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Transference of Vector-valued Multipliers on Weighted L^p-spaces

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Abstract. New transference results for Fourier multiplier operators defined by regulated symbols are presented. We prove restriction and extension of multipliers between weighted Lebesgue spaces with two different weights, which belong to a class more general than periodic weights, and two different exponents of integrability that can be below one.

We also develop some ad-hoc methods that apply to weights defined by the product of periodic weights with functions of power type. Our vector-valued approach allows us to extend our results to transference of maximal multipliers and provide transference of Littlewood–Paley inequalities.

1 Introduction

This paper is devoted to the study of various methods for transference of Fourier multipliers between the groups $\mathbb{R}^n$ and $\mathbb{T}^n$. For that reason, we start by giving a unified definition of Fourier multiplier operators over both groups.

Let X^n denote either the n-dimensional euclidean space $\mathbb{R}^n$ or the n-dimensional torus $\mathbb{T}^n = \Pi_{k=1}^n[-1/2, 1/2)$. We will call a *weight* any measurable, nonnegative, and locally integrable function defined in X^n. For each $0 < p < \infty$ and w a weight in X^n, we denote by $L^p(X^n, w)$ the space of measurable functions defined in X^n such that

$$\|f\|_{L^p(X^n,w)} = \left(\int_{X^n} \big| f(x)\big|^p w(x)dx \right)^{1/p} < \infty.$$

As usual, when $w(x) = 1$, the notation is abbreviated to $L^p(X^n)$.

The Fourier transform of a function in $L^1(X^n)$ is the function defined in $\widehat{X^n}$ (corresponding to $\widehat{\mathbb{R}^n} = \mathbb{R}^n$ and $\widehat{\mathbb{T}^n} = \mathbb{Z}^n$) by

$$\widehat{f}(y) = \int_{X^n} e^{-2\pi i x\cdot y} f(x)dx.$$

Given a bounded measurable function ϕ with domain in $\widehat{X^n}$, we define the operator

$$\widehat{T_\phi(f)}(y) = \phi(y)\widehat{f}(y), \quad y \in \widehat{X^n}$$

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with f in a suitable dense subset of $L^p(X^n, w)$ for which $\widehat{f}$ is defined. Then, we say that ϕ is a *Fourier multiplier* on $L^p(X^n, w)$ if T_ϕ extends to a bounded operator on $L^p(X^n, w)$. The space of multipliers on $L^p(X^n, w)$ will be denoted by $M_{p,w}(X^n)$ equipped with the norm $\|\phi\|_{M_{p,w}} = \|T_\phi\|_{L^p(\mathbb{R}^n,w)\to L^p(\mathbb{R}^n,w)}$.

Sometimes we will call the operator just defined a *multiplier*, while using the term *symbol* for the function ϕ.

The first transference result was the celebrated restriction theorem of K. De Leeuw in 1965 [12]. See also Jodeit's paper [9] for an alternative proof that develops many of the ideas we employ in this paper. The result reads as follows.

Theorem A *Let $1 \le p < \infty$ and ϕ be a function defined in $\mathbb{R}$ which is continuous at the integers $\mathbb{Z}$. If ϕ defines a bounded multiplier in $M_p(\mathbb{R})$ then, $\phi|_{\mathbb{Z}}$ defines a multiplier in $M_p(\mathbb{T})$. Moreover the p-multiplier norm of $\phi|_{\mathbb{Z}}$ does not exceed that of ϕ.*

In the 1980s, C. Kenig and P. Tomas (see [10]) extended this result to maximal operators associated with a family of multipliers defined by dilations of a given function.

More recently, the interest in transference shifted to the setting of weighted spaces with weights given by periodic functions. The first result in this setting was obtained by E. Berkson and A. T. Gillespie (see [2]) in 2003 by means of the well-known A_p-theory. We recall that for any $1 < p < \infty$, a weight W is in the Muckenhoupt $A_p(\mathbb{R})$ class if its characteristic $[W]_{A_p}$ is finite, where,

$$[W]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}$$

with the supremum taken over all cubes in $\mathbb{R}^d$ with sides parallel to the axes. This way, their result can be stated as follows.

Theorem B (see [2, Theorem 1.2]) *Let $1 < p < \infty$. Let W be a 1-periodic function in $A_p(\mathbb{R})$ and let w be its restriction to $\mathbb{T}$. Let ϕ be a continuous function. If ϕ is a multiplier in $L^p(\mathbb{R}, W)$, then its restriction to the integers, $\phi|_{\mathbb{Z}}$, defines a multiplier in $L^p(\mathbb{T}, w)$.*

In 2009, K. F. Andersen and P. Mohanty (see [1]) showed that this transference result can be proved without the A_p-theory, and therefore, it holds for any periodic weight.

Theorem C (see [1, Theorem 1.1]) *Let $1 < p < \infty$ and W be a nonnegative function of period one defined in $\mathbb{R}^n$ such that its restriction to $\mathbb{T}^n$, say w, belongs to $L^1(\mathbb{T}^n)$. Let ϕ be a continuous function defined in $\mathbb{R}^n$. If ϕ is a Fourier multiplier for $L^p(\mathbb{R}^n, W)$, then $\phi|_{\mathbb{Z}^n}$ defines a multiplier in $L^p(\mathbb{T}^n, w)$.*

Notice that, even though the A_p theory is not used, their result does not hold for the endpoints $p = 1$ or $p = \infty$.

Recently, M. J. Carro and S. Rodriguez (see [5]) have extended this result to the setting of maximal Fourier multipliers, to normalized symbols ϕ, and to weak type inequalities for $1 \le p < \infty$. In particular, they have shown the following result.

Theorem D (see [5, Theorem 3.11]) *Let $1 \leq p < \infty$ and let W be a nonnegative function of period one defined in $\mathbb{R}^n$ such that its restriction w to $\mathbb{T}^n$ belongs to $L^1(\mathbb{T}^n)$.*

Let $(\phi_j)_j$ be a sequence of continuous functions in $\mathbb{R}^n$ and denote

$$T^*_\phi(f) = \sup_{j\in\mathbb{N}} |T_{\phi_j}(f)|,$$

defined in a suitable dense subspace of $L^p(\mathbb{R}^n, W)$. If T^*_ϕ is bounded on $L^p(\mathbb{R}^n, W)$, then

$$T^*_{\phi|Z^n}(f) = \sup_{j\in\mathbb{N}} |T_{\phi_j|_{Z^n}}(f)|$$

is also bounded on $L^p(\mathbb{T}^n, w)$.

Furthermore, the same cited papers point out generalizations of the transference results in two new directions: to the setting of nonperiodic weights and to transference between multipliers not necessarily related by the restriction/extension relationship. We first mention the following recent result by K. Andersen and P. Mohanty.

Theorem E *Let $1 < p < \infty$ and $w(x) = |x|^\gamma$ for $-n < \gamma < n(p-1)$.*

(i) *If $\phi \in M_{p,w}(\mathbb{R}^n)$, then $\phi\, |_{\mathbb{Z}^n} \in M_p(\mathbb{T}^n)$* ([1, Theorem 1.2]).

(ii) *If $\phi \in M_{p,w}(\mathbb{R}^n)$, then $\phi \in M_p(\mathbb{R}^n)$* ([1, Corollary 1.3]).

Moreover, Carro and Rodriguez managed to adapt the Coifman–Weiss transference method to the weighted case and thus provide new conditions that guarantee restriction results with nonperiodic weights.

