optimization [2], [4], [5], [15], [22], [35]. Distance functions are of this type and many results of game theory and optimal control theory rely on their study (*e.g.*, [24], [42]); moreover they play an important role for bilevel programming (*e.g.*, [25], [43]) and for the solvability of Hamilton-Jacobi equations (*e.g.*, [3], [40]). Value functions are seldom differentiable. For this reason, strong assumptions have to be made in order to get differentiability or subdifferentiability properties. The usual ones are compactness or coercivity assumptions [5], [10], [12], [39]. Here our assumptions are of a different kind: as in [14], [32], we use the simple idea that stability or smoothness properties for the value function are ensured by a certain coherence of the variations of the objective function as the parameter changes. Of course, we have to be careful in imposing such a condition; otherwise some situations would be ruled out. Moreover, we have to take into account the one-sided character of the problem. For subdifferentiability properties (as for semicontinuity or calmness properties) the case of marginal functions and the case of performance functions must be distinguished. They are the functions obtained as

$$m(w) := \sup_{x \in X} f(w,x), \quad w \in W$$

$$p(w) := \inf_{x \in X} f(w,x) \quad w \in W$$

respectively. Hereafter the decision variable x belongs to an arbitrary set X, the parameter variable w belongs to a normed vector space W and $f\colon W \times X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is a function called the *perturbation* function. A distinctive feature

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of our work is that we do not assume attainment, an assumption which would much simplify the question, but which is not always satisfied.

Our objective in the present paper is to devise conditions ensuring strict differentiability of m and p at a given point u of W. This notion is still more demanding than differentiability at u (in the presence of differentiability around u, it amounts to continuity at u of the derivative). It is an important concept which is appropriate for inverse mapping theorems (*e.g.*, [7], [11], [26], [28]). It plays a crucial role in recent studies about generalized equations ([34]) which are connected with optimality conditions for mathematical programming problems. Let us also recall that a Lipschitzian function on W is strictly differentiable at u iff its Clarke subdifferential at u is a singleton.

Our study has been prompted by genericity results about the existence of optimal solutions to the problem of minimizing $f_w := f(w, \cdot)$ which require strict differentiability of the performance function p; differentiability properties would not suffice. These results ([31]) are in the line of the work by Ekeland and Lebourg ([14]); see also [2], [13], [21], [23], [29], [36], [37], [45]. Our proofs here also closely follow the methods of [14]; however, we deal with unilateral properties using one-sided assumptions and, as mentioned above, we focus most of our efforts on (sub)differentiability at a specific point u. In order to get genericity properties we rely on deep results of Preiss [33]. The uses of new notions of tameness and of the minimizing grill of a function introduced in [32] also represent new features of our approach. These concepts are recalled in the next section. Section 3 is devoted to subdifferentiability properties. Differentiability properties are obtained in sections 4 and 5. An application to the regularization of functions is presented in section 6.

For the applications of the strict differentiability results of section 5 to existence and genericity properties, we refer to [31], and, for previous results of this kind, to [2], [4], [5], [14], [23], [45], [46] and their bibliographies; they might also be relevant to the methods of [20] which concerns such questions and to classical topics of mathematical economics such as the customer problem. Here we present another application; it concerns the regularization of nonsmooth functions. We consider a Moreau type regularization; for simplicity, we do not take a general regularization kernel as in [6], [8], [9] but limit our illustration to an infimal convolution process as in [30]. We also present some illustrations in the study of best approximations and of the Fenchel transform.

We use standard notation. In particular, the open ball with center w and radius r in W is denoted by $B(w, r)$ and if $h\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is a function and $\lambda \in \mathbb{R}$, $[h \leq \lambda]$ stands for $\{x \in X : h(x) \leq \lambda\}$. In the following sections but the last one, u is a given point of W at which p is finite; $\mathcal{N}(u)$ stands for the family of neighborhoods of u in W.

2 Preliminaries

Let us recall some notions which have been coined in [32]. We first observe that the behavior of the performance function p around the given point $u \in W$ is more influenced by the behavior of f at points (w, x) where x is an approximate solution than by the behavior of f at any other point. The precise meaning of the *approximate*

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solution set is as follows: for $w \in W, \alpha > 0$

$$S(w,\alpha) := S_\alpha(w) := \{x \in X : f(w,x) \leq p(w) \widehat{+} \alpha\},$$

where the addition is extended to $\overline{\mathbb{R}} \times (0,+\infty)$ by setting $r \widehat{+} \alpha := -1/\alpha$ for $r = -\infty$, $r \widehat{+} \alpha := +\infty$ if $r = +\infty$. The set $S(w,\alpha)$ is also called the set of α-minimizers of $f_w := f(w,\cdot)$; its definition is devised in such a way that it is always nonempty. Moreover the set $S(w)$ of minimizers of f_w satisfies $S(w) = \bigcap_{\alpha>0} S(w,\alpha)$.

In fact, instead of controlling the functions $f_x := f(\cdot,x)$ for any $x \in S(w,\alpha)$, it would suffice to control these functions for x in a sufficiently representative subset of $S(w,\alpha)$. In order to give a precise meaning to this idea, we introduced in [32] the *minimizing grill* of f_w as the family

$$\mathcal{M}_w := \{M \subset X : \inf f_w(M) = p(w)\} = \{M \subset X : \forall \alpha > 0 \; M \cap S(w,\alpha) \neq \varnothing\}.$$

For $w = u$, we simplify the notation by setting $\mathcal{M} := \mathcal{M}_u$. Of course, any member of the family $\mathcal{A}_w := \{S(w,\alpha) : \alpha > 0\}$ of approximate solution sets of f_w is a member of $\mathcal{M}_w$ but $\mathcal{M}_w$ is a much larger family (so that many assumptions below are less stringent than assumptions formulated in terms of the family $\mathcal{A}_w$). Both families play a natural role in minimization problems: M belongs to $\mathcal{M}_w$ iff M contains a minimizing sequence of f_w. In making assumptions about a family $(f_x)_{x\in M} := (f(\cdot,x))_{x\in M}$ one is willing to take $M \in \mathcal{M}$ as small as possible. The best case occurs when the set $S(u)$ of minimizers of f_u is nonempty: then one can take for M a singleton $\{x\}$, where $x \in S(u)$. However, we endeavour to avoid the assumption that $S(u)$ is nonempty.

As for calmness properties, a strong control of the approximate solution sets is required in order to get differentiability properties. Let us recall appropriate concepts partly introduced in [32] which have some similarities with the notion of tame perturbation of [36] but are different.

Definition 2.1 The perturbation f is said to be *compliant at* $u \in W$ or that it is a C-perturbation at u if $p(u) := \inf f_u$ is finite and if for any $\alpha > 0$ there exist $\beta > 0$ and $V \in \mathcal{N}(u)$ such that for each $v \in V$ one has

$$S(v,\beta) \subset S(u,\alpha).$$

It is said that f is *docile at* u or that it is a D-perturbation at u if $p(u) := \inf f_u$ is finite and if for any $\alpha > 0$ there exists $V \in \mathcal{N}(u)$ such that $S(u,\alpha) \in \mathcal{M}_v$ for each $v \in V$, or, in other terms,

$$\forall \alpha > 0,\ \exists V \in \mathcal{N}(u),\ \forall v \in V,\ \forall \beta > 0, \quad S(v,\beta) \cap S(u,\alpha) \neq \varnothing.$$

It is compliant (resp. docile) with respect to some subset M of X if one can replace in the preceding conditions $S(u,\alpha)$ and $S(v,\beta)$ by $S(u,\alpha) \cap M$ and $S(v,\beta) \cap M$ respectively.

