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CONFORMALLY FLAT CONTACT THREE-MANIFOLDS

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Abstract

In this paper, we consider contact metric three-manifolds $(M; \eta, g, \varphi, \xi)$ which satisfy the condition $\nabla_\xi h = \mu h\varphi + \nu h$ for some smooth functions μ and ν, where $2h = \pounds_\xi \varphi$. We prove that if M is conformally flat and if μ is constant, then M is either a flat manifold or a Sasakian manifold of constant curvature $+1$. We cannot extend this result for a smooth function μ. Indeed, we give such an example of a conformally flat contact three-manifold which is not of constant curvature.

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1. Introduction

First, we briefly review several of the results on conformally flat contact metric manifolds. In 1962, Okumura [10] proved that a conformally flat Sasakian manifold of dimension greater than or equal to five is of constant curvature $+1$. Tanno [13] extended the result to the K-contact case and to the three-dimensional case. Around 30 years later, a remarkable development was achieved by Calvaruso *et al.* [4]. They showed that a conformally flat H-contact three-manifold is of constant curvature 0 or $+1$. Here, an H-contact structure means that the Reeb vector field ξ is a harmonic vector field. Perrone [12] introduced the notion and proved that such a structure is characterized by the property that ξ is an eigenvector of the Ricci operator. Recently, Gouli-Andreou and Tsolakidou [8], and Bang and Blair [1] independently proved that a conformally flat H-contact manifold is of constant curvature $+1$ for dimensions greater than or equal to five. Integrating these results, gives the following theorem.

Theorem 1. *A conformally flat H-contact manifold is either a three-dimensional and flat manifold or a space of constant curvature $+1$.*

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At least in dimension three, we cannot remove the H-contact condition in the above result. Indeed, Blair [2] shows an example of a three-dimensional conformally flat contact metric space which is not of constant curvature.

On the other hand, for a contact metric structure (η, g, φ, ξ), we have another fundamental structure tensor field $h = \frac{1}{2}\pounds_\xi\varphi$, where $\pounds_\xi$ denotes the Lie differentiation along Reeb flow ξ. We first prove that a non-Sasakian contact metric three-manifold satisfies the condition $\nabla_\xi h = \mu h\varphi + \nu h$ for some smooth functions μ and ν, where $h \neq 0$ (Lemma 5). Then, we consider a class of contact metric three-manifolds $(M; \eta, g, \varphi, \xi)$ that satisfy the following condition:

(∗) $\nabla_\xi h = \mu h\varphi + \nu h$ for a constant μ and a smooth function ν on M.

Then we prove the following theorem.

Theorem 2. *A conformally flat contact metric three-manifold M which satisfies the condition* (∗) *is either a flat manifold or a Sasakian manifold of constant curvature* +1.

Also, it is notable that we cannot extend our result for a smooth function μ. In fact, Blair's example mentioned above satisfies $\nabla_\xi h = \mu h\varphi$ for a smooth function μ. In Remark 1, we examine relationships between the H-contact condition and our condition (∗). Then we remark that there are examples satisfying (∗), but they are not H-contact. Finally, we note that our result extends Calvaruso's result [3, Theorem 3]. Indeed, his result corresponds to the case $\mu \neq 4$ and $\nu = 0$.

Recently, Ghosh and Sharma [7] introduced the so-called Jacobi (k, μ)-contact manifolds whose characteristic Jacobi operator ℓ satisfies

$$\ell = -k\varphi^2 + \mu h \tag{1}$$

for $(k, \mu) \in \mathbb{R}^2$. Then we find $\nabla_\xi h = \mu h\varphi$, and deduce the following corollary.

Corollary 3. *A conformally flat Jacobi (k, μ)-contact three-manifold is either a flat manifold or a Sasakian manifold of constant curvature* +1.

2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. First, we briefly review conformal flatness in three-dimensional Riemannian manifolds (M^3, g). A Riemannian manifold is said to be *conformally flat* if it is conformally related to the Euclidean metric in the local sense. Denote by R its Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for any vector fields X, Y, Z on M. The *Schouten tensor* of type (1, 1) is defined by

$$LX = SX - \frac{r}{4}X$$

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for any vector fields X on M, where S denotes the Ricci operator and r the scalar curvature. Then it is well known that the manifold is conformally flat if and only if the Schouten tensor L is a Codazzi tensor, that is, if

$$(\nabla_X L)Y = (\nabla_Y L)X$$

for any vector fields X, Y on M.

