BULL. AUSTRAL. MATH. SOC. 90D05, 34D20
VOL. 21 (1980), 207-210.

STABILITY IN THE GAMING EQUATION

JASON GAIT

We discuss a computationally stable numerical method for the solution of linear programs and games. The method is useful in obtaining approximate solutions to large numerically unstable linear programs.

Symmetric games were first represented in the form of differential equations by Brown and von Neumann in 1950 [1]; they point out that the numerical solution of the differential equation they construct is a computationally stable method converging to a solution of the corresponding game. But for this there are two caveats that Brown and von Neumann neglect to mention, although they are self-evident. One is that the solution so obtained depends on the initial conditions selected, and the second is that the solution arrived at in this way need not be unique.

We give in this note a proof of asymptotic stability in the sense of Lyapunov, which is equivalent to a convergence proof, and at the same time implies that the process is computationally stable, and hence a computationally reliable method for solving intractable games and large linear programs with computers.

Lastly, we describe two examples which illustrate the importance of proper selection of initial conditions.

Since any game, as well as any linear program, can be represented as a symmetric game [3], it suffices to consider the case of a symmetric game

$$A = (a_{ij}) = (-a_{ji}) .$$

Received 3 September 1979.

https://doi.org/10.1017/S000497270000602X Published online by Cambridge University Press

Following Brown and von Neumann, and using the ket-vector notation $|x\rangle = (x_i)$, we introduce the notations

$$|u\rangle = (u_i) = A|x\rangle ,$$

$$|\phi\rangle = (\phi_i) = (\max\{0, u_i\}) , \tag{1}$$

$$\Phi = \langle 1|\phi\rangle ,$$

where the bra-vector $\langle 1|$ has each of its entries equal to one. The differential equation whose stability we are interested in is

$$\frac{d}{dt}|x\rangle = |\phi\rangle - \Phi|x\rangle . \tag{2}$$

In (2) the ket-vector $|x\rangle$ is interpreted as the strategy vector, and the vector $|u\rangle = (u_i)$ is the corresponding payoff vector. The equation (2) describes an action on probability vector space whose fixed points are the solutions to the game that corresponds to (2). The set of solutions is a compact connected set in phase space equal to probability vector space and the Lyapunov stability of this set will now be demonstrated.

We rely on La Salle and Lefshetz [4] for the requisite definitions and theorems involving Lyapunov stability without further comment. A suitable Lyapunov function for (2) is

$$L = \langle\phi|\phi\rangle .$$

Brown and von Neumann have already shown that the functions ϕ_i go to zero as t goes to infinity, so L is necessarily zero at the equilibrium points. The time derivative of L can easily be found to be

$$\frac{dL}{dt} = -2\Phi L ,$$

which is identically negative since both L and Φ are positive. This is enough to show that L is a Lyapunov function and thus the system (2) is asymptotically stable. The set of game solutions, then, acts as a stable node or a stable focus, just as though it were a single point. We have established the following:

THEOREM 1. *The system* (2) *converges to a game solution, perhaps not unique.*

REMARK 2. The system (2) is computationally stable, that is, numerical methods tend to converge reliably (but perhaps slowly) to a true solution.

We now describe the two examples which illustrate the use of (2) as a numerical method. For the first example consider the game

$$\begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{matrix}$$

and notice that a trajectory started at (0, 0, 0, 1) converges to the solution (0, 0, 1, 0) , but the trajectory started at (0, 1, 0, 0) converges to the solution (1, 0, 0, 0) . Both solutions are stable nodes for the system (2). In fact the solution set

$$\{(x, 0, 1-x, 0);\ x \text{ in } [0, 1]\}$$

acts as though it were a stable node.

The second example is the well-known game of Morra [2]. For this game the solution set has four extreme points, which happen to be very close together in phase space. The game has only three active strategies and the solution of (2) spirals down on the solution set, which of course contains infinitely many points, as though it were a stable focus. Clearly no trajectory, regardless of initial conditions, converges to a unique solution.

We had little difficulty in discovering an example in which the solution set is a focus rather than a node. Such examples are probably common. Examples in which solution sets are infinite are also probably common. Thus the method must be used with some caution with regard to the uniqueness and degree of approximation of the resulting answers. Despite these convergence and uniqueness problems we feel that the computational method (2) is worth further attention as a possibly useful tool in the solution of very large, computationally unstable linear programs when they prove intractible using other methods.

References

[1] G.W. Brown and J. von Neumann, "Solutions of games by differential equations", *Contributions to the theory of games*, 73-79 (Annals of Mathematics Studies, **24**. Princeton University Press, Princeton, New Jersey, 1950).

[2] Melvin Dresher, *Games of strategy. Theory and applications* (Prentice-Hall, Englewood Cliffs, New Jersey, 1961).

[3] D. Gale, H.W. Kuhn and A.W. Tucker, "On symmetric games", *Contributions to the theory of games*, 81-87 (Annals of Mathematics Studies, **24**. Princeton University Press, Princeton, New Jersey, 1950).

[4] Jospeh La Salle and Solomon Lefschetz, *Stability by Lyapunov's direct method with applications* (Mathematics in Science and Engineering, **4**. Academic Press, New York, London, 1961).

National Bureau of Standards,
Gaithersburg,
Maryland 20760,
USA

and

Carrier Hall,
University of Mississippi,
University,
Mississippi 38677,
USA.