Thus a compliant perturbation is docile. Before recalling compliance criteria, let us present an example.

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Example 2.1 Let X be a closed nonempty subset of a normed vector space W and let f be given by $f(w,x) := \|w-x\|$. For any $u \in W$ and any $\alpha, \beta, \rho > 0$ such that $\beta + 2\rho \leq \alpha$ one has $S(v,\beta) \subset S(u,\alpha)$ for each $v \in B(u,\rho)$ since $p = d(\cdot, X)$ is Lipschitzian with rate 1. Thus f is compliant.

Example 2.2 More generally, if X is an arbitrary set and if for each $x \in X$ the function f_x is Lipschitzian with rate k (*i.e.*, $|f_x(w) - f_x(w')| \leq kd(w,w')$ for any $w,\ w' \in W$, $x \in X$), then the perturbation f is compliant at each point u of W since for any $\alpha, \beta, \rho > 0$ such that $\beta + 2k\rho \leq \alpha$ one has $S(v,\beta) \subset S(u,\alpha)$ for each $v \in B(u,\rho)$.

Lemma 2.2 ([32]) *Suppose X is a topological space, f is lower semicontinuous at (u,x) and f_u is continuous at x for any $x \in X_0$, where X_0 is a subset of X such that for any sequences $(\varepsilon_n) \to 0_+$, $(u_n) \to u$ in W, (x_n) in X with $x_n \in S(u_n, \varepsilon_n)$ for each n, the sequence (x_n) has a cluster point in X_0. Then, if p is upper semicontinuous at u, f is compliant at u.*

Proposition 2.3 ([14], [32]) *Suppose that for each $x \in X$ the function f_x is lower semicontinuous on W and bounded below. Suppose there exist some $\lambda > p(u)$, $k \in \mathbb{R}_+$ and $V \in \mathcal{N}(u)$ such that for each $v \in V$ and each $x \in [f_v \leq \lambda]$ there exists $\sigma > 0$ for which*

$$f_x(w) \leq f_x(v) + kd(v,w) \qquad \forall w \in B(v,\sigma). \tag{2.1}$$

Then f is compliant at u and p is Lipschitzian with rate k on some neighborhood V' of u.

3 Subdifferentiability of the Optimal Value Functions

Recall that a function $h : W \to \overline{\mathbb{R}}$ finite at $u \in W$ is said to be *Fréchet* (or *firmly*) *subdifferentiable at u* if for some $u^* \in W^*$ one has

$$\liminf_{w \to 0,\ w \neq 0} \frac{1}{\|w\|} \big(h(u+w) - h(u) - \langle u^*, w\rangle\big) \geq 0.$$

We denote by $\partial^- h(u)$ the set of $u^* \in W^*$ satisfying this condition and we call it the *firm* (or *Fréchet*) *subdifferential* of h at u. We observe that $\partial^- h(u) = \bigcap_{\varepsilon>0} \partial_\varepsilon^- h(u) = \partial_0^- h(u)$, where, for $\varepsilon \in \mathbb{R}_+$, $\partial_\varepsilon^- h(u)$ is the set of $u^* \in W^*$ such that for each $\varepsilon' > \varepsilon$ the function $w \mapsto f(w) + \langle u^*, w\rangle + \varepsilon' \|w-u\|$ attains a local minimum at $w = u$. The *firm* (or *Fréchet*) *superdifferential* of h at u is the set

$$\partial^+ h(u) := -\partial^-(-h)(u).$$

Similarly, $\partial^+ h(u) = \bigcap_{\varepsilon>0} \partial_\varepsilon^+ h(u)$, where $\partial_\varepsilon^+ h(u) := -\partial_\varepsilon^-(-h)(u)$. In particular, $\partial^- m(u) = -\partial^+ p(u)$. We say that h is *subdifferentiable* (resp. *superdifferentiable*) at u if $\partial^- h(u)$ (resp. $\partial^+ h(u)$) is nonempty. Obviously, h is (Fréchet) differentiable at u iff

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it is subdifferentiable and superdifferentiable at u. Then one has $\partial^- h(u) = \partial^+ h(u) = \{Dh(u)\}$.

Using the preceding notions, two compliance criteria can be deduced from Proposition 2.3.

Corollary 3.1 *Suppose that for each $x \in X$ the function f_x is lower semicontinuous and bounded below on W and one of the following assumptions holds:*

(i) *there exist some $\lambda > p(u)$, $c \in \mathbb{R}_+$ and $V \in \mathcal{N}(u)$ such that for each $v \in V$ and each $x \in [f_v \leq \lambda]$ the function f_x is finite and differentiable at v with $\|Df_x(v)\| \leq c$.*

(ii) *there exist some $\varepsilon > 0$, $\lambda > p(u)$, $c \in \mathbb{R}_+$ and $V \in \mathcal{N}(u)$ such that for each $v \in V$ and each $x \in [f_v \leq \lambda]$ the function f_x is finite at v and $\partial^+_\varepsilon f_x(v) \cap B(0, c) \neq \varnothing$.*

Then f is compliant at u and, for each $k \geq c + \varepsilon$, p is Lipschitzian with rate k on some neighborhood V' of u.

Proof Assume (ii) holds. Given $k \geq c + \varepsilon$, for each $v \in V$ and each $x \in [f_v \leq \lambda]$, picking $v^* \in \partial^+_\varepsilon f_x(v) \cap B(0, c)$, one can find $\sigma > 0$ such that

$$f_x(w) \leq f_x(v) + \langle v^*, w - v\rangle + \varepsilon \|w - v\| \leq f_x(v) + kd(v, w) \qquad \forall w \in B(v, \sigma).$$

Thus Proposition 2.3 applies.

Superdifferentiability of the performance function p (and subdifferentiability of m) is easy to obtain.

Proposition 3.2 *Suppose the following condition is satisfied for some $u^* \in W^*$:*

(s^+) *for any $\varepsilon > 0$ there exists $\eta > 0$ such that for any $v \in B(u, \eta)$, $\alpha > 0$, there are $x \in S(u, \alpha)$ and $w^* \in B(u^*, \varepsilon)$ with*

$$f(v, x) \leq f(u, x) + \langle w^*, v - u\rangle + \varepsilon \|v - u\|. \tag{3.1}$$

Then p is superdifferentiable at u and $u^* \in \partial^+ p(u)$.

When the set $S(u)$ of minimizers of f_u is nonempty and when for some $x \in S(u)$ the function f_x is superdifferentiable at u, one gets that $\partial^+ f_x(u) \subset \partial^+ p(u)$, an obvious fact. Here we do not assume $S(u) \neq \varnothing$.