Next, we review contact Riemannian three-manifolds. A three-dimensional manifold M^3 is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta) \neq 0$ everywhere. Given a contact form η, there is a unique vector field ξ, which is called the *Reeb vector field* or the *characteristic vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well known that there exist a Riemannian metric g and a $(1, 1)$-tensor field φ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi, \tag{2}$$

where X and Y are vector fields on M. From (2), it follows that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

A Riemannian manifold M^3 equipped with structure tensors (η, g, φ, ξ) satisfying (2) is said to be a *contact metric three-manifold or a contact Riemannian three-manifold* and is denoted by $M = (M^3; \eta, g, \varphi, \xi)$. Given a contact Riemannian manifold M, we define a $(1, 1)$-tensor field h by $h = \frac{1}{2}\pounds_\xi \varphi$, where $\pounds_\xi$ denotes Lie differentiation for ξ. Then we may observe that h is a self-adjoint operator and satisfies

$$h\xi = 0 \quad \text{and} \quad h\varphi = -\varphi h, \tag{3}$$

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{4}$$

where ∇ is the Levi-Civita connection. From (3) and (4), we see that each trajectory of ξ is a geodesic. Along a trajectory of ξ, the Jacobi operator $\ell = R(\cdot, \xi)\xi$ is a symmetrical $(1, 1)$-tensor field.

$$\varphi\ell\varphi - \ell = 2(h^2 + \varphi^2), \tag{5}$$

$$\nabla_\xi h = \varphi - \varphi\ell - \varphi h^2. \tag{6}$$

A contact Riemannian manifold for which ξ is Killing is called a *K-contact manifold*. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For a contact Riemannian manifold M, one may define naturally an almost complex structure J on $M \times \mathbb{R}$: that is,

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\varphi X - f\xi, \eta(X)\frac{d}{dt}\Big),$$

where X is a vector field tangential to M, t is the coordinate of $\mathbb{R}$ and f is a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be *normal* or *Sasakian*. It is known that M is normal if and only if M satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A Sasakian manifold is characterized by a condition

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{7}$$

for all vector fields X and Y on the manifold. For more details about contact Riemannian manifolds, we refer the reader to [2].

3. Conformally flat contact three-manifolds

In this section, we prove our main theorem. For contact metric three-manifolds M, the associated CR-structure is integrable. Then we know that M always satisfies

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{8}$$

The following lemma is immediate from (7) and (8).

Lemma 4. *A three-dimensional contact Riemannian manifold is Sasakian if and only if* $h = 0$.

Proof of Theorem 2. Let $M = (M^3; \eta, g, \varphi, \xi)$ be a contact metric three-manifold. Then the curvature tensor R is expressed by

$$\begin{aligned} R(Y, X)Z = &\rho(X, Z)Y - \rho(Y, Z)X + g(X, Z)SY - g(Y, Z)SX \\ &- \frac{r}{2}\{g(X, Z)Y - g(Y, Z)X\} \end{aligned} \tag{9}$$

for all vector fields X, Y, Z on the manifold. If $h = 0$ on M, then, from Lemma 4, we see that M is Sasakian. Then, by a theorem due to Tanno [13], we see that a conformally flat M is of constant curvature one. So we consider on M the maximal open subset U_1 on which $h \neq 0$ and the maximal open subset U_2 on which h is identically zero. (U_2 is the union of all points p in M such that $h = 0$ in a neighborhood of p). $U_1 \cup U_2$ is open and dense in M. Suppose that M is non-Sasakian. Then U_1 is nonempty and there is a local orthonormal frame field $\{e_1 = e, e_2 = \varphi e, e_3 = \xi\}$ on U_1 such that $he_1 = \lambda e_1$, $he_2 = -\lambda e_2$ for some positive function λ.

Lemma 5. *On* U_1,

$$\nabla_\xi h = \mu h\varphi + \nu h \tag{10}$$

for some smooth functions μ *and* ν *on* M, *where* $\xi(\lambda) = \nu\lambda$.

Proof. We compute on U_1

$$\begin{aligned} (\nabla_\xi h)e_1 &= (\xi\lambda)e_1 + (\lambda - h)\nabla_\xi e_1 = (\xi\lambda)e_1 - \lambda\mu e_2, \\ (\nabla_\xi h)e_2 &= -(\xi\lambda)e_2 + (\lambda - h)\nabla_\xi e_2 = -(\xi\lambda)e_2 - \lambda\mu e_1, \end{aligned}$$

where we have put $\mu = -2g(\nabla_\xi e_1, e_2)$. If we put $\nu = \xi(\lambda)/\lambda$, then we obtain (10). Thus, we have proved Lemma 5. □