Theorem F (see [5, Theorem 3.13]) *Let $1 \leq p < \infty$. Let $V = UW$, where W is a nonnegative locally integrable function of period one whose restriction to $\mathbb{T}^n$ is denoted by w and U is a nonnegative weight that satisfies*

$$1/C \leq \frac{U(x)}{U(x+y)} \leq C, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{T}^n,$$

and

$$\lim_{s\to\infty} \Big(\int_{\{|x|\leq r+s\}} U(x)dx \Big) \Big(\int_{\{|x|\leq s\}} U(x)dx \Big)^{-1} = 1, \quad r > 0.$$

Let ϕ be a continuous function. If ϕ defines a Fourier multiplier in $L^p(\mathbb{R}^n, V)$, then $\phi|_{\mathbb{Z}^n}$, its restriction to $\mathbb{Z}^n$, defines a multiplier in $L^p(\mathbb{T}^n, w)$.

We recall the standard notation $D^p_\lambda f(x) = \lambda^{-n/p} f(\lambda^{-1}x)$, for $\lambda > 0$. This satisfies $\|D^p_\lambda f\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}$, $\tau_y f(x) = f(x-y)$, and $M_y f(x) = e^{2\pi i x y} f(x)$. Throughout the paper we write $|x|$ and $|x|_\infty$ for the ℓ_2 and ℓ_∞ norms of $x \in \mathbb{R}^n$, and we say that a function $f\colon \mathbb{R}^n \to \mathbb{C}$ is ∞-radial if it satisfies $f(x) = f(|x|_\infty)$. We also note that $|I|$ denotes the Lebesgue measure of $I \subset \mathbb{R}^n$, $W(I) = \int_I W(x)dx$, $\lambda I = \{\lambda x : x \in I\}$, $1/p + 1/p' = 1$ for $1 \leq p < \infty$, and C denotes a constant whose value may vary from line to line.

2 Main Results

The purpose of this paper is to present a systematic treatment of transference for Fourier multipliers in the double-weighted setting with nonperiodic weights. Our results hold for continuous and regulated symbols; vector-valued symbols; nonhomogenous transference (that is, the transference of multipliers between weighted Lebesgue spaces with two different exponents of integrability) and the case of quasi Banach spaces, in particular Lebesgue spaces with exponents $0 < p \leq 1$.

With this work, we improve all theorems mentioned in the introduction by increasing the range of applicability to new exponents and new weights and by obtaining smaller constants of transference, which in many instances are optimal.

Because of our interest in transference of maximal multipliers, we have chosen an approach different from those in [1] and [5], which is based on the use of vector-valued multipliers. So, we shall work in the setting of vector-valued Lebesgue spaces with multipliers given by vector-valued functions $\phi\colon \widehat{X^n} \to E$, where E is a Banach space.

Let $(E, \|\cdot\|)$ be a complex Banach space. We denote by $L^p_E(X^n, w)$ the space of E-valued measurable functions defined in X^n such that

$$\|f\|_{L^p_E(X^n,w)} = (\int_{X^n} \|f(x)\|^p w(x)dx)^{1/p} < \infty.$$

Throughout the paper we use the notation W for weights defined in $\mathbb{R}^n$, which are always assumed to be locally integrable and w for weights defined in $\mathbb{T}^n$. We write $\mathcal{C}^\infty_c(\mathbb{R}^n, E)$ for the space of E-valued functions in $C^\infty(\mathbb{R}^n)$ that are compactly supported, while $\mathcal{P}(\mathbb{T}^n)$ (respectively $\mathcal{P}(\mathbb{T}^n, E)$) stands for the space of trigonometric polynomials (respectively E-valued trigonometric polynomials) in $\mathbb{T}^n$. We recall that an E-valued trigonometric polynomial is a trigonometric polynomial with coefficients in the Banach space E.

The property of W being locally integrable implies that w belongs to $L^1(\mathbb{T}^n)$. Therefore, one has that $\mathcal{P}(\mathbb{T}^n) \subset L^p(\mathbb{T}^n, w)$ and $\mathcal{C}^\infty_c(\mathbb{R}^n, E) \subset L^p_E(\mathbb{R}^n, W)$ for any $0 < p < \infty$. Moreover, the E-valued polynomials, $\mathcal{P}(\mathbb{T}^n, E)$, are dense in $L^p_E(\mathbb{T}^n, w)$ for any $0 < p < \infty$. This follows from the scalar-valued result combined with the fact that the subspace $L^p(\mathbb{T}^n, w) \otimes E$ given by finite combinations $\sum_{k=1}^m x_k\phi_k$ where $x_k \in E$ and $\phi_k \in L^p(\mathbb{T}^n, w)$ is dense in $L^p_E(\mathbb{T}^n, w)$. A similar argument shows that $\mathcal{C}^\infty_c(\mathbb{R}^n, E)$ is dense in $L^p_E(\mathbb{R}^n, W)$.

Since we extend our results to regulated symbols, we rephrase their definition in the vector-valued setting. We shall say that a locally integrable function $\phi\colon \mathbb{R}^n \to E$ is regulated if

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^n} \int_{x+\varepsilon\mathbb{T}^n} \phi(y)dy = \phi(x), \quad x \in \mathbb{R}^n.$$

Or equivalently,

$$\lim_{\epsilon\to 0} \phi * D^1_\epsilon \varphi(x) = \phi(x), \quad x \in \mathbb{R}^n,$$

where $\varphi \in C^\infty_c(\mathbb{R}^n)$, $\varphi \geq 0$ and $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$.

We now define the Fourier multiplier operators in the vector-valued nonhomogeneous double-weighted settting:

Definition 2.1 Let $\overline{p} = (p_1, p_2)$ be a pair of positive numbers and $\overline{w} = (w_1, w_2)$ a pair of weights. A bounded measurable function $\phi\colon \widehat{X^n} \to E$ is called a *multiplier* from $L^{p_1}(X^n, w_1)$ into $L_E^{p_2}(X^n, w_2)$ if the operator

$$\widehat{(T_\phi f)}(y) = \phi(y)\widehat{f}(y), \qquad y \in \widehat{X^n}$$

defined on a suitable dense subset of $L^{p_1}(X^n, w_1)$, extends to a bounded operator from $L^{p_1}(X^n, w_1)$ to $L_E^{p_2}(X^n, w_2)$.

We will denote by $M_{\overline{p},\overline{w}}(X^n, E)$ the space of multipliers from $L^{p_1}(X^n, w_1)$ to $L_E^{p_2}(X^n, w_2)$ and write

$$\|\phi\|_{M_{\overline{p},\overline{w}}(X^n,E)} = \|T_\phi\|_{L^{p_1}(X^n,w_1)\to L_E^{p_2}(X^n,w_2)}.$$

Whenever $p_1 = p_2 = p$ and $w_1 = w_2 = w$, we will drop the vectorial notation, writing $M_{p,w}(X^n, E)$; $M_{p,w}(X^n)$ for $E = \mathbb{C}$ and $M_p(X^n)$ for $w = 1$ and $E = \mathbb{C}$.

Our main concern here is to determine conditions over the weight functions $W_i\colon \mathbb{R}^n \to [0, \infty)$ and $w_i\colon \mathbb{T}^n \to [0, \infty)$ for $i = 1, 2$ so that functions ϕ belonging to $M_{\overline{p},\overline{W}}(\mathbb{R}^n)$ satisfy that $\phi|_{\mathbb{Z}^n} \in M_{\overline{p},\overline{w}}(\mathbb{T}^n)$ and vice versa.

We divide the theorems in the paper into three different groups: restriction, extension, and ad-hoc methods. In the first group, we find Theorem 2.4 and its several applications to transference of maximal multipliers (Corollary 3.1) and Littlewood–Paley (Corollary 3.2).

The main result related to extension of multipliers is Theorem 2.5, together with Corollaries 3.4 and 3.5, which describe the periodic and power extensions respectively. We also state a characterization result showing when a multiplier can be restricted and then recovered by extension (Corollary 2.6).

The third set of results contains two different ad-hoc methods stated in Theorems 2.10 and 2.12 together with several applications given in Corollaries 3.6, 3.7, and 3.9 among others.