Proof Given $\varepsilon > 0$, let $\eta > 0$ be as in condition (s^+). Then, for any $v \in B(u, \eta)$ and any $\alpha > 0$, we pick $x \in S(u, \alpha)$ and $w^* \in B(u^*, \varepsilon)$ such that (3.1) is satisfied. Then, from the inequalities $p(v) \leq f(v, x)$, $f(u, x) \leq p(u) + \alpha$, $\langle w^*, v - u\rangle \leq \langle u^*, v - u\rangle + \varepsilon \|v - u\|$ we deduce from (3.1) that

$$p(v) \leq p(u) + \alpha + \langle u^*, v - u\rangle + 2\varepsilon \|v - u\| \qquad \forall v \in B(u, \eta).$$

Since $\alpha > 0$ is arbitrarily small, we obtain

$$p(v) \leq p(u) + \langle u^*, v - u\rangle + 2\varepsilon \|v - u\| \qquad \forall v \in B(u, \eta),$$

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so that $u^* \in \partial^+ p(u)$. ■

In the following corollary we use the notion of *limit superior of a family* $\mathcal{B}$ of subsets of W^*: $u^* \in \limsup \mathcal{B}$ if for any $\varepsilon > 0$, $B \in \mathcal{B}$, the set $B \cap B(u^*, \varepsilon)$ is nonempty. Thus $\limsup \mathcal{B} = \bigcap_{B \in \mathcal{B}} \overline{B}$, where $\overline{B}$ is the closure of B. One also says that u^* is a cluster point of $\mathcal{B}$. When $\mathcal{B} = \{F(t) : t > 0\}$, where $(F(t))_{t>0}$ is an increasing family of subsets of W^*, *i.e.*, $F(s) \subset F(t)$ for $s < t$, as is the case below, this notion coincides with the familiar concept of $\limsup_{t \to 0_+} F(t)$. In the following corollary we take for $\mathcal{B}$ the family $\mathcal{D}^+ := \{D^+_\alpha : \alpha > 0\}$, where, for some given $M \in \mathcal{M}_u$,

$$D^+_\alpha := D^+_{u,\alpha} := \{w^* \in W^* : \ \exists x \in S(u, \alpha) \cap M, \ w^* \in \partial^+ f_x(u)\}.$$

Corollary 3.3 *Suppose the following conditions bearing on some member M of $\mathcal{M}_u$ hold:*

(a^+) *for each $x \in M$ the function f_x is superdifferentiable at u;*
(b^+) $\limsup \mathcal{D}^+$ *is nonempty;*
(e^+) *for any $\varepsilon > 0$ there exist $\alpha, \eta > 0$ such that for any $x \in M \cap S(u, \alpha)$, $w^* \in \partial^+ f_x(u)$ one has*

$$f_x(v) \le f_x(u) + \langle w^*, v - u\rangle + \varepsilon \|v - u\| \qquad \forall v \in B(u, \eta). \tag{3.2}$$

Then p is superdifferentiable at u and $\limsup \mathcal{D}^+ \subset \partial^+ p(u)$.

Condition (e^+) is obviously satisfied when f_x is concave for each $x \in M$. In that case a number of results ensuring equality under appropriate assumptions are known (see [19] p. 201, [41] p. 66 for instance). This assumption is a weakened form of the following condition (which can be called equi-superdifferentiability at u of the family $(f_x)_{x \in M}$):

(e'^+) for any $\varepsilon > 0$ there exists $\eta > 0$ such that for any $x \in M$, $w^* \in \partial^+ f_x(u)$ (3.2) holds.

Proof Given $\varepsilon > 0$, we take $\eta > 0$, $\alpha > 0$ as in condition (e^+). Then, we use condition (b^+) to pick $u^* \in \limsup \mathcal{D}^+$, so that for any $\beta > 0$, there exists $x \in M \cap S(u, \gamma)$ with $\gamma = \min(\alpha, \beta)$ and $w^* \in \partial^+ f_x(u) \cap B(u^*, \varepsilon)$ satisfying (3.2); that ensures that condition (s^+) is satisfied. ■

Of course, usual differentiability and equi-differentiability can be substituted to their one-sided counterparts used in the preceding corollary, observing that under assumption (a) below, one has $\partial^+ f_x(u) = \{Df_x(u)\}$.

Corollary 3.4 *Suppose the following conditions hold for some $M \in \mathcal{M}_u$:*

(a) *for each $x \in M$ the function f_x is differentiable at u;*
(b) $\limsup \mathcal{D}$ *is nonempty, where $\mathcal{D} := \{D_\alpha : \alpha > 0\}$ with $D_\alpha := \{Df_x(u) : \ x \in M \cap S(u, \alpha)\}$;*

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(e) *the family* $(f_x)_{x \in M}$ *is eventually equi-differentiable at* u *in the following sense: for any* $\varepsilon > 0$ *there exist* $\eta > 0$, $\alpha > 0$ *such that for any* $x \in M \cap S(u, \alpha)$ *one has*

$$|f_x(v) - f_x(u) - \langle Df_x(u), v - u\rangle| \le \varepsilon \|v - u\| \qquad \forall v \in B(u, \eta). \tag{3.3}$$

Then p *is superdifferentiable at* u *and* $\limsup \mathcal{D} \subset \partial^+ p(u)$.

The following example presents a case in which (e) is automatically satisfied. A more general criterion ensuring equi-differentiability will be recalled in Lemma 4.3 below. When the set $S(u)$ of minimizers of f_u is nonempty and f_x is differentiable at u for any $x \in S(u)$, conditions (a), (b), (e) are satisfied for the choice $M := \{x\}$ for any $x \in S(u)$ and, denoting by $\overline{\text{co}}A^*$ the weakly closed convex hull of a subset A^* of W^*, one gets

$$\overline{\text{co}}\{Df_x(u) : x \in S(u)\} \subset \partial^+ p(u),$$

an easy and classical result (see for instance [3, p. 44] where equality is shown under some uniform continuity assumptions, W being finite dimensional and X being compact).

The preceding result can be translated into a subdifferentiability criterion for the marginal function m. A classical example concerns the case of the Fenchel transform of a function (see [1], [45] for related results dealing with Fréchet differentiability under additional well-posedness assumptions).

Example 3.1 Suppose X is a normed vector space, $W = X^*$, $u = 0$ and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is an arbitrary function such that $\inf_X \varphi$ is finite. Taking $f(w, x) = \varphi(x) - \langle w, x\rangle$, setting $D_\alpha := \{-x : \varphi(x) \le \inf_X \varphi + \alpha\}$, $\mathcal{D} := \{D_\alpha : \alpha > 0\}$ and observing that the family $(f_x)_{x \in X}$ being composed of affine continuous functions is equi-differentiable at 0, we get that the (Fenchel or Fréchet) subdifferential at 0 of the Fenchel transform $\varphi^* := -p$ of φ contains $-\overline{\text{co}}(\limsup \mathcal{D})$. In particular, the set S of minimizers of φ satisfies $-\overline{\text{co}}S \subset \partial \varphi^*(0)$, a well known fact.