Lemma 6. *On U_1,*

$$
\begin{aligned}
\nabla_{\xi} e_1 &= -\frac{\mu}{2} e_2, & \nabla_{\xi} e_2 &= \frac{\mu}{2} e_1,\\
\nabla_{e_1} \xi &= -(\lambda+1) e_2, & \nabla_{e_2} \xi &= -(\lambda-1) e_1,\\
\nabla_{e_1} e_1 &= \frac{1}{2\lambda}\{e_2(\lambda)+\rho_{13}\} e_2, & \nabla_{e_2} e_2 &= \frac{1}{2\lambda}\{e_1(\lambda)+\rho_{23}\} e_1,\\
\nabla_{e_1} e_2 &= -\frac{1}{2\lambda}\{e_2(\lambda)+\rho_{13}\} e_1 + (\lambda+1)\xi, & &\\
\nabla_{e_2} e_1 &= -\frac{1}{2\lambda}\{e_1(\lambda)+\rho_{23}\} e_2 + (\lambda-1)\xi. & &
\end{aligned}
\tag{11}
$$

Proof. The proof is in [4]. □

From (11), we have the Ricci operator S, where

$$
\begin{aligned}
Se_1 &= \Big(\frac{r}{2} - 1 + \lambda^2 + \mu\lambda\Big) e_1 + (\xi\lambda) e_2 + \rho_{13}\xi,\\
Se_2 &= (\xi\lambda) e_1 + \Big(\frac{r}{2} - 1 + \lambda^2 - \mu\lambda\Big) e_2 + \rho_{23}\xi,\\
S\xi &= \rho_{13} e_1 + \rho_{23} e_2 + 2(1-\lambda^2)\xi,
\end{aligned}
\tag{12}
$$

and where we denote $\rho_{ij} = \rho(e_i, e_j)$ for $i, j = 1, 2, 3$.

Using (11) and $\xi(\lambda) = \nu\lambda$,

$$
\begin{aligned}
\xi(e_1(\lambda)) &= [\xi, e_1](\lambda) + e_1(\xi\lambda) = \Big(-\frac{\mu}{2} + \lambda + 1\Big) e_2(\lambda) + \nu(e_1\lambda) + \lambda(e_1\nu),\\
\xi(e_2(\lambda)) &= [\xi, e_2](\lambda) + e_2(\xi\lambda) = \Big(\frac{\mu}{2} + \lambda - 1\Big) e_1(\lambda) + \nu(e_2\lambda) + \lambda(e_2\nu).
\end{aligned}
\tag{13}
$$

We compute the Jacobi identity

$$[e_1, [e_2, \xi]] + [e_2, [\xi, e_1]] + [\xi, [e_1, e_2]] = 0.$$

Using (11), first we find that $[e_1, [e_2, \xi]] = [e_2, [\xi, e_1]] = 0$, and then, together with (13), we compute

$$
\begin{aligned}
0 &= 2\lambda[\xi, [e_1, e_2]]\\
&= \Big\{\Big(\frac{\mu}{2} + \lambda - 1\Big)\rho_{23} + \nu\rho_{13} - \lambda(e_2\nu) - \xi(\rho_{13})\Big\} e_1\\
&\quad + \Big\{\Big(-\frac{\mu}{2} + \lambda + 1\Big)\rho_{13} + \nu\rho_{23} - \lambda(e_1\nu) - \xi(\rho_{23})\Big\} e_2,
\end{aligned}
$$

from which

$$\xi(\rho_{13}) = \Big(\frac{\mu}{2} + \lambda - 1\Big)\rho_{23} + \nu\rho_{13} - \lambda(e_2\nu) \tag{14}$$

and

$$\xi(\rho_{23}) = \Big(-\frac{\mu}{2} + \lambda + 1\Big)\rho_{13} + \nu\rho_{23} - \lambda(e_1\nu). \tag{15}$$

Now, we suppose that M is conformally flat. As stated in Section 2, a three-dimensional Riemannian manifold is conformally flat if and only if its Schouten tensor L is a Codazzi tensor. Hence, we get the following lemma.

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LEMMA 7. *On U_1,*

$$\nabla_k\rho_{ij} - \nabla_i\rho_{kj} = \tfrac{1}{4}(\delta_{ij}\nabla_k r - \delta_{kj}\nabla_i r), \tag{16}$$

where we have used $\nabla_i\rho_{jk} = (\nabla_{e_i}\rho)(e_j, e_k)$, $\nabla_i r = \nabla_{e_i} r$, $i = 1, 2, 3$.