We now explain in detail the main contributions in each group of results.

2.1 Restriction

Concerning restriction of multipliers over weighted Lebesgue spaces, we shall consider a new class of weights including not only periodic functions but also the class of almost periodic functions introduced by Bohr and also its generalization given by Besicovitch.

We introduce some of the notions defining the weights that arise naturally when dealing with transference of Fourier multipliers in the weighted setting. Given a locally integrable function $f\colon \mathbb{R}^n \to \mathbb{C}$ we will assume that there exists $M(f) \in \mathbb{C}$ such that

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} f(x)dx = M(f), \tag{2.1}$$

and furthermore, that there exists a sequence of values $M_k(f) \in \mathbb{C}$, $k \in \mathbb{Z}^n$, such that

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} f(x)e^{-2\pi ikx}dx = M_k(f). \tag{2.2}$$

In both cases we will also assume that

$$\limsup_{\lambda\to\infty} (\frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} |f(x)|^p dx)^{1/p} = B < \infty \tag{2.3}$$

for some $1 \le p < \infty$.

Any locally integrable periodic function f (not necessarily bounded) with period 1 trivially satisfies conditions (2.2) and (2.3) with

$$M_k(f) = \int_{\mathbb{T}^n} f(x)e^{-2\pi ikx}dx = \widehat{(f\chi_{\mathbb{T}^n})}(k) \tag{2.4}$$

for all $k \in \mathbb{Z}^n$. We will also show that many other nonperiodic functions belong to this class, for instance, almost periodic functions in the Bohr and Besicovitch senses.

For our restriction theorems we shall use weights so that $W(x)e^{-2\pi ikx}$ satisfy the above conditions for any $k \in \mathbb{Z}$. Let us give a name to such a class where the restriction of multipliers can be extended. We shall denote this class by $\mathcal{RW}(\mathbb{R}^n)$.

Definition 2.2 Let W be a weight. We say $W \in \mathcal{RW}(\mathbb{R}^n)$ if there exists a sequence $(A_k)_{k\in\mathbb{Z}^n} \subset \mathbb{R}$ such that

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} W(x)e^{-2\pi ikx}dx = A_k$$

for all $k \in \mathbb{Z}$.

Note that for such weights, (2.3) holds for $p = 1$ with $B = A_0$.

Let $W \in \mathcal{RW}(\mathbb{R}^n)$ with associated sequence (A_k). Then, for any trigonometric polynomial $P(x) = \sum_{|j|\le N} \alpha_j e^{2\pi ijx}$ defined in $\mathbb{T}^n$, we have

$$\Big| \sum_{|j|\le N} \alpha_j \overline{A_j} \Big| = \lim_{\lambda\to\infty} \frac{1}{\lambda^n} \Big| \int_{\lambda\mathbb{T}^n} W(x) \Big(\sum_{|j|\le N} \alpha_j e^{2\pi ijx} \Big) dx \Big| \le B\|P\|_{L^\infty(\mathbb{T}^n)}.$$

Then, using that trigonometric polynomials in $\mathbb{T}^n$ are dense in $C(\mathbb{T}^n)$ and $(C(\mathbb{T}^n))^* = M(\mathbb{T}^n)$, we have that there is a Borel measure $\mu \in M(\mathbb{T}^n)$ (which we will later show is positive) such that

$$A_k = \int_{\mathbb{T}^n} e^{-2\pi ikx} d\mu(x) = \widehat{\mu}(k). \tag{2.5}$$

Definition 2.3 For each $W \in \mathcal{RW}(\mathbb{R}^n)$ with associated sequence (A_k), the measure satisfying (2.5) is said to be a restriction measure of W and is sometimes denoted by μ_W.

Given $\overline{\mu} = (\mu_1, \mu_2)$, where μ_i are nonnegative measures in $M(X^n)$, the notation $\phi \in M_{\overline{p},\overline{\mu}}(X^n, E)$ means that the operator T_ϕ is bounded from $L^{p_1}(X^n, \mu_1)$ into $L_E^{p_2}(X^n, \mu_2)$.

Theorem 2.4 (Restriction) *Let $0 < p < \infty$, $\overline{p} = (p, p)$ and let E be a Banach space. Let $W_1, W_2 \in \mathcal{RW}(\mathbb{R}^n)$ and let μ_1, μ_2, be their restrictions to $\mathbb{T}^n$ as defined in (2.5).*

Let $\phi\colon \mathbb{R}^n \to E$ be a bounded regulated function. If $\phi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n, E)$, then $\phi|_{\mathbb{Z}^n} \in M_{\overline{p},\overline{\mu}}(\mathbb{T}^n, E)$. Moreover, there exists K_p for which

$$\|\phi|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{\mu}}(\mathbb{T}^n,E)} \leq K_p \|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}$$

and the constant $K_p = 1$ whenever $p \geq 1$ or ϕ is continuous at $\mathbb{Z}^n$.

We notice that Theorem 2.4 deals only with the homogeneous case $p_1 = p_2$.

2.2 Extension

We also study the converse of Theorem 2.4. The reader is referred to [14, Thm. 3.18] for the unweighted case.

If we denote $D_\lambda f(x) = f(\lambda^{-1}x)$, then, by using Fourier transform, we have that

$$T_{D_\lambda \phi}(f) = D_{1/\lambda} T_\phi(D_\lambda f) \tag{2.6}$$

for smooth functions f. Moreover, it is immediate to show that

$$\|D_{1/\lambda} T_\phi D_\lambda\|_{L^{p_1}(\mathbb{R}^n,W_1)\to L_E^{p_2}(\mathbb{R}^n,W_2)} = \|T_\phi\|_{L^{p_1}(\mathbb{R}^n,D_\lambda^1 W_1)\to L_E^{p_2}(\mathbb{R}^n,D_\lambda^1 W_2)},$$

which implies

$$\|D_\lambda \phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} = \|\phi\|_{M_{\overline{p},D_\lambda^1\overline{W}}(\mathbb{R}^n,E)}, \tag{2.7}$$

and so,

$$\|D_\lambda \phi\|_{M_{\overline{p},D_{1/\lambda}^1\overline{W}}(\mathbb{R}^n,E)} = \|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}.$$

Therefore, $\phi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n, E)$ if and only if $D_\lambda \phi \in M_{\overline{p},D_{1/\lambda}^1\overline{W}}(\mathbb{R}^n, E)$. This and Theorem 2.4 imply that if $\phi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n, E)$, then $(D_\lambda\phi)_{|\mathbb{Z}^n} \in M_{\overline{p},\overline{\mu}_\lambda}(\mathbb{T}^n, E)$ for all λ where $\overline{\mu}_\lambda$ are the restriction measures corresponding to $D_{1/\lambda}^1\overline{W}$. In this section, we analyze a possible converse statement of the last fact.

For the statement of the next theorem, we remind the reader that $\widetilde{f}$ denotes the periodization of a function. Despite this notation being standard, we give the formal definition in Section 5.1.

Theorem 2.5 (Extension) *Let $0 < p_1, p_2 < \infty$. Let W_i be weights for $i = 1, 2$. We define the following family of periodic weights in $\mathbb{T}^n$: $w_{m,i} = ((D_{1/m}^1 W_i)\chi_{\mathbb{T}^n})^\sim$ for all $m \in \mathbb{N}$ and $i = 1, 2$. Let $\phi\colon \mathbb{R}^n \to E$ be a bounded regulated function such that*

$(D_m\phi)|_{\mathbb{Z}^n} \in M_{\overline{p},\overline{w}_m}(\mathbb{T}^n, E)$, *where* $\overline{w}_m = (w_{m,1}, w_{m,2})$, *uniformly for* $m \in \mathbb{N}$, *that is, satisfying*

$$\sup_{m\in\mathbb{N}} \|(D_m\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}_m}(\mathbb{T}^n,E)}. < \infty$$

Then $\phi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n, E)$ *with*

$$\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} \leq \sup_{m\in\mathbb{N}} \|(D_m\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}_m}(\mathbb{T}^n,E)}.$$

2.3 Characterization

By combining Theorems 2.4 and 2.5, we get the following characterization of transferable multipliers for periodic weights and $p_1 = p_2$. The reader should note that $D^1_{1/m}W_i$ are also periodic, and using (2.4) one has that $d\mu_i = (D_{1/m}W_i)\chi_{\mathbb{T}^n})\tilde{}\,dx$ for $i = 1, 2$.