Subdifferentiability of p requires more stringent assumptions. One of them uses the notion of limit inferior of a family $(F(s))_{s \in S}$ of subsets of a metric space Z parametrized by a subset S of a topological space T as $s \to \bar{t}$, where $\bar{t}$ is some point in the closure of S : $u^* \in \liminf_{s \to \bar{t}} F(s)$ iff $d(u^*, F(s)) \to 0$ as $s \to \bar{t}$ in S. We will use this concept with $Z := W^*$ and $S := W \times (0, +\infty)$, $T := W \times \mathbb{R}$, $\bar{t} := (u, 0)$, and for some $M \subset X$, $s := (v, \beta) \in S$, $F(s) := A_{v,\beta}$ where

$$A_{v,\beta} := \{v^* \in W^* : \ \exists x \in S(v, \beta) \cap M, \ v^* \in \partial^- f_x(u)\}.$$

Since $A_{v,\beta} \subset A_{v,\gamma}$ for $\beta < \gamma$, one has $u^* \in \liminf_{(v,\beta) \to (u,0_+)} A_{v,\beta}$ iff

$$\forall \varepsilon > 0 \ \exists \eta > 0 : \forall v \in B(u, \eta), \ \forall \beta > 0, \qquad A_{v,\beta} \cap B(u^*, \varepsilon) \neq \varnothing.$$

Proposition 3.5 *Suppose the following conditions hold for some* $V \in \mathcal{N}(u)$ *and some* $M \subset X$ *such that* $M \in \mathcal{M}_v$ *for each* $v \in V$:

(a^-) f_x *is subdifferentiable at* u *for each* $x \in M$;

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(b$^-$) *the set* $\liminf_{(v,\beta)\to(u,0_+)} A_{v,\beta}$ *is nonempty;*
(c) *the perturbation* f *is compliant with respect to* M*;*
(e$^-$) *the family* $(f_x)_{x\in M}$ *is eventually equi-subdifferentiable at* u *in the following sense: for any* $\varepsilon > 0$ *there exist* $\alpha, \eta > 0$ *such that for any* $x \in S(u,\alpha) \cap M$, $w^* \in \partial^- f_x(u)$, *one has*

$$f_x(v) - f_x(u) - \langle w^*, v - u\rangle \geq -\varepsilon \|v - u\| \qquad \forall v \in B(u, \eta). \tag{3.4}$$

Then p *is subdifferentiable at* u *and* $\liminf_{(v,\alpha)\to(u,0_+)} A_{v,\alpha} \subset \partial^- p(u)$.

Remarks

(1) Condition (e$^-$) is clearly satisfied when for each $x \in M$ the function f_x is convex.

(2) Assumption (e$^-$) is a weakened form of the following equi-subdifferentiability condition (or uniform with respect to $x \in M$ subdifferentiability condition):

(e$'^-$) for any $\varepsilon > 0$ there exists $\eta > 0$ such that for any $x \in M$, $w^* \in \partial^- f_x(u)$ one has (3.4).

(3) Assumption (b$^-$) is rather stringent. It is not satisfied for $W = X = \mathbb{R}$, $f(w,x) = |w - x|$, $u = 0$, although $p = 0$ is differentiable with derivative 0. However, it is satisfied when $f(w,x) = (w-x)^2$. In the next corollary we give a criterion ensuring it.

Proof Given $\varepsilon > 0$, replacing ε by $\varepsilon/2$ in condition (e$^-$), we get some $\alpha, \eta > 0$ such that for any $x \in S(u,\alpha) \cap M$, $v \in B(u,\eta)$, $w^* \in \partial^- f_x(u)$ we have

$$f_x(v) - f_x(u) - \langle w^*, v - u\rangle \geq -(\varepsilon/2) \|v - u\|.$$

Let $u^* \in \liminf_{(v,\beta)\to(u,0_+)} A_{v,\beta}$ so that, taking a smaller η if necessary, for any $v \in B(u,\eta)$ and any $\beta > 0$ there exist $x \in S(v,\beta) \cap M$ and $w^* \in \partial^- f_x(u) \cap B(u^*, \varepsilon/2)$. Since f is compliant, taking β small enough and replacing η by a smaller number if necessary, we have $S(v,\beta) \cap M \subset S(u,\alpha) \cap M$. Then, using the preceding inequality, we get

$$f_x(v) \geq f_x(u) + \langle u^*, v - u\rangle - \varepsilon \|v - u\|,$$

for each $v \in B(u,\eta)$ and some $x \in S(v,\beta) \cap M$, hence

$$p(v) \hat{+} \beta \geq f_x(v) \geq p(u) + \langle u^*, v - u\rangle - \varepsilon \|v - u\|.$$

Passing to the infimum over β, we get

$$p(v) \geq p(u) + \langle u^*, v - u\rangle - \varepsilon \|v - u\|. \tag{3.5}$$

This shows that p is subdifferentiable at u, and $u^* \in \partial^- p(u)$ ■

In the next statement, we replace the compliance condition by the weaker one that f is docile at u, but we have to reinforce condition (b$^-$). For doing so, we recall the notion of lower limit of a family $(F(x))_{x\in X}$ of subsets of a metric space Z parametrized

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by a set X as $e(x) \to \bar{t}$, where e is a map from X to some topological space T and $\bar{t}$ is some point in T : $u^* \in \liminf_{e(x)\to\bar{t}} F(x)$ iff $d(u^*, F(x)) \to 0$ as $e(x) \to \bar{t}$ in T. This notion encompasses the usual notion of $\liminf_{s\to\bar{t}} F(s)$ used above: taking $X = S$, $e(x) = x$, we recover the preceding notion. Here we take $e(x) := f(u,x) - p(u)$ and $F(x) := \partial^- f_x(u)$. Thus, $u^* \in \liminf_{e(x)\to 0} F(x)$ if for any $\varepsilon > 0$ there exists some $\beta > 0$ such that for any $x \in S(u,\beta) \cap M$ one has $\partial^- f_x(u) \cap B(u^*, \varepsilon) \neq \varnothing$.

Proposition 3.6 *Suppose that for some $V \in \mathcal{N}(u)$ and some $M \subset X$ such that $M \in \mathcal{M}_v$ for each $v \in V$ the conditions* (a$^-$), (e$^-$) *of the preceding proposition hold and that the conditions* (b$^-$) *and* (c) *are replaced by the following ones:*

(b$'^-$) *the set* $\liminf_{e(x)\to 0} F(x)$ *is nonempty;*
(d) *the perturbation f is docile with respect to M;*

Then p is subdifferentiable at u and $\liminf_{e(x)\to 0} F(x) \subset \partial^- p(u)$.