Using (11) and (12), we compute

$$\begin{aligned}\nabla_1\rho_{22} &= \tfrac{1}{2}(e_1 r) + (2\lambda - \mu)(e_1\lambda) - 2(\lambda + 1)\rho_{23} + \nu(e_2\lambda + \rho_{13}) - \lambda(e_1\mu),\\ \nabla_2\rho_{12} &= \nu(e_2\lambda) + \lambda(e_2\nu) - \mu(e_1\lambda) + (-\mu - \lambda + 1)\rho_{23}.\end{aligned} \tag{17}$$

By Lemma 7,

$$\nabla_1\rho_{22} - \nabla_2\rho_{12} = \tfrac{1}{4}(e_1 r). \tag{18}$$

From (17) and (18),

$$\tfrac{1}{4}(e_1 r) = \lambda(e_2\nu) - 2\lambda(e_1\lambda) + (-\mu + \lambda + 3)\rho_{23} - \nu\rho_{13} + \lambda(e_1\mu). \tag{19}$$

This time, we compute

$$\begin{aligned}\nabla_1\rho_{33} &= -4\lambda(e_1\lambda) + 2(\lambda + 1)\rho_{23},\\ \nabla_3\rho_{13} &= (\mu + \lambda - 1)\rho_{23} + \nu\rho_{13} - \lambda(e_2\nu),\end{aligned}$$

and then, using $\nabla_1\rho_{33} - \nabla_3\rho_{13} = \tfrac{1}{4}(e_1 r)$,

$$\tfrac{1}{4}(e_1 r) = \lambda(e_2\nu) - 4\lambda(e_1\lambda) + (-\mu + \lambda + 3)\rho_{23} - \nu\rho_{13}. \tag{20}$$

From (19) and (20),

$$2(e_1\lambda) + e_1(\mu) = 0. \tag{21}$$

In a similar way, using $\nabla_2\rho_{11} - \nabla_1\rho_{12} = \tfrac{1}{4}(e_2 r)$,

$$\tfrac{1}{4}(e_2 r) = \lambda(e_1\nu) - 2\lambda(e_2\lambda) + (\mu + \lambda - 3)\rho_{13} - \nu\rho_{23} - \lambda(e_2\mu) \tag{22}$$

and, using $\nabla_2\rho_{33} - \nabla_3\rho_{23} = \tfrac{1}{4}(e_2 r)$,

$$\tfrac{1}{4}(e_2 r) = -4\lambda(e_2\lambda) + (\mu + \lambda - 3)\rho_{13} - \nu\rho_{23} + \lambda(e_1\nu). \tag{23}$$

From (22) and (23),

$$2(e_2\lambda) - e_2(\mu) = 0. \tag{24}$$

Assume that μ is constant. Then, from (21) and (24), $e_1(\lambda) = e_2(\lambda) = 0$. Moreover, using (11),

$$0 = [e_1, e_2](\lambda) = 2(\xi\lambda).$$

Since M is connected, we get the following lemma.

LEMMA 8. *λ is constant and $\nu = 0$ on M.*

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Then, from (14) and (15), respectively,

$$\begin{aligned} \xi(\rho_{13}) &= \Big(\frac{\mu}{2} + \lambda - 1\Big)\rho_{23}, \\ \xi(\rho_{23}) &= \Big(-\frac{\mu}{2} + \lambda + 1\Big)\rho_{13}. \end{aligned} \tag{25}$$

From (19) and (22), respectively,

$$\begin{aligned} e_1(r) &= 4(-\mu + \lambda + 3)\rho_{23}, \\ e_2(r) &= 4(\mu + \lambda - 3)\rho_{13}. \end{aligned} \tag{26}$$

Equation (11) in Lemma 6 simplifies to

$$\begin{aligned} &\nabla_\xi e_1 = -\frac{\mu}{2}e_2, && \nabla_\xi e_2 = \frac{\mu}{2}e_1, \\ &\nabla_{e_1}\xi = -(\lambda+1)e_2, && \nabla_{e_2}\xi = -(\lambda-1)e_1, \\ &\nabla_{e_1}e_1 = \frac{1}{2\lambda}\rho_{13}e_2, && \nabla_{e_2}e_2 = \frac{1}{2\lambda}\rho_{23}e_1, \\ &\nabla_{e_1}e_2 = -\frac{1}{2\lambda}\rho_{13}e_1 + (\lambda+1)\xi, \\ &\nabla_{e_2}e_1 = -\frac{1}{2\lambda}\rho_{23}e_2 + (\lambda-1)\xi. \end{aligned} \tag{27}$$

Similarly, equation (12) simplifies to

$$\begin{aligned} Se_1 &= \Big(\frac{r}{2} - 1 + \lambda^2 + \mu\lambda\Big)e_1 + \rho_{13}\xi, \\ Se_2 &= \Big(\frac{r}{2} - 1 + \lambda^2 - \mu\lambda\Big)e_2 + \rho_{23}\xi, \\ S\xi &= \rho_{13}e_1 + \rho_{23}e_2 + 2(1-\lambda^2)\xi. \end{aligned} \tag{28}$$