Corollary 2.6 (Characterization) *Let* $0 < p < \infty$, $\overline{p} = (p, p)$ *and let* E *be a Banach space. Let* W_1, W_2 *be periodic weights, and let the family of weights* $w_{m,i} = ((D^1_{1/m}W_i)\chi_{\mathbb{T}^n})\tilde{}$ *in* $\mathbb{T}^n$, $m \in \mathbb{N}, i = 1, 2.$

Let $\phi\colon \mathbb{R}^n \to E$ *be a bounded regulated function. Then* $\phi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n, E)$ *if and only if* $D_m\phi|_{\mathbb{Z}^n} \in M_{\overline{p},\overline{w}_m}(\mathbb{T}^n, E)$ *with*

$$\sup_{m\in\mathbb{N}} \|(D_m\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}_m}(\mathbb{T}^n,E)} < \infty.$$

Moreover,

$$\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} = \sup_{m\in\mathbb{N}} \|(D_m\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}_m}(\mathbb{T}^n,E)}.$$

2.4 Other Transference Results

In this section we develop other methods for transference of multipliers between weighted Lebesgue spaces. As in Theorems E and F, the basic idea of these new methods is the use of known boundedness of multipliers with respect the product of two weights to deduce the analogue boundedness of the same multiplier with respect only one of the weight factors.

Our results hold for nonhomogeneous transference ($p_1 \neq p_2$) including exponents below one. In both cases we cover power weights and improve Theorems E and F.

Because of the proof methods employed, the notion that plays a relevant role for this type of transference is the norm of the dilation operator $D_\lambda f(x) = f(x/\lambda)$, for $\lambda > 0, x \in \mathbb{R}^n$ on $L^p(\mathbb{R}^n, W)$.

Here is a basic notion on weights to be used in this section to handle the action of the dilation operator. We call this class $\mathcal{DW}(\mathbb{R}^n)$.

Definition 2.7 Let W be a weight. We say that $W \in \mathcal{DW}(\mathbb{R}^n)$ if for any $\lambda > 0$ there exists $C = C(\lambda) > 0$ such that

$$W(\lambda I) \leq C(\lambda)W(I) \tag{2.8}$$

for any interval $I \subset \mathbb{R}^n$.

Let us denote by $\|D_\lambda\|_{p,W} = \|D_\lambda\|_{L^p_E(\mathbb{R}^n,W)\to L^p_E(\mathbb{R}^n,W)}$, *i.e.*,

$$\sup\Big\{ \Big(\int_{\mathbb{R}^n} \|f(x/\lambda)\|^p W(x)dx \Big)^{1/p} : \|f\|_{L^p_E(\mathbb{R}^n,W)} = 1 \Big\}.$$

Notice that in $\|D_\lambda\|_{p,W}$ we will not denote the dependence on the Banach space E.

It is clear that

$$\|D_\lambda\|_{p,W} = \|Id\|_{L^p_E(\mathbb{R}^n,W)\to L^p_E(\mathbb{R}^n,D^1_{1/\lambda}W)} \quad \text{and} \quad \|D_\lambda\|_{p,W} = \|D_\lambda\|_{1,W}^{1/p}$$

for all $p > 0$, and so, everything reduces to $\|D_\lambda\|_{1,W}$.

Proposition 2.8 *A weight W satisfies (2.8) if and only if D_λ is bounded in $L^p_E(\mathbb{R}^n, W)$ for any $\lambda > 0$, $p > 0$ and any Banach space E.*

Proof It follows from the following formula

$$\|D_\lambda\|_{1,W} = \sup_I \frac{W(\lambda I)}{W(I)},$$

where the supremum is taken over all intervals in $\mathbb{R}^n$. One implication follows selecting $f = \chi_I$. For the converse, assume condition (2.8). Now given a step function $f = \sum_{j=1}^n x_j \chi_{I_j}$, one has

$$\|D_\lambda(f)\|_{1,W} = \sum_{j=1}^n \|x_j\| W(\lambda I_j) \le C \sum_{j=1}^n \|x_j\| W(I_j) = C\|f\|_{1,W}.$$

The result now follows using that step functions supported over rectangles are dense in $L^p(\mathbb{R}^n, W)$. ■

Definition 2.9 Let $\overline{W} = (W_1, W_2)$ a couple of weights in $\mathcal{DW}(\mathbb{R}^n)$ and let $\overline{p} = (p_1, p_2)$. For each $\lambda > 0$ we denote

$$D(\lambda, \overline{p}, \overline{W}) = \|D_{1/\lambda}\|_{p_1,W_1} \|D_\lambda\|_{p_2,W_2}$$

and

$$D^1(\lambda, \overline{p}, \overline{W}) = \|D^1_{1/\lambda}\|_{p_1,W_1} \|D^1_\lambda\|_{p_2,W_2} = \lambda^{-n(\frac{1}{p_2}-\frac{1}{p_1})} D(\lambda, \overline{p}, \overline{W}).$$

As seen, we have

$$D(\lambda, \overline{p}, \overline{W}) = \Big(\sup_I \frac{W_1(I)}{W_1(\lambda I)} \Big)^{1/p_1} \Big(\sup_I \frac{W_2(\lambda I)}{W_2(I)} \Big)^{1/p_2}.$$

In particular, for power weights $W(x) = |x|^\gamma$ (including $\gamma = 0$), one has that $\|D_\lambda\|_{p,W} = \lambda^{(n+\gamma)/p}$ and for $\overline{W} = (W_1, W_2)$ with $W_i(x) = |x|^{\gamma_i}$, one has that

$$D(\lambda, \overline{p}, \overline{W}) = \lambda^{(n+\gamma_2)/p_2 - (n+\gamma_1)/p_1}.$$

For $W_1(x) = W_2(x) = 1$, we have $D(\lambda, \overline{p}, \overline{W}) = \lambda^{n(1/p_2 - 1/p_1)}$ and $D^1(\lambda, \overline{p}, \overline{W}) = 1$.

Moreover, if $W_1(\lambda I) \geq C_1 \lambda^{\delta_1} W_1(I)$ and $W_2(\lambda I) \leq C_1 \lambda^{\delta_2} W_2(I)$ for some $C_i, \delta_i > 0$ for $i = 1, 2$, then $D(\lambda, \overline{p}, \overline{W}) \leq C_2/C_1 \lambda^{\delta_2/p_2 - \delta_1/p_1}$.

Theorem 2.10 *Let $0 < p_1, p_2 < \infty$. Let $W_i \in \mathcal{RW}(\mathbb{R}^n) \cap \mathcal{DW}(\mathbb{R}^n)$ with restriction measures μ_i. Let U_i be weight functions and we define $V_i = U_i W_i$. Let $\phi\colon \mathbb{R}^n \to E$ be a bounded regulated function such that $\phi \in M_{\overline{p}, D^1_\lambda \overline{V}}(\mathbb{R}^n, E)$ for all $\lambda \geq 1$ and*

$$A(\phi) = \sup_{\lambda \geq 1} D^1(\lambda, \overline{p}, \overline{W}) \|\phi\|_{M_{\overline{p}, D^1_\lambda \overline{V}}(\mathbb{R}^n, E)} < \infty.$$

(i) *If there exist $0 < \delta < \rho < 1$ such that*

$$\sup_{\{\delta \leq |y| \leq \rho\}} U_1(y) \leq K \inf_{\{\delta \leq |y| \leq \rho\}} U_2(y), \tag{2.9}$$

then $\phi|_{\mathbb{Z}^n} \in M_{\overline{p}, \overline{\mu}}(\mathbb{T}^n, E)$.