Proof Given $\varepsilon > 0$, we get again from (e$^-$) some $\alpha, \eta > 0$ such that condition (3.4) is satisfied with $\varepsilon/2$ instead of ε for any $x \in S(u,\alpha) \cap M$, $v \in B(u,\eta)$, $w^* \in \partial^- f_x(u)$. Given $u^* \in \liminf_{e(x)\to 0} F(x)$, taking a smaller α if necessary, we may suppose that for any $x \in S(u,\alpha) \cap M$ there exists some $w^* \in \partial^- f_x(u) \cap B(u^*, \varepsilon/2)$. Then, using the preceding inequality, we get

$$f_x(v) \geq f_x(u) + \langle u^*, v - u \rangle - \varepsilon \|v - u\|,$$

for each $v \in B(u,\eta)$ and any $x \in S(u,\alpha) \cap M$. Since f is docile with respect to M, taking β small enough and replacing η by a smaller number if necessary, we have $S(v,\beta) \cap S(u,\alpha) \cap M \neq \varnothing$ for any $v \in B(u,\eta)$. Thus, in the preceding relation we may pick $x \in S(v,\beta) \cap S(u,\alpha) \cap M$. It follows that

$$p(v)\hat{+}\beta \geq f_x(v) \geq p(u) + \langle u^*, v - u \rangle - \varepsilon \|v - u\|.$$

Passing to the infimum over β, we obtain again relation (3.5) which shows that p is subdifferentiable at u, and that $u^* \in \partial^- p(u)$. ■

Remark One may observe that, when f is docile with respect to M, one has

$$\liminf_{e(x)\to 0} F(x) \subset \liminf_{(v,\alpha)\to(u,0_+)} A_{v,\alpha}.$$

Let $u^* \in \liminf_{e(x)\to 0} F(x)$. By definition, given $\varepsilon > 0$, we can find some $\theta > 0$ such that for any $x \in S(u,\theta) \cap M$, we have $B(u^*, \varepsilon) \cap \partial^- f_x(u) \neq \varnothing$. Since f is docile at u there exists $\eta > 0$ such that $S(u,\theta) \cap M \in \mathcal{M}_v$ for each $v \in B(u,\eta)$. Therefore, for any $v \in B(u,\eta)$, $\alpha > 0$ there exists some $x \in S(v,\alpha) \cap S(u,\theta) \cap M$ and, as $B(u^*, \varepsilon) \cap \partial^- f_x(u) \neq \varnothing$, we get $B(u^*, \varepsilon) \cap A_{v,\alpha} \neq \varnothing$. ■

Example 3.2 Let X be a nonempty closed subset of a normed vector space W and let $u \in W$ be such that u has a best approximation $z \in X$ and such that $(x_n) \to z$ whenever $x_n \in X$ and $(d(u,x_n)) \to d_X(u) := \inf_{x\in X} d(u,x)$. Then, if the norm of W

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is Fréchet differentiable at $z-u$, with derivative $J(z-u)$, one has $J(z-u) \in \partial^- d_X(u)$. In fact, when the norm is Fréchet differentiable at $z - u$, by [17, Theorem 4, p. 148] or [44, Theorem 2.4.11], the multifunction $J = \partial \|\cdot\|$ is lower semicontinuous at $z - u$. Thus, our well-posedness assumption ensures that $J(z - u) \subset \liminf \mathcal{D}^-$ for $f(w, x) = \|w - x\|$, $M = W$. Since f is convex continuous, assumptions (a$^-$) and (e$^-$) are satisfied. Moreover, f is compliant by Proposition 2.3. This fact is a variant of differentiability results of [16, Corollary 3.5], [27, Prop. 1.5], [30, Corollary 2.10], [45, Corollary 2] given under additional smoothness properties of the norm or additional assumptions on X. Let us note that since the conditions of Corollary 3.3 are satisfied with $M = \{z\}$, we get that $J(z - u) \in \partial^- d_X(u) \cap \partial^+ d_X(u)$, so that d_X is Fréchet differentiable at u. ■

4 Differentiability Properties

Gathering the previous results, we obtain a differentiability property. Recall that a family $\mathcal{B}$ of nonempty subsets of W^* *converges* to some $u^* \in W^*$ if for any $\varepsilon > 0$ there exists $B \in \mathcal{B}$ such that $B \subset B(u^*, \varepsilon)$. When the members of $\mathcal{B}$ are of the form $B := B_\alpha := F(S_\alpha)$ for some map $F : X \to W^*$ and some family $(S_\alpha)_{\alpha>0}$ of subsets of X indexed by $\alpha \in (0, +\infty)$ and such that $S_\alpha \subset S_\beta$ for $\alpha < \beta$, the family $\mathcal{B}$ converges to u^* if, and only if $\liminf F(S_\alpha) = \limsup F(S_\alpha) = \{u^*\}$. This situation occurs in the next statement in which $S_\alpha = S(\alpha, u)$, $F(x) = Df_x(u)$, the derivative of f_x at u, the functions f_x being supposed to be differentiable at u. Then, conditions (b$^+$) and (b$'^-$) above are consequences of condition (b) below, so that the result is a direct consequence of Corollary 3.4 and Proposition 3.6.

Proposition 4.1 *Suppose the following conditions hold for some $M \in \mathcal{M}_u$:*

(a) *for each $x \in M$, f_x is differentiable at u;*
(b) *the family $\mathcal{D} := \{D_\alpha : \alpha > 0\}$ with $D_\alpha := \{Df_x(u) : \ x \in S(u, \alpha) \cap M\}$ converges;*
(d) *f is docile at u with respect to M.*
(e) *the family $(f_x)_{x\in M}$ is eventually equi-differentiable at u in the sense given in Corollary 3.4;*

Then p is Fréchet differentiable at u.

When f is compliant (or just docile) at u, a natural choice for M is $M = S(u, \theta)$ for some $\theta > 0$. Using the methods of the previous proofs, we can obtain a strict differentiability result using this choice for M.

Proposition 4.2 *Suppose the following conditions hold for some $\theta > 0$:*

(a) *for each $x \in M := S(u, \theta)$ the function f_x is differentiable at u;*
(b) *the family $\mathcal{D} := \{D_\alpha : \alpha \in (0, \theta]\}$, with $D_\alpha := \{Df_x(u) : \ x \in S(u, \alpha)\}$, converges;*
(d) *f is docile at u with respect to $S(u, \theta)$.*

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(e_s) *the family $(f_x)_{x\in M}$ is eventually strictly equi-differentiable at u: for any $\varepsilon > 0$ there exist $\alpha \in (0, \theta], \eta > 0$ such that for any $x \in S(u, \alpha)$ one has*

$$|f_x(v) - f_x(w) - \langle Df_x(u), v - w\rangle| \le \varepsilon \|v - w\| \qquad \forall v, w \in B(u, \eta).$$

Then p is strictly differentiable at u.

Let us note that assumption (e_s) is a weakened form of the assumption that the family $(f_x)_{x\in M}$ is strictly equi-differentiable at u in the following sense: for any $\varepsilon > 0$ there exists $\eta > 0$ such that for any $x \in M$ one has

$$|f_x(v) - f_x(w) - \langle Df_x(u), v - w\rangle| \le \varepsilon \|v - w\| \qquad \forall v, w \in B(u, \eta).$$

Proof Let u^* be the limit of $\mathcal{D}$: given $\varepsilon > 0$ there exists some $\alpha_\varepsilon \in]0, \theta]$ such that for any $x \in S(u, \alpha_\varepsilon)$ one has

$$\|Df_x(u) - u^*\| \le \varepsilon/2.$$

Replacing ε by $\varepsilon/2$ in condition (e_s), we get some $\alpha \in]0, \alpha_\varepsilon], \eta > 0$ such that for any $x \in S(u, \alpha)$, $v, w \in B(u, \eta)$ we have

$$|f_x(v) - f_x(w) - \langle Df_x(u), v - w\rangle| \le (\varepsilon/2) \|v - w\|$$

hence

$$p(v) \le f_x(w) + \langle u^*, v - w\rangle + \varepsilon \|v - w\|.$$

Now, since f is docile at u, we can find $\delta \in]0, \eta]$ such that for any $w \in B(u, \delta)$ we have $S(u, \alpha) \in \mathcal{M}_w$. Then, taking the infimum over $x \in S(u, \alpha)$, we get for $v, w \in B(u, \delta)$

$$p(v) \le p(w) + \langle u^*, v - w\rangle + \varepsilon \|v - w\|.$$

Since the roles of v and w can be interchanged, we obtain that for $v, w \in B(u, \delta)$ we have

$$|p(v) - p(w) - \langle u^*, v - w\rangle| \le \varepsilon \|v - w\|.$$

This proves that p is strictly differentiable at u, with derivative u^*. ■

Now let us give the announced criterion for strict equi-differentiability.