A straightforward computation using (25), (27), (28) and Lemma 8 yields

$$\begin{aligned} &\nabla_1\rho_{11} = \tfrac{1}{2}(e_1 r), \\ &\nabla_1\rho_{12} = \nabla_1\rho_{21} = (\mu - \lambda - 1)\rho_{13}, \\ &\nabla_1\rho_{13} = \nabla_1\rho_{31} = e_1(\rho_{13}) - \frac{1}{2\lambda}\rho_{13}\rho_{23}, \\ &\nabla_1\rho_{22} = \tfrac{1}{2}(e_1 r) - 2(\lambda+1)\rho_{23}, \\ &\nabla_1\rho_{23} = \nabla_1\rho_{32} = e_1(\rho_{23}) + \frac{1}{2\lambda}(\rho_{13})^2 + (\lambda+1)\Big(\frac{r}{2} - \mu\lambda - 3(1-\lambda^2)\Big), \\ &\nabla_1\rho_{33} = 2(\lambda+1)\rho_{23}, \\ &\nabla_2\rho_{11} = \tfrac{1}{2}(e_2 r) - 2(\lambda-1)\rho_{13}, \\ &\nabla_2\rho_{12} = \nabla_2\rho_{21} = (-\mu - \lambda + 1)\rho_{23}, \\ &\nabla_2\rho_{13} = \nabla_2\rho_{31} = e_2(\rho_{13}) + \frac{1}{2\lambda}(\rho_{23})^2 + (\lambda-1)\Big(\frac{r}{2} + \mu\lambda - 3(1-\lambda^2)\Big), \end{aligned}$$

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$$\nabla_2\rho_{22} = \tfrac{1}{2}(e_2 r),$$
$$\nabla_2\rho_{23} = \nabla_2\rho_{32} = e_2(\rho_{23}) - \tfrac{1}{2\lambda}\rho_{23}\rho_{13},$$
$$\nabla_2\rho_{33} = 2(\lambda - 1)\rho_{13},$$
$$\nabla_3\rho_{11} = \nabla_3\rho_{22} = \tfrac{1}{2}(\xi r),$$
$$\nabla_3\rho_{12} = \nabla_3\rho_{21} = -\mu^2\lambda,$$
$$\nabla_3\rho_{13} = \nabla_3\rho_{31} = (\mu + \lambda - 1)\rho_{23},$$
$$\nabla_3\rho_{23} = \nabla_3\rho_{32} = (-\mu + \lambda + 1)\rho_{13},$$
$$\nabla_3\rho_{33} = 0.$$

From (16) in Lemma 7, we compute the equations

$$\begin{aligned}
\nabla_1\rho_{13} - \nabla_3\rho_{11} &= \tfrac{1}{4}\{(e_1 r)g_{13} - (\xi r)g_{11}\},\\
\nabla_2\rho_{23} - \nabla_3\rho_{22} &= \tfrac{1}{4}\{(e_2 r)g_{23} - (\xi r)g_{22}\},\\
\nabla_3\rho_{12} - \nabla_1\rho_{23} &= \tfrac{1}{4}\{(\xi r)g_{12} - (e_1 r)g_{23}\},\\
\nabla_3\rho_{12} - \nabla_2\rho_{13} &= \tfrac{1}{4}\{(\xi r)g_{12} - (e_2 r)g_{13}\}.
\end{aligned}$$

Then

$$e_1(\rho_{13}) = e_2(\rho_{23}) = \frac{1}{2\lambda}\rho_{13}\rho_{23} + \frac{1}{4}\xi(r), \tag{29}$$
$$e_1(\rho_{23}) = -\mu^2\lambda - \frac{1}{2\lambda}(\rho_{13})^2 - (\lambda + 1)\Big(\frac{r}{2} - \mu\lambda - 3(1 - \lambda^2)\Big), \tag{30}$$
$$e_2(\rho_{13}) = -\mu^2\lambda - \frac{1}{2\lambda}(\rho_{23})^2 - (\lambda - 1)\Big(\frac{r}{2} + \mu\lambda - 3(1 - \lambda^2)\Big). \tag{31}$$

We compute $[e_1, e_2](r)$ in two ways and compare them. First, using (26) and (27),

$$\begin{aligned}
[e_1, e_2](r) &= (\nabla_{e_1} e_2 - \nabla e_2 e_1)(r)\\
&= \frac{4}{\lambda}(\mu - 3)\rho_{13}\rho_{23} + 2\xi(r).
\end{aligned} \tag{32}$$