(ii) *If we further assume that W_i are periodic weights and*

$$\sup_{m \in \mathbb{N}} \frac{\sup_{\{m \leq |y| \leq 2m\}} U_1(y)}{\inf_{\{m \leq |y| \leq 2m\}} U_2(y)} < \infty, \tag{2.10}$$

then $\phi \in M_{\overline{p}, \overline{W}}(\mathbb{R}^n, E)$.

We would like to get assumptions on U_i for $i = 1, 2$ weaker than (2.9) and (2.10) to be able to cover more examples in Theorem 2.10. This can be done, at least for $p_i > 1$, by imposing a little extra assumption on W_i for $i = 1, 2$, and so considering a subclass in $\mathcal{RW}(\mathbb{R}^n)$.

Definition 2.11 Let $1 < q \leq \infty$. We say that a weight $W \in \mathcal{RW}^q(\mathbb{R}^n)$ if there exists a sequence $(A_k)_{k \in \mathbb{Z}}$ such that

$$\lim_{\lambda \to \infty} \frac{1}{\lambda^n} \int_{\lambda \mathbb{T}^n} W(x) e^{-2\pi i k x} dx = A_k$$

and

$$\limsup_{\lambda \to \infty} \Big(\frac{1}{\lambda^n} \int_{\lambda \mathbb{T}^n} W(x)^q dx\Big)^{1/q} = B_q < \infty.$$

With this, we state the following theorem.

Theorem 2.12 *Let $1 < p_1, p_2 < \infty$. Let $1 < q_i \leq \infty$ and $W_i \in \mathcal{RW}^{q_i}(\mathbb{R}^n) \cap \mathcal{DW}$ and μ_i are their restriction measures for $i = 1, 2$.*

Let U_i be ∞-radial weight functions in $\mathbb{R}^n$ for $i = 1, 2$ such that $U_1 \in L^{q_1'}_{loc}(\mathbb{R}^n)$ and $U_2^{1-p_2'} \in L^{q_2'}_{loc}(\mathbb{R}^n)$. Let V_1, V_2 be weights in $\mathbb{R}^n$ defined by $V_i = U_i W_i$.

Let $\phi\colon \mathbb{R}^n \to E$ be a bounded regulated function such that $\phi \in M_{\overline{p}, D^1_\lambda \overline{V}}(\mathbb{R}^n, E)$ for all $\lambda \geq 1$ and

$$A(\phi) = \sup_{\lambda \geq 1} D^1(\lambda, \overline{p}, \overline{W}) \|\phi\|_{M_{\overline{p}, D^1_\lambda \overline{V}}(\mathbb{R}^n, E)} < \infty. \tag{2.11}$$

(i) *Then* $\phi\,|_{\mathbb{Z}^n} \in M_{\overline{p},\overline{\mu}}(\mathbb{T}^n, E)$ *and* $\|\phi|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{\mu}}(\mathbb{T}^n,E)} \leq CA(\phi)$, *where*

$$C = \|U_1^{1/p_1}\chi_{\{|x|<1\}}\|_{L^{p_1}(\mathbb{R}^n)}\|U_2^{-1/p_2}\chi_{\{|x|<1\}}\|_{L^{p_2'}(\mathbb{R}^n)}.$$

(ii) *If* W_i *are periodic weights and there exists* $C > 0$ *such that for all* $m \in \mathbb{N}$

$$\Big(\frac{1}{m^n}\int_{|x|<m} U_1(x)dx\Big)^{1/p_1}\Big(\frac{1}{m^n}\int_{|x|<m} U_2(x)^{1-p_2'}dx\Big)^{1/p_2'} \leq C,$$

then $\phi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n, E)$ *and* $\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} \leq CA(\phi)$.

3 Applications

The applications hereby provided are organized in the same three groups as before. Since most of them are a direct consequence of the results above, we do not include any proofs in this section.

3.1 Restriction

We shall consider particular examples of Banach spaces E to get different applications. For instance, given the family of multipliers $(\phi_j(x))_{j\in\mathbb{N}}$ described in Theorem D, we can take $E = \ell^\infty$ and $\phi\colon \widehat{X^n} \to \ell^\infty$ defined by $\phi(x) = (\phi_j(x))_{j\in\mathbb{N}}$ for which we have

$$\|T_\phi(f)(x)\|_{\ell^\infty} = T^*_\phi(f)(x) \quad \text{and} \quad \|T_{\phi|_{\mathbb{Z}^n}}(f)(x)\|_{\ell^\infty} = T^*_{\phi|_{\mathbb{Z}^n}}(f)(x).$$

Then, by Theorem 2.4, we automatically obtain the following extension to maximal multipliers. The result improves Theorem D, since it covers the remaining cases of exponents, enlarges the class of weight, and in some cases provides a smaller constant of transference.

Corollary 3.1 *Let* $0 < p < \infty$, $W_1, W_2 \in \mathcal{RW}(\mathbb{R}^n)$, *and let* μ_1, μ_2 *be their restrictions to* $\mathbb{T}^n$.

Let (ϕ_j) *be a sequence of functions uniformly regulated in* $\mathbb{R}^n$, i.e.,

$$\lim_{\varepsilon\to 0}\varepsilon^{-n}\sup_{j\in\mathbb{N}}\Big|\int_{\varepsilon\mathbb{T}^n}\big(\phi_j(x+y)-\phi_j(x)\big)\,dy\Big| = 0.$$

If T^*_ϕ *is bounded from* $L^p(\mathbb{R}^n, W_1)$ *to* $L^p(\mathbb{R}^n, W_2)$, *then* $T^*_{\phi|\mathbb{Z}^n}$ *is also bounded on* $L^p(\mathbb{T}^n, \mu_1)$ *to* $L^p(\mathbb{T}^n, \mu_2)$.

On the other hand, we could also select $E = \ell^q$ with $1 \leq q < \infty$, $\phi\colon \widehat{X^n} \to \ell^p$ given by $\phi(x) = (\phi_j(x))_{j\in\mathbb{N}}$ and define

$$S^{(q)}_\phi(f)(x) = \Big(\sum_{j\in\mathbb{N}}|T_{\phi_j}(f)(x)|^q\Big)^{1/q} = \|T_\phi(f)(x)\|_{\ell^q}.$$

This way, we can apply Theorem 2.4 to get the following corollary.

Corollary 3.2 *Let $0 < p < \infty$ and $1 \le q < \infty$. Let $W_i \in \mathcal{RW}(\mathbb{R}^n)$ and let μ_i be its restriction to $\mathbb{T}^n$.*

Let $(\phi_j)_j$ be a sequence of regulated functions such that

$$\lim_{\varepsilon\to 0} \varepsilon^{-n}\Big(\sum_{j\in\mathbb{N}}\Big|\int_{\varepsilon\mathbb{T}^n}(\phi_j(x+y)-\phi_j(x))dy\Big|^q\Big)^{1/q} = 0 \tag{3.1}$$

for which $S_\phi^{(q)}$ is bounded from $L^p(\mathbb{R}^n, W_1)$ into $L^p(\mathbb{R}^n, W_2)$. Then $S^{(q)}_{\phi|\mathbb{Z}^n}$ is bounded from $L^p(\mathbb{T}^n, \mu_1)$ into $L^p(\mathbb{T}^n, \mu_2)$.