Lemma 4.3 *The following assumptions ensure that condition* (e_s) *holds:*

(a_s) *there exists some $\theta > 0$ such that, for each $x \in S(u, \theta)$, f_x is differentiable on $B(u, \theta)$;*

(e'_s) *for any $\varepsilon > 0$ there exist $\alpha, \eta \in (0, \theta)$ such that for $v \in B(u, \eta)$, $x \in S(u, \alpha)$ one has $\|Df_x(v) - Df_x(u)\| \le \varepsilon$.*

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Note that if in condition (e'_s) the relation $x \in S(u, \alpha)$ is replaced by the relation $x \in M$ (where M is a given subset of X) we get that the family $(f_x)_{x \in M}$ is strictly equi-differentiable at u.

Proof Given $\varepsilon > 0$ we take $\alpha, \eta > 0$ as in (e'_s). Then, the mean value theorem applied to the functions $w \mapsto f_x(w) - Df_x(u)(w)$ with $x \in S(u, \alpha)$ ensures that for $v, w \in B(u, \eta)$ one has

$$|f_x(v) - f_x(w) - \langle Df_x(u), v - w\rangle| \le \varepsilon \|v - w\| . \qquad \blacksquare$$

Taking into account Corollary 3.1 and this criterion, we get the following corollary.

Corollary 4.4 *Suppose that for each $x \in X$ the function f_x is lower semicontinuous and bounded below on W, and for some $\theta > 0$ one has*

- (a_s) *for each $x \in S(u, \theta)$ the function f_x is differentiable on $B(u, \theta)$;*
- (b) *the family $\{Df_x(u) : x \in S(u, \alpha)\}_{\alpha \in]0,\theta]}$ converges as $\alpha \to 0_+$;*
- (e'_s) *for any $\varepsilon > 0$ there exist $\alpha, \eta \in (0, \theta)$ such that for $v \in B(u, \eta)$, $x \in S(u, \alpha)$ one has $\|Df_x(v) - Df_x(u)\| \le \varepsilon$.*
- (f) *there exists $c \in \mathbb{R}_+$ and $V \in \mathcal{N}(u)$ such that for each $v \in V$ and each $x \in [f_v \le p(u) + \theta]$ the function f_x is finite and differentiable at v with $\|Df_x(v)\| \le c$.*

Then f is compliant at u and p is strictly differentiable at u.

5 Density and Genericity Results

In order to prove that under appropriate conditions the set of points where the performance function p is strictly differentiable is large enough, we will use a deep result of [33] and two results of [14]. The first one is similar to [14] Prop. 2.2. Since in this section the point u is no longer fixed, we use the notation $\mathcal{M}_u$, $\mathcal{D}^+_u$, $D^+_{u,\alpha}$ instead of the simplified notation $\mathcal{M}$, $\mathcal{D}^+$, D^+_α. Recall that $D^+_{u,\alpha} := \{w^* \in \partial^+ f_x(u) : x \in S(u, \alpha)\}$.

Lemma 5.1 *Let $\varepsilon \ge 0$ and $u \in W$, $u^* \in W^*$ be such that $u^* \in \partial^-_\varepsilon p(u)$. Suppose condition ($e^+$) is satisfied for some $M \in \mathcal{M}_u$:*

(e^+) *for any $\gamma > 0$ there exist $\eta, \alpha > 0$ such that for any $x \in S(u, \alpha) \cap M$, $w^* \in \partial^+ f_x(u)$, one has*

$$f_x(v) - f_x(u) - \langle w^*, v - u\rangle \le \gamma \|v - u\| \qquad \forall v \in B(u, \eta). \tag{5.1}$$

Then, for any $\delta > \varepsilon$, there exists some $\beta > 0$ such that $D^+_{u,\beta} \subset B(u^*, \delta)$. In particular, if $u^* \in \partial^- p(u)$ and if (e^+) holds, then the family $\mathcal{D}^+_u := \{D^+_{u,\alpha} : \alpha > 0\}$ converges to u^* when $D^+_{u,\alpha}$ is non empty for each $\alpha > 0$.

Thus, under the assumptions of the last assertion, the set $\partial^- p(u)$ is at most a singleton. As in Corollary 4.4, a natural choice for M is $M = S(u, \theta)$ for some $\theta > 0$. When the set $S(u)$ of minimizers of f_u is nonempty, one can take $M = \{x\}$ for some

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$x \in S(u)$ and one gets that $\partial^+ f_x(u) \subset B(u^*, \delta)$ for any $\delta > \varepsilon$ if $u^* \in \partial^-_\varepsilon p(u)$; thus, we recover the elementary fact that when $S(u)$ is nonempty and when $u^* \in \partial^- p(u)$, then for any $x \in S(u)$ one has $\partial^+ f_x(u) \subset \{u^*\}$. More generally, we obtain that when $D^+_{u,\alpha}$ is non empty for each $\alpha > 0$ and when (e$^+$) holds, the diameter of $\partial^-_\varepsilon p(u)$ is not larger than 2ε.

Proof Let $\varepsilon' \in (\varepsilon, \delta)$ and let $u^* \in \partial^-_\varepsilon p(u)$. Let ρ be such that

$$p(v) \geq p(u) + \langle u^*, v - u\rangle - \varepsilon' \|v - u\| \qquad \forall v \in B(u, \rho). \tag{5.2}$$

Let $M \in \mathcal{M}_u$ be as in condition (e$^+$) and let us take $\gamma < \frac{1}{2}(\delta - \varepsilon')$ in condition (e$^+$). Then we can find $\eta \in (0, \rho]$, $\alpha > 0$ such that relation (5.1) holds for any $x \in S(u, \alpha) \cap M$, $w^* \in \partial^+ f_x(u)$. We may assume $\beta := \gamma\eta < \alpha$. Taking into account relation (5.2), and, for $x \in S(u, \beta) \cap M$, $v \in B(u, \eta)$, the inequalities $p(v) \leq f_x(v)$, $f_x(u) - \beta \leq p(u)$, we get for any $w^* \in \partial^+ f_x(u)$,

$$\langle u^*, v - u\rangle - \varepsilon' \|v - u\| \leq f_x(v) - (f_x(u) - \beta) \leq \langle w^*, v - u\rangle + \gamma \|v - u\| + \gamma\eta$$

hence $\|u^* - w^*\| \leq \varepsilon' + 2\gamma < \delta$.