Next, we compute

$$\begin{aligned}
[e_1, e_2](r) &= e_1(e_2 r) - e_2(e_1 r)\\
&= \frac{4}{\lambda}(\mu - 3)\rho_{13}\rho_{23} + 2(\mu - 3)\xi(r),
\end{aligned} \tag{33}$$

where we have used (26) and (29). Comparing (32) and (33) gives

$$\mu = 4 \quad \text{or} \quad \xi(r) = 0. \tag{34}$$

On the one hand, using (27), we compute

$$\begin{aligned}
R(e_1, e_2)e_1 &= \nabla_{e_1}(\nabla_{e_2} e_1) - \nabla_{e_2}(\nabla_{e_1} e_1) - \nabla_{[e_1,e_2]} e_1\\
&= \frac{1}{2\lambda^2}\{(\rho_{13})^2 + (\rho_{23})^2 + \lambda^2 r - 4(1 - \lambda^2)\lambda^2 + 2\mu^2\lambda^2\}e_2 - \rho_{23}\xi.
\end{aligned} \tag{35}$$

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But, from (9) and (28),

$$R(e_1, e_2)e_1 = \Big(-\frac{r}{2} + 2 - 2\lambda^2\Big)e_2 - \rho_{23}\xi. \tag{36}$$

Comparing (35) and (36) gives

$$(\rho_{13})^2 + (\rho_{23})^2 + 2\lambda^2 r - 8\lambda^2(1-\lambda^2) + 2\mu^2\lambda^2 = 0. \tag{37}$$

Differentiating (37) in the direction ξ and using (25) gives

$$\xi(r) = -\frac{2}{\lambda}\rho_{13}\rho_{23}. \tag{38}$$

Now, we consider two possible open sets: (I) $\xi(r) = 0$ or (II) $\xi(r) \neq 0$.

Case I. From (38), we see that $\rho_{13} = 0$ or $\rho_{23} = 0$. If both $\rho_{13} = 0$ and $\rho_{23} = 0$, then we find that ξ is an eigenvector of the Ricci operator S. Then, by a theorem due to Calvaruso *et al.* [4] we know that M is of constant curvature zero. So we suppose that $\rho_{13} = 0$ and $\rho_{23} \neq 0$. Then, from (25), it follows that

$$\mu = 2(1-\lambda). \tag{39}$$

Also, from (31),

$$(\rho_{23})^2 = -2\mu^2\lambda^2 - 2\lambda(\lambda-1)\Big(\frac{r}{2} + \mu\lambda - 3(1-\lambda^2)\Big).$$

Differentiating the last equation with respect to e_1, then using (26), gives

$$e_1(\rho_{23}) = 2\lambda(1-\lambda)(-\mu+\lambda+3). \tag{40}$$

Comparing (30) and (40) gives

$$(\lambda+1)r = 2(1-\lambda)(3\lambda^2+2\lambda+3),$$

where we have used (39). Since $\lambda > 0$, from the last equation,

$$r = \frac{2(1-\lambda)(3\lambda^2+2\lambda+3)}{\lambda+1},$$

and hence the scalar curvature r is constant. Then, from (26), $\lambda = -\frac{1}{3}$, which is a contradiction. Similarly, in the case when $\rho_{23} = 0$ and $\rho_{13} \neq 0$, $\mu = 2(1+\lambda)$ and

$$(\rho_{13})^2 = -2\mu^2\lambda^2 - 2\lambda(\lambda+1)\Big(\frac{r}{2} - \mu\lambda - 3(1-\lambda^2)\Big). \tag{41}$$

Differentiating the last equation with respect to e_2, then using (26), gives

$$e_2(\rho_{13}) = -2\lambda(1+\lambda)(\mu+\lambda-3). \tag{42}$$

Comparing (31) and (42) gives

$$(\lambda-1)r = -2(1+\lambda)(3\lambda^2-2\lambda+3).$$

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From this we are aware that $\lambda \neq 1$. So

$$r = \frac{-2(1+\lambda)(3\lambda^2 - 2\lambda + 3)}{\lambda - 1}.$$

Then, from (26), $\lambda = \frac{1}{3}$ and $r = \frac{32}{3}$. But, from (41),

$$(\rho_{13})^2 = -\frac{256}{81} < 0,$$

which is a contradiction.

Case II. Suppose that $\xi(r) \neq 0$. Then, from (34), $\mu = 4$. Also, from (38), both $\rho_{13} \neq 0$ and $\rho_{23} \neq 0$. Combining (29) with (38) gives

$$e_1(\rho_{13}) = e_2(\rho_{23}) = 0. \tag{43}$$

From (30) and (31), using (26) and (43),

$$e_2(e_2\rho_{13}) = 2(1-\lambda^2)\rho_{13}, \tag{44}$$

$$e_1(e_1\rho_{23}) = 2(1-\lambda^2)\rho_{23}, \tag{45}$$

$$e_2(e_1\rho_{23}) = \left\{-2(1+\lambda)^2 - \frac{1}{\lambda}e_2(\rho_{13})\right\}\rho_{13}, \tag{46}$$

$$e_1(e_2\rho_{13}) = \left\{-2(1-\lambda)^2 - \frac{1}{\lambda}e_1(\rho_{23})\right\}\rho_{23}. \tag{47}$$