The reader is referred to [2] for similar result in the cases $q = 2$ and $1 < p < \infty$, where $W_i = W \in A_p(\mathbb{R}^n)$ and it is periodic.

Finally, let us also recall the particular case $\phi_j(x) = \chi_{\Delta_j}(x)$, where

$$\Delta_j = [k_j^{(1)}, k_{j+1}^{(1)}) \times \cdots \times [k_j^{(n)}, k_{j+1}^{(n)}) \quad \text{and} \quad k_{j+1}^{(l)}/k_j^{(l)} \ge \alpha$$

with $\alpha > 1$ for all $l \in \{1, \dots, n\}$. For $q = 2$ one obtains that $S_\phi^{(q)}$ corresponds to the classical Littlewood–Paley square function $S_\Delta(f)$. This operator was shown to be bounded on $L^p(\mathbb{R}^n, W)$ for $1 < p < \infty$ and $W \in A_p(\mathbb{R}^n)$ by D. Kurtz (see [11]).

We can now consider the case $n = 1$, $k_j = 2^j$, and $I_j = (2^j, 2^{j+1})$. Notice that the mapping $\phi\colon \mathbb{R} \to \ell^2(\mathbb{N})$ given by

$$\phi(x) = \sum_{j=0}^{\infty} \chi_{I_{j-1}}(x) e_j \quad \text{for } x \neq \pm 2^{j-1},$$

$$\phi(\pm 2^{j-1}) = \frac{1}{2}(e_j + e_{j-1}),$$

$j \in \mathbb{N}$, is not continuous but satisfies (3.1). Then we have the following corollary.

Corollary 3.3 *Let $1 < p < \infty$, let $W \in A_p(\mathbb{R})$ be a periodic weight, and let w be its restriction to $\mathbb{T}$. Let*

$$\widetilde{S}_\Delta(f)(x) = \Big(\sum_j \Big|\sum_{2^j \le |k| < 2^{j+1}} \widehat{f}(k) e^{2\pi i k x}\Big|^2\Big)^{1/2}, \quad f \in L^p(\mathbb{T}, w).$$

Then there exists $C_p > 0$ such that $\|\widetilde{S}_\Delta(f)\|_{L^p(\mathbb{T}^n, w)} \le C_p \|f\|_{L^p(\mathbb{T}^n, w)}$.

3.2 Extension

We now see applications of Theorem 2.5 when considering particular cases. The first one essentially states the fact that any multiplier in $M_{\overline{p},\overline{w}}(\mathbb{T}^n)$ is also a multiplier in $M_{\overline{p},\overline{w}}(\mathbb{R}^n)$.

Given $w \in L^1(\mathbb{T}^n)$ we define its periodic extension as $W(x) = w(x - [x])$ for $x \in \mathbb{R}^n$, where $[x]$ stands for the unique value $k \in \mathbb{Z}^n$ such that $x \in k + \mathbb{T}^n$. Note that for the periodic extension of w one has that

$$D^1_{1/m} W(x) \chi_{\mathbb{T}^n}(x) = m^n W(mx) = m^n w(x), \ m \in \mathbb{N}, x \in \mathbb{T}^n,$$

and so,

$$\|\phi\|_{M_{\bar{p},D^1_{1/m}\overline{W}\chi_{\mathbb{T}^n}}(\mathbb{T}^n,E)} \le m^{n(1/p_2-1/p_1)}\|\phi\|_{M_{\overline{p},\overline{w}}(\mathbb{T}^n,E)}.$$

This allows us to state the following corollary.

Corollary 3.4 (Periodic extension) *Let $0 < p_1, p_2 < \infty$. Let $\overline{w} = (w_1, w_2)$ be two weight functions in $\mathbb{T}^n$ and let $\overline{W} = (W_1, W_2)$ be their periodic extensions to $\mathbb{R}^n$.*

Let $\phi\colon \mathbb{R}^n \to E$ be a bounded regulated function. If $(D_m\phi)|_{\mathbb{Z}^n} \in M_{\overline{p},\overline{w}}(\mathbb{T}^n, E)$ for all $m \in \mathbb{N}$ and

$$\sup_{m\in\mathbb{N}} m^{n(1/p_2-1/p_1)}\|(D_m\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}}(\mathbb{T}^n,E)}, < \infty$$

then $\phi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n, E)$ with

$$\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} \le \sup_{m\in\mathbb{N}} m^{n(1/p_2-1/p_1)}\|(D_m\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}}(\mathbb{T}^n,E)}.$$

Notice that in this case there is no need of a family of weights in $\mathbb{T}^n$. A single function is enough.

Furthermore, in the case of weights given by power functions $W(x) = |x|^\gamma$ with $\gamma > 0$, we note that $w_\lambda = ((D^1_{1/\lambda}W)\chi_{\mathbb{T}^n})\tilde{} = \lambda^{\gamma+n}(W\chi_{\mathbb{T}^n})\tilde{}$, and so, given two power weights $W_i(x) = |x|^{\gamma_i}$ we have

$$\|(D_\lambda\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}_\lambda}(\mathbb{T}^n,E)} = \lambda^{(\gamma_2+n)/p_2-(\gamma_1+n)/p_1}\|(D_\lambda\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}}(\mathbb{T}^n,E)}.$$

As before, this allows us to state the extension result without the need of a whole family of weights in $\mathbb{T}^n$.

Corollary 3.5 (Power extension) *Let $0 < p_1, p_2 < \infty$. Let W_1, W_2 be two power weight functions in $\mathbb{R}^n$ and let w_1, w_2 be their restrictions to $\mathbb{T}^n$.*

Let $\phi\colon \mathbb{R}^n \to E$ be a bounded regulated function. If $(D_\lambda\phi)|_{\mathbb{Z}^n} \in M_{\overline{p},\overline{w}}(\mathbb{T}^n, E)$ for all $\lambda \ge 1$ such that

$$\sup_{\lambda\ge1} \lambda^{(\gamma_2+n)/p_2-(\gamma_1+n)/p_1}\|(D_\lambda\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}}(\mathbb{T}^n,E)} < \infty,$$

then $\phi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n, E)$ with

$$\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} \le \sup_{\lambda\ge1} \lambda^{(\gamma_2+n)/p_2-(\gamma_1+n)/p_1}\|(D_\lambda\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}}(\mathbb{T}^n,E)}.$$

3.3 Other Transferences

In this section, we first simplify the transference criteria of Theorems 2.12 and 2.10 by using conditions that allow us to remove the dependence on the dilations of the symbol. Then we apply the result in different scenarios.

3.3.1 Removing the Dilations over the Symbol

We remind the reader that the conditions for transference in Theorems 2.12 and 2.10 above were that $\phi \in M_{\overline{p},D^1_\lambda\overline{V}}(\mathbb{R}^n,E)$ for all $\lambda \geq 1$ and the criterium (2.11):

$$\sup_{\lambda\geq 1} D^1(\lambda,\overline{p},\overline{W})\|\phi\|_{M_{\overline{p},D^1_\lambda\overline{V}}(\mathbb{R}^n,E)} < \infty.$$

This last inequality depends strongly on the dilations of the multiplier ϕ.

However, whenever a weight W is such that the dilation operator D_λ is bounded on $L^p(\mathbb{R}^n,W)$, we can use (2.6) and (2.7) to deduce that

$$\begin{aligned}\|\phi\|_{M_{\overline{p},D^1_\lambda\overline{W}}(\mathbb{R}^n,E)} &= \|D_\lambda\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} \leq \|D_{1/\lambda}\|_{p_2,W_2}\|D_\lambda\|_{p_1,W_1}\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}\\ &= D(\lambda^{-1},\overline{p},\overline{W})\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}.\end{aligned}$$

For this reason, in Theorems 2.10 and 2.12, if we add the hypothesis that the dilation operators associated with the weights V_i are also bounded, then we can remove the dependence of the previous criteria upon ϕ, and so replace the previous conditions by $\phi \in M_{\overline{p},\overline{V}}(\mathbb{R}^n,E)$ and

$$\sup_{\lambda\geq 1} D^1(\lambda,\overline{p},\overline{W})D(\lambda^{-1},\overline{p},\overline{V}) < \infty.$$

Although Theorem 2.10 can be simplified in such a case, we only state the corollary of Theorem 2.12.