The last assertion follows from the fact that when $u^* \in \partial^- p(u)$, δ can be arbitrarily small. ∎

We are now in a position to state our first result about strict differentiability. We recall that a subset G of a metric space is said to be *generic* if it contains a $\mathcal{G}_\delta$ subset *i.e.*, a countable intersection of open subsets) which is dense. We also recall that a Banach space W is an *Asplund space* if for any open convex subset W_0 of W and for any convex continuous function f on W_0 there exists a generic subset of W_0 at each point of which f is Fréchet differentiable. This class of spaces is large and is of classical use in nonsmooth analysis: any reflexive Banach space is Asplund and any separable Banach space whose dual is separable is an Asplund space.

Theorem 5.2 *Suppose W is an Asplund space, W_0 is an open subset of W, the perturbation $f\colon W \times X \to \overline{\mathbb{R}}$ is such that for each $x \in X$ the function f_x is finite and differentiable on W_0 and that for any $u \in W_0$ the performance function p is finite at u and conditions* (e$'_s$), (f) *of Corollary 4.4 are satisfied. Then, there exists a dense subset W_∞ of W_0 such that the performance function p is strictly differentiable at each point of W_∞.*

Proof By Corollary 3.1, f is compliant on W_0 and p is locally Lipschitzian at each point of W_0. By a result of Preiss [33], the set D of points of W_0 where p is Fréchet differentiable is dense in W_0. Given $\varepsilon > 0$, let $D_{u,\alpha} := \{Df_x(u) : x \in S(u, \alpha)\}$,

$$W_\varepsilon := \{u \in W_0 : \exists u^* \in W^*, \alpha > 0,\ D_{u,\alpha} \subset B(u^*, \varepsilon)\}.$$

By Lemma 5.1, for each $\varepsilon > 0$, D is contained in W_ε. Therefore $W_\infty := \bigcap_{n\geq 1} W_{1/n}$ is a dense subset of W_0. For each $u \in W_\infty$ and each $n \in \mathbb{N}\backslash\{0\}$, the family $\mathcal{D}_u :=$

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$\{D_{u,\alpha} : \alpha > 0\}$ has some member $D_{u,\alpha} := \{Df_x(u) : \ x \in S(u,\alpha)\}$ with diameter at most $2/n$. Since W^* is complete, this means that $\mathcal{D}_u$ converges: condition (b) of Corollary 4.4 is satisfied and strict differentiability of p at u follows. ■

Remark In the preceding proof, one may replace the use of the Preiss' theorem by using the easier fact that in an Asplund space the set of points where a lower semicontinuous function is subdifferentiable is dense.

A refinement of the preceding method similar to the one in [14] yields a genericity result. As above, we suppose that for some open subset W_0 of W, and for each $u \in W_0$ condition (f) of Corollary 4.4 is satisfied and for $\eta, \alpha > 0$ small enough we set

$$G_{u,\eta,\alpha} := \{Df_x(v) : x \in S(u,\alpha), v \in B(u,\eta)\}.$$

Lemma 5.3 *Let $\varepsilon \geq 0$, $u \in W_0$, $u^* \in W^*$ be such that $u^* \in \partial_\varepsilon^- p(u)$. Assume conditions* ($\mathrm{e}_s'$), (f) *of Corollary 4.4. Then, for any $\delta > \varepsilon$, there exist $\eta, \alpha > 0$ such that $G_{u,\eta,\alpha} \subset B(u^*,\delta)$.*

Proof Setting $M = S(u,\theta)$, where $\theta > 0$ is as in condition (f) of Corollary 4.4, the family $(f_x)_{x\in M}$ is eventually strictly equi-differentiable at u *i.e.*, condition (e_s) of Proposition 4.2 is satisfied. Let $\delta > \varepsilon$ and let $\delta' \in (\varepsilon,\delta)$. By Lemma 5.1 there exists some $\beta \in (0,\theta]$ such that $D_{u,\beta} \subset B(u^*,\delta')$. Then the equicontinuity assumption (e_s') provides some $\alpha \in (0,\beta)$, $\eta > 0$ such that for each $v \in B(u,\eta)$ and each $x \in S(u,\alpha)$ one has $\|Df_x(v) - Df_x(u)\| \leq \delta - \delta'$. It follows that $G_{u,\eta,\alpha} \subset B(u^*,\delta)$. ■

As in [14], given $\varepsilon > 0$ and assuming condition (f) of Corollary 4.4, we set

$$T_\varepsilon := \{u \in W_0 : \exists \eta, \alpha > 0, \ \operatorname{diam} G_{u,\eta,\alpha} < \varepsilon\}.$$

The following lemma incorporates the contents of [14], Prop. 2.3. It shows the interest of the notion of compliant perturbation.

Lemma 5.4 *Suppose condition* (f) *of Corollary 4.4 holds and let $u \in T_\varepsilon$ for some $\varepsilon > 0$. If f is compliant at u, then T_ε is a neighborhood of u. In particular, if for each $x \in X$ the function f_x is lower semicontinuous and condition* (f) *of Corollary 4.4 holds, then T_ε is a neighborhood of u.*

Proof By assumption, there are $\eta, \alpha > 0$ such that $\operatorname{diam} G_{u,\eta,\alpha} < \varepsilon$. Since f is compliant at u, there are $\delta \in (0,\eta)$, $\beta \in (0,\alpha)$ such that $S(v,\beta) \subset S(u,\alpha)$ for each $v \in B(u,\delta)$. Let $\xi := \eta - \delta$. Since $B(v,\xi) \subset B(u,\eta)$ for each $v \in B(u,\delta)$, we get $G_{v,\xi,\beta} \subset G_{u,\eta,\alpha}$ and $\operatorname{diam} G_{v,\xi,\beta} < \varepsilon$. Thus $B(u,\delta) \subset T_\varepsilon$. ■

Theorem 5.5 *Suppose W is an Asplund space, W_0 is an open subset of W, the perturbation $f\colon W \times X \to \overline{\mathbb{R}}$ is such that for each $x \in X$ the function f_x is finite and lower semicontinuous on W_0 and that for each $u \in W_0$ the performance function p is finite at u and conditions* (e_s'), (f) *of Corollary 4.4 are satisfied. Then, there exists a generic subset G of W_0 such that the performance function p is strictly differentiable at each point of G.*

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Proof By Lemma 5.3, for each $\varepsilon > 0$, the set D^- of subdifferentiability points of p is contained in T_ε. Moreover, it is dense. Now, by Lemma 5.4, T_ε is open. Therefore $G := \bigcap_{n\geq 1} T_{1/n}$ is a generic subset of W_0. Since for any $u \in W, \alpha > 0$ one has $D_{u,\alpha} \subset G_{u,\eta,\alpha}$, hence $T_\varepsilon \subset W_{2\varepsilon}$ for each $\varepsilon > 0$, one has $G \subset \bigcap_{n\geq 1} W_{1/n} = W_\infty$, so that p is strictly differentiable at each point of G. ■

6 Application to Regularization

Let us show that the preceding results can be applied to a classical regularization process of Moreau type. Let X be a normed vector space on which some function $k\colon X \to \mathbb{R}_+$ is defined with the following properties:

(r1) k is coercive, Lipschitzian on bounded subsets and $k(0) = 0$;
(r2) for any $c \in (0, 1)$, $r > 0$ there exists some $m \in \mathbb{R}$ such that

$$k(x - w) \geq ck(x) - m \quad \forall w \in B(0, r), \qquad \forall x \in X;$$

(r3) k is continuously differentiable on X and either X is complete or the derivative of k is uniformly continuous on bounded subsets of X.
(r4) k is uniformly convex on bounded subsets: for any $r > 0$ there exists some nondecreasing function $\gamma\colon [0, r] \to \mathbb{R}$ such that $\gamma(t) > 0$ for $t > 0$ and

$$\frac{1}{2}k(x) + \frac{1}{2}k(x') - k\Big(\frac{1}{2}x + \frac{1}{2}x'\Big) \geq \gamma(\|x - x'\|) \qquad \forall x, x' \in B(0, r).$$

These conditions are satisfied when $k(\cdot) = s^{-1}\|\cdot\|^s$ with $s > 1$ when $(X, \|\cdot\|)$ is uniformly convex and uniformly smooth (see [44] for instance).