On the other hand, using (25) and (27),

$$\begin{aligned} e_2(e_1\rho_{23}) &= [e_2, e_1](\rho_{23}) + e_1(e_2\rho_{23}) \\ &= (\nabla_{e_2}e_1 - \nabla_{e_1}e_2)(\rho_{23}) \\ &= \left\{\frac{1}{2\lambda}e_1\rho_{23} + 2(1-\lambda)\right\}\rho_{13}, \end{aligned} \tag{48}$$

$$\begin{aligned} e_1(e_2\rho_{13}) &= [e_1, e_2](\rho_{13}) + e_2(e_1\rho_{13}) \\ &= (\nabla_{e_1}e_2 - \nabla_{e_2}e_1)(\rho_{13}) \\ &= \left\{\frac{1}{2\lambda}e_2\rho_{13} + 2(1+\lambda)\right\}\rho_{23}. \end{aligned} \tag{49}$$

Comparing (46) and (48) gives

$$\frac{1}{2\lambda}e_1(\rho_{23}) + 2(1-\lambda) = -2(1+\lambda)^2 - \frac{1}{\lambda}e_2(\rho_{13}). \tag{50}$$

Differentiating (50) with respect to e_1, then using (45), gives

$$e_1(e_2\rho_{13}) = -(1-\lambda^2)\rho_{23}.$$

This, together with (47), gives

$$e_1(\rho_{23}) = \lambda(1-\lambda)(3\lambda - 1).$$

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Similarly, comparing (47) and (49),

$$e_2(\rho_{13}) = -\lambda(1+\lambda)(3\lambda+1). \tag{51}$$

Hence, $e_1(e_1\rho_{23}) = e_2(e_2\rho_{13}) = 0$. Thus, from (44) (or (45)), $\lambda = 1$. But, if $\lambda = 1$, from (31) using (51),

$$(\rho_{23})^2 = -2\mu^2\lambda^2 - 2\lambda(\lambda-1)\Big(\frac{r}{2} + \mu\lambda - 3(1-\lambda^2)\Big) - 2\lambda e_2(\rho_{13}) = -16,$$

which is a contradiction. That is, we see that the case $\mu = 4$ cannot occur.

All in all, we conclude that M is either a Sasakian manifold of constant curvature $+1$ or a flat manifold. Thus, we have completed the proof. □

Let M be a Jacobi (k,μ)-contact manifold with characteristic Jacobi operator $\ell = -k\varphi^2 + \mu h$ for $(k,\mu) \in \mathbb{R}^2$. Then, from (1) and (5), $h^2 = (k-1)\varphi^2$. Moreover, from (6), $\nabla_\xi h = \mu h\varphi$. This leads to Corollary 3.

The following example due to Blair [2] gives a conformally flat contact three-manifold with $\ell = -k\varphi^2 + \mu h$ for smooth functions k and μ.

Example 9. We consider $\mathbb{R}^3$ with cylindrical coordinates (r,θ,z). Let $\eta = \frac{1}{2}(\beta r\, d\theta + \gamma\, dz)$ be a contact form on $\mathbb{R}^3$ such that both β and γ are smooth functions depending only on r that satisfy

$$\frac{1}{r}\beta + \beta' = \gamma\sqrt{\beta^2+\gamma^2}, \quad -\gamma' = \beta\sqrt{\beta^2+\gamma^2}. \tag{52}$$

Indeed, Blair proved the existence of a regular solution of the system of ordinary differential equations (52) in $[0,\infty)$. Then $d\eta = \frac{1}{2}\{(\beta + r\beta')\, dr\wedge d\theta + \gamma' dr \wedge dz\}$. Assume that the metric g is conformally flat. Then we may write it as

$$ds^2 = \tfrac{1}{4}\exp(2\sigma)(dr^2 + r^2 d\theta^2 + dz^2).$$

If g is also an associated metric to the contact form η, then, from $g(X,\xi) = \eta(X)$, the characteristic vector field is given by

$$\xi = 2\exp(-2\sigma)\Big(\beta\frac{1}{r}\partial_\theta + \gamma\partial_z\Big),$$

where $\partial_\theta = \partial/\partial\theta$, $\partial_z = \partial/\partial z$. Also, from $\eta(\xi) = 1$, $\exp(2\sigma) = \beta^2 + \gamma^2$. Moreover, using (52), $d\eta(\cdot,\xi) = 0$. Compute

$$\begin{aligned}\varphi = g^{-1}d\eta &= \exp(-2\sigma)\begin{pmatrix}1&0&0\\0&\frac{1}{r^2}&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&\beta+r\beta'&\gamma'\\-\beta-r\beta'&0&0\\-\gamma'&0&0\end{pmatrix}\\ &= \exp(-2\sigma)\begin{pmatrix}0&\beta+r\beta'&\gamma'\\-\frac{1}{r^2}(\beta+r\beta')&0&0\\-\gamma'&0&0\end{pmatrix}.\end{aligned} \tag{53}$$