Corollary 3.6 *Let $1 < p_1, p_2 < \infty$ and let $\overline{p} = (p_1,p_2)$. Let $W_i \in \mathcal{RW}^{q_i}(\mathbb{R}^n)\cap\mathcal{DW}$ and let w_i be their restrictions to $\mathbb{T}^n$.*

Let U_i be ∞-radial weights $U_1 \in L^{q_1'}_{loc}(\mathbb{R}^n)$ and $U_2^{1-p_2'} \in L^{q_2'}_{loc}(\mathbb{R}^n)$. We further assume that $V_i = U_iW_i \in \mathcal{DW}(\mathbb{R}^n)$ for $i = 1,2$ and

$$A = \sup_{\lambda\geq 1} D^1(\lambda,\overline{p},\overline{W})D(\lambda^{-1},\overline{p},\overline{V}) < \infty.$$

Let $\phi\colon \mathbb{R}^n \to E$ be a regulated function. If $\phi \in M_{\overline{p},\overline{V}}(\mathbb{R}^n,E)$, then

$$\phi\,|_{\mathbb{Z}^n} \in M_{\overline{p},\overline{w}}(\mathbb{T}^n,E) \quad \textit{and} \quad \|\phi|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}}(\mathbb{T}^n,E)} \leq CA\|\phi\|_{M_{\overline{p},\overline{V}}(\mathbb{R}^n,E)},$$

where $C = \|U_1^{1/p_1}\chi_{\{|x|<1\}}\|_{L^{p_1}(\mathbb{R}^n)}\|U_2^{-1/p_2}\chi_{\{|x|<1\}}\|_{L^{p_2'}(\mathbb{R}^n)}$.

3.3.2 Improving Theorem E

Applying Corollary 3.6 for $p_1 = p_2$, $W_1 = W_2 = 1$ and $U_1 = U_2 = U$, we can obtain the following corollary, which recovers Theorem E.

Corollary 3.7 *Let $1 < p < \infty$ and let U be a ∞-radial weight in $\mathcal{DW}(\mathbb{R}^n)$,*

$$C = \sup_{m\in\mathbb{N}} \frac{1}{m^n}\left(\int_{|x|<m} U(x)dx\right)^{1/p}\left(\int_{|x|<m} U(x)^{1-p'}dx\right)^{1/p'} < \infty \tag{3.2}$$

and

$$\sup_{\lambda<1} D(\lambda,(p,p),(U,U)) < \infty. \tag{3.3}$$

Let $\phi\colon \mathbb{R}^n \to E$ be a bounded regulated function. If $\phi \in M_{p,U}(\mathbb{R}^n, E)$, then $\phi \in M_p(\mathbb{R}^n, E)$ and $\phi|_{\mathbb{Z}^n} \in M_p(\mathbb{T}^n, E)$ with

$$\max\big\{\, \|\phi\|_{M_p(\mathbb{R}^n,E)}, \|\phi|_{\mathbb{Z}^n}\|_{M_p(\mathbb{T}^n,E)}\big\} \le C \sup_{\lambda>1} D(\lambda, p, U)^{-1}\|\phi\|_{M_{p,U}(\mathbb{R}^n,E)}.$$

Remark 3.8 Observe that Theorem E follows from Corollary 3.7, by selecting $U(x) = |x|^\gamma$. Notice that the condition (3.2) holds if $-n < \gamma < n(p-1)$ and the obvious condition $D(\lambda,(p,p),(U,U)) = 1$.

When $W_1 = W_2 = 1$, the statement of Theorem 2.10 reads as follows.

Corollary 3.9 *Let $\overline{p} = (p_1, p_2)$ with $0 < p_1, p_2 < \infty$.*
Let U_i be nonnegative locally integrable functions with $U_2(x) > 0$ if $x > 0$ and

$$\sup_{\{m\le|y|\le 2m\}} U_1(y) \le K \inf_{\{m\le|y|\le 2m\}} U_2(y). \tag{3.4}$$

Let $\phi\colon \mathbb{R}^n \to E$ be a bounded regulated function such that $\phi \in M_{\overline{p},D^1_m\overline{U}}(\mathbb{R}^n, E)$ for all $m \in \mathbb{N}$ and

$$A = \sup_{m\in\mathbb{N}} \|\phi\|_{M_{\overline{p},D^1_m\overline{U}}(\mathbb{R}^n,E)} < \infty, \tag{3.5}$$

then $\phi|_{Z^n} \in M_{\overline{p}}(\mathbb{T}^n, E)$ and $\phi \in M_{\overline{p}}(\mathbb{R}^n, E)$.
Moreover, $\max\{\|\phi\|_{M_{\overline{p}}(\mathbb{R}^n,E)}, \|\phi|_{Z^n}\|_{M_{\overline{p}}(\mathbb{T}^n,E)}\} \le KA$.

This last result shows that the family of power weights and the values of p in Theorem E can be improved. Note that if $U(x) = |x|^\gamma$, then $D^1_m U(x) = m^{-(n+\gamma)}U(x)$ and for all $m \in \mathbb{N}$,

$$\|\phi\|_{M_{\overline{p},D^1_m\overline{U}}(\mathbb{R}^n,E)} = m^{-(n+\gamma_2)/p_2+(n+\gamma_1)/p_1}\|\phi\|_{M_{\overline{p},\overline{U}}(\mathbb{R}^n,E)}.$$

Notice that conditions (3.4) and (3.5) hold for power weights if $\gamma_1 = \gamma_2$ and $p_1 = p_2$. In particular, only the local integrability of the weight is required for the result, and so we obtain the following improvement.

Corollary 3.10 *Let $0 < p < \infty$ and $U(x) = |x|^\gamma$ for $-n < \gamma$.*

(i) *If $\phi \in M_{p,U}(\mathbb{R}^n)$, then $\phi\,|_{\mathbb{Z}^n} \in M_p(\mathbb{T}^n)$.*
(ii) *If $\phi \in M_{p,U}(\mathbb{R}^n)$, then $\phi \in M_p(\mathbb{R}^n)$.*

3.3.3 Other Applications

Note that we can use Corollary 3.7 to conclude some results about multipliers on weighted L^p spaces. Using Fefferman's result on the ball multiplier, one obtains the following consequence.

Corollary 3.11 *Let $1 < p < \infty$, $n \geq 2$ and let U be a locally integrable weight in $\mathbb{R}^n$ that satisfies (3.2) and (3.3). Then $\phi = \chi_{\{|x|<1\}} \notin M_{p,U}(\mathbb{R}^n)$ for $p \neq 2$. In particular $\chi_{\{|x|<1\}} \notin M_{p,|x|^\gamma}(\mathbb{R}^n)$ for $p \neq 2$ and $-n < \gamma < (p-1)n$.*

Recalling that $M_{\overline{p},\overline{W}}(\mathbb{R}^n) = \{0\}$ when and $W_1 = W_2 = 1$ and $p_1 > p_2$, we can now also apply Corollary 3.6 and Theorem 3.9 for $p_2 < p_1$ and $W_1 = W_2 = 1$ to obtain the following result.