In the sequel we approach a given function g by a more regular function g_t by using an infimal convolution procedure. Since here the parameter $t \in (0, +\infty)$ is considered as fixed, we do not mention it in the expression for f, so that the relationships with what precedes are clearer; however, the value function p is now denoted by g_t.

Theorem 6.1 *Suppose conditions* (r1)–(r4) *hold. Suppose $g\colon X \to \mathbb{R}\cup\{\infty\}$ takes at least one finite value, and is such that for some $a \in \mathbb{R}_+$ the function $g(\cdot) + ak(\cdot)$ is convex and bounded below. Then, for $t > a$, the regularized function g_t of g given by*

$$g_t(w) = \inf_{x\in X} f(w, x), \text{ where } f(w, x) := g(x) + tk(x - w), w \in W := X$$

is of class C^1 on $W = X$ and $g_t \leq g$.

Proof Clearly, by assumption (r1), we have $g_t \leq g$, and, taking $x_0 \in g^{-1}(\mathbb{R})$, for each $w \in W := X$, we have $g_t(w) \leq g(x_0) + tk(x_0 - w) < +\infty$. Given $t > a$, $r > 0$, we have shown in [32] that g_t is Lipschitzian on the ball $B(0, r)$. Here we focus on differentiability properties. Taking $c \in (0, 1)$ such that $ct > a$ and taking

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m associated with c and r as in assumption (r2) above, for $w \in B(0,r)$, $x \in X$ and $b := \inf\big(g(\cdot) + ak(\cdot)\big)$, we have

$$f(w,x) \geq b - ak(x) - mt + ctk(x) \geq b - mt.$$

Then, $g_t(w) \geq b - mt$ for $w \in B(0,r)$ and it follows from the preceding estimate that for any $\beta \in (0,1]$, $w \in B(0,r)$, $x \in S(w,\beta)$, we have

$$(ct-a)k(x) + b - mt \leq f(w,x) \leq g_t(w) + \beta \leq g(x_0) + tk(x_0 - w) + 1,$$

so that the coercivity of k entails the existence of some $r_1 > 0$ such that $S(w,\beta) \subset S(w,1) \subset M := B(0,r_1)$ for any $w \in B(0,r)$. The first of the preceding inequalities also shows that, setting $\theta := 1$, $\lambda := p(u) + \theta$, increasing r_1 if necessary, we may suppose that $[f_w \leq \lambda] \subset B(0,r_1)$ for any $w \in B(0,r)$. Let us first suppose Dk is uniformly continuous on bounded subsets. Then Dk is bounded on bounded subsets, and since $Df_x(w) = -tDk(x-w)$, conditions (a_s), (e'_s) and (f) of Corollary 4.4 are satisfied. Let us deal with condition (b).

For $u \in X$, $\alpha > 0$ and $x, x' \in S(u,\alpha)$ we have, with $x'' := \frac{1}{2}x + \frac{1}{2}x'$,

$$\begin{aligned}\frac{1}{2}g(x) + \frac{1}{2}tk(u-x) + \frac{1}{2}g(x') + \frac{1}{2}tk(u-x') &\leq \frac{1}{2}\big(g_t(u)+\alpha\big) + \frac{1}{2}\big(g_t(u)+\alpha\big)\\ &= g_t(u) + \alpha \leq g(x'') + tk(u-x'') + \alpha\end{aligned}$$

hence, using (r4) and the convexity of $g + ak$,

$$\begin{aligned}&(t-a)\gamma(\|x-x'\|)\\ &\leq (t-a)\Big(\frac{1}{2}k(u-x) + \frac{1}{2}k(u-x') - k(u-x'')\Big)\\ &\leq g(x'') + ak(u-x'') + \alpha - \frac{1}{2}\big(g(x) + ak(u-x)\big) - \frac{1}{2}\big(g(x') + ak(u-x')\big)\\ &\leq \alpha.\end{aligned}$$

It follows that the diameter $\operatorname{diam} S(u,\alpha)$ of $S(u,\alpha)$ tends to 0 when $\alpha \to 0$. Since Dk is uniformly continuous on bounded sets and since $S(u,\alpha) \subset B(0,r_1)$ for $\alpha \in (0,1]$, we get that $\operatorname{diam}\{Dk(u-x) : x \in S(u,\alpha)\} \to 0$ as $\alpha \to 0$. Thus, X^* being complete, condition (b) of Corollary 4.4 is satisfied. It follows from this corollary that g_t is strictly differentiable at each point u of X, hence, by a classical result, g_t is of class C^1.

When k is just of class C^1 but X is complete, the family $(S(u,\alpha))_{\alpha>0}$ converges to some point $P(u)$ in X. Then, when $w \to u$, $\alpha \to 0_+$, we have

$$\sup\{\|(w-x) - (u-Pu)\| : x \in S(u,\alpha)\} \to 0.$$

Since $Df_x(w) = -tDk(x-w)$ is continuous in (w,x) at $(u,P(u))$, we get that

$$\{Df_x(w) \colon x \in S(u,\alpha)\} \to -tDk(Pu-u)$$

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and conditions (a_s), (b) and (e_s') of Corollary 4.4 are satisfied. Let us check condition (f) of Corollary 4.4, with $p = g_t$, as mentioned above. Given $\theta \in (0,1], \eta > 0$, we observed that $[f_v \le p(u) + \theta]$ is bounded, uniformly for $v \in B(u,\eta)$. If c is the Lipschitz rate of k on this set, for any $x \in [f_v \le p(u) + \theta]$ we have

$$g(x) + tk(x-u) \le g(x) + tk(x-v) + tc\,\|v-u\| \le p(u) + \theta + tc\eta,$$

hence $x \in S(u,\alpha)$ for $\alpha := \theta + tc\eta$. Let $\rho > 0$ be such that Dk is bounded on $B(P(u) - u, \rho)$. When θ and η are small enough, α is so small that we have $\|(v-x) - (u - Pu)\| < \rho$ for each $v \in B(u,\eta)$. Thus $Df_x(v)$ is bounded for $v \in B(u,\eta), x \in [f_v \le p(u) + \theta]$ and condition (f) of Corollary 4.4 holds. Again, the fact that g is of class C^1 ensues from that corollary. ∎

For other results about regularization processes in Banach spaces, see [8], [9], [38]. In these references X is complete and g is supposed to be bounded below; on the other hand, as in the Lasry-Lions method for Hilbert spaces, an iteration of the regularization process enables one to get rid of the convexity condition made above.

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