Then, from $\varphi^2 = -I + \eta\otimes\xi$ using (53), it follows that $(\gamma'/\beta)^2 = \exp(2\sigma)$, where $\beta \neq 0$. Indeed, from $\eta\wedge d\eta \neq 0$, we see that $\beta \neq 0$ and $\gamma \neq 0$. Assume that $\exp(\sigma) = -\gamma'/\beta$

and let $X = 2\exp(-2\sigma)(\gamma/r\partial_\theta - \beta\partial_z)$. Then $\varphi X = 2\exp(-\sigma)\partial_r$ and $\{X, \varphi X, \xi\}$ is an orthonormal frame field. Moreover, X is an eigenvector of $h = \frac{1}{2}\pounds_\xi\varphi$, that is, $hX = \lambda X$, where $\lambda = (2\beta\gamma/r\exp(3\sigma)) - 1$. Then, $\nabla_\xi h = 2(\lambda+1)h\varphi$. By computing ℓ, we find that X and φX are also eigenvectors of $\ell = R(\cdot, \xi)\xi$, that is, $\ell(X) = (\lambda+1)^2 X$ and $\ell(\varphi X) = (-3\lambda^2 - 2\lambda + 1)\varphi X$. These data give

$$\ell = -(1-\lambda^2)\varphi^2 + 2(1+\lambda)h.$$

That is, it is a so-called *generalized Jacobi* (k,μ)*-contact space* [5] for $k = 1-\lambda^2$ and $\mu = 2(1+\lambda)$, which is not a Jacobi (k,μ)-contact space. In addition, by using a tedious computation, we can show that $\eta(SX) = (4\beta^3\gamma/r^2\exp(6\sigma)) \neq 0$, where we have used (52). Hence, it is not an H-contact space.

Remark 10. H-contact spaces and contact metric spaces satisfying $(*)$ $\nabla_\xi h = \mu h\varphi + \nu h$ with a constant μ are considered as two natural generalizations of K-contact spaces. It is interesting to examine their relationships. At least for dimension three, we have no inclusion relationship between them. There are some examples that share both properties. Indeed, unimodular Lie groups with left-invariant contact metric structure are H-contact and at the same time they satisfy $(*)$ with $\nu = 0$ (compare with [6]). Also, [9, Example 4.3] shows such an example that is nonhomogeneous and satisfies $(*)$ (with a smooth function ν). However, Examples 4.1 and 4.2 in the same paper [9] are H-contact, but their μ is not a constant. On the contrary, nonunimodular Lie groups and Perrone's nonhomogeneous example in [11] satisfy $(*)$ (with $\nu = 0$), but they are not H-contact (see Example 12).

Example 11. Let M be a three-dimensional nonunimodular Lie group with left-invariant contact metric structure. Then we know that (compare with [12]) there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\} \in \mathfrak{m}$ such that

$$[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = \gamma e_2, \tag{54}$$

where $\alpha \neq 0$. Moreover, M is Sasakian if and only if $\gamma = 0$. From (54), $h = \gamma/2(\omega^1 \otimes e_1 - \omega^2 \otimes e_2)$ and $\nabla_\xi h = (2-\gamma)h\varphi$, where ω^i is the dual 1-form of e_i, $i = 1, 2$. M is not an H-contact manifold. Indeed, we compute $\eta(Se_2) = \alpha\gamma \neq 0$.

Example 12 (Perrone's example in [11]). Let M be the open submanifold $\{(x,y,z) \in \mathbf{R}^3 \mid x \neq 0\}$ of Cartesian three-space together with a contact form $\eta = xy\,dx + dz$. The characteristic vector field of this contact three-manifold is $\xi = \partial/\partial z$. Take a global frame field

$$e_1 = -\frac{2}{x}\frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}, \quad e_3 = \xi$$

and define a Riemannian metric g with respect to $\{e_1, e_2, e_3\}$ to be an orthonormal frame. Moreover, we define an endomorphism field φ by $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$ and $\varphi\xi = 0$. Then (g, φ, ξ) is an associated almost contact metric structure for η. In addition, $h = \omega^1 \otimes e_1 - \omega^2 \otimes e_2$ and $\nabla_\xi h = 4h\varphi$ $(\mu = 4)$. We compute $\eta(Se_1) = -2/x \neq 0$, which implies that M is not H-contact.

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