Corollary 3.12 *Let $\overline{p} = (p_1, p_2)$ with $0 < p_1, p_2 < \infty$ and let U_1, U_2 be ∞-radial weight functions with*

$$\sup_{0<\lambda<1} D(\lambda, \overline{p}, \overline{U}) > 0, \tag{3.6}$$

which satisfy that either $1 < p_2 < p_1 < \infty$ and

$$\sup_{m\in\mathbb{N}} \left(\frac{1}{m^n} \int_{|x|<m} U_1(x)dx \right)^{1/p_1} \left(\frac{1}{m^n} \int_{|x|<m} U_2^{1-p_2'}(x)dx \right)^{1/p_2'} < \infty \tag{3.7}$$

or $0 < p_2 < p_1 < \infty$ and

$$\sup_{m\in\mathbb{N}} \frac{\sup_{m\leq|x|\leq 2m} U_1(x)dx}{\inf_{m\leq|x|\leq 2m} U_2(x)dx} < \infty. \tag{3.8}$$

Then $M_{\overline{p},\overline{U}}(\mathbb{R}^n) = \{0\}$.

The last corollary is an application of the previous result.

Corollary 3.13 *There is no nonzero multiplier from $L^{p_1}(\mathbb{R}^n, |x|^{\gamma_1})$ to $L^{p_2}(\mathbb{R}^n, |x|^{\gamma_2})$ whenever either*

$$p_1 > p_2 \quad \textit{and} \quad -n < \gamma_1 \leq \gamma_2$$

or

$$1 < p_2 < p_1, -n < \gamma_2 < n(p_2 - 1) \quad \textit{and} \quad -n < \gamma_2 < \gamma_1 \leq \frac{p_1}{p_2}\gamma_2.$$

In this case, $W_i = 1$, $U_i(x) = |x|^{\gamma_i}$, $V_i = U_i$ and so we have $D(\lambda, \overline{p}, \overline{U}) = \lambda^{\frac{n+\gamma_2}{p_2} - \frac{n+\gamma_1}{p_1}}$.

Moreover, the local integrability means $\min\{\gamma_1, \gamma_2\} > -n$, while condition (3.7) becomes

$$\frac{\gamma_2}{p_2} \geq \frac{\gamma_1}{p_1}, \ \gamma_1 > -n, \ \gamma_2 < n(p_2 - 1)$$

and condition (3.8) becomes $\gamma_1 \leq \gamma_2$. In both cases condition (3.6), corresponding to $\frac{n+\gamma_2}{p_2} \geq \frac{n+\gamma_1}{p_1}$, automatically holds.

4 Restriction of Multipliers

4.1 A New Class of Weights

We start this section by showing that the new class of weights $\mathcal{RW}(\mathbb{R}^n)$ presented in Definition 2.2 contains the classical classes of periodic functions and Bohr and Besicovich almost periodic functions.

We recall that an almost periodic function in the real line has been shown to coincide with uniform limits of trigonometric polynomials, $F(x) = \sum_{j=1}^{N} \alpha_j e^{2\pi i \lambda_j x}$, where $\lambda_j \in \mathbb{R}$ (see for instance [11]). It is easy to show that almost periodic functions are uniformly continuous, bounded, and that the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)dx$$

exists.

Therefore, we say that a locally integrable function $f : \mathbb{R}^n \to \mathbb{C}$ is almost periodic (in the sense of Bohr), $f \in AP(\mathbb{R}^n)$, if f can be approximated by trigonometric polynomials $f_N(x) = \sum_{j=1}^{N} \alpha_j e^{2\pi i \lambda_j \cdot x}$, where $\alpha_j \in \mathbb{C}$ and $\lambda_j \in \mathbb{R}^n$ under the norm in $L^\infty(\mathbb{R}^n)$.

Somewhat later Besicovitch considered the space of functions that can be approximated by trigonometric polynomials under the norm

$$\|f\|_{B_1} = \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x)|dx$$

and described them in more intrinsic way (see [3, 4, 6]).

We then say that a locally integrable function $f : \mathbb{R}^n \to \mathbb{C}$ belongs to $B^1(\mathbb{R}^n)$ if f can be approximated by almost periodic functions under the norm

$$\|f\|_{B^1(\mathbb{R}^n)} = \limsup_{\lambda\to\infty} \frac{1}{|\lambda\mathbb{T}^n|} \int_{\lambda\mathbb{T}^n} |f(x)|dx.$$

Condition (2.3) is trivial to check either for almost periodic functions and for functions in the Besicovitch class $B^1(\mathbb{R}^n)$. Let us show that weights belonging to these classes are included in $\mathcal{RW}(\mathbb{R}^n)$.

Theorem 4.1 (i) *If $f \in AP(\mathbb{R}^n)$, then there exists a sequence $(A_k)_{k\in\mathbb{Z}^n}$ such that for any interval $I \subset \mathbb{R}^n$, $|I| \neq 0$,*

$$\lim_{\lambda\to\infty} \frac{1}{|\lambda I|} \int_{\lambda I} f(x) e^{-2\pi i k x} dx = A_k.$$

(ii) *If $f \in B^1(\mathbb{R}^n)$, then there exists a sequence $(A_k)_{k\in\mathbb{Z}^n}$ such that*

$$\lim_{\lambda\to\infty} \frac{1}{|\lambda\mathbb{T}^n|} \int_{\lambda\mathbb{T}^n} f(x) e^{-2\pi i k x} dx = A_k.$$

Proof (i) We first notice that for $a < b$ and a real number $\gamma \neq 0$ one has that

$$\lim_{T\to\infty} \frac{1}{T(b-a)} \int_{aT}^{bT} e^{2\pi i\gamma x} dx = 0. \tag{4.1}$$

Hence for a trigonometric polynomial $f(x) = \sum_{j=0}^{N} \alpha_j e^{2\pi i\gamma^j x}$, where $\alpha_j \in \mathbb{C}$ and $\gamma^j \in \mathbb{R}^n$, one concludes that for any interval $I = \prod_{r=1}^{n} I_r$, any $j \in \{0, \cdots, N\}$, and any given $k \in \mathbb{Z}^n$,

$$\lim_{\lambda\to\infty} \frac{1}{|\lambda I|} \int_{\lambda I} e^{2\pi i\gamma^j x} e^{-2\pi ikx} dx = \prod_{r=1}^{n} \lim_{\lambda\to\infty} \frac{1}{|\lambda I_r|} \int_{\lambda I_r} e^{2\pi i(\gamma_r^j - k_r)x_r} dx.$$

Therefore, from (4.1) one obtains

$$\lim_{\lambda\to\infty} \frac{1}{|\lambda I|} \int_{\lambda I} f(x), e^{-2\pi ikx} dx = A_k(f)$$

where $A_k(f) = \alpha_j$ whenever $\gamma^j = k$, and 0 otherwise.

For a general almost periodic f, given $\varepsilon > 0$, take f_N polynomial with

$$\|f - f_N\|_{L^\infty(\mathbb{R}^n)} < \varepsilon/4$$

and λ_0 such that if $\lambda \geq \lambda_0$ one has that

$$|A_k(f_N) - \frac{1}{|\lambda I|} \int_{\lambda I} f_N(x) e^{-2\pi ikx} dx| < \varepsilon/4.$$

Therefore, if $\lambda_1, \lambda_2 \geq \lambda_0$, then

$$\begin{aligned}
&\Big| \frac{1}{|\lambda_1 I|} \int_{\lambda_1 I} f(x) e^{-2\pi ikx} dx - \frac{1}{|\lambda_2 I|} \int_{\lambda_2 I} f(x) e^{-2\pi ikx} dx \Big| \\
&\quad \leq \frac{1}{|\lambda_1 I|} \int_{\lambda_1 I} |f(x) - f_N(x)| dx + \frac{1}{|\lambda_2 I|} \int_{\lambda_2 I} |f(x) - f_N(x)| dx \\
&\qquad + \Big| \frac{1}{|\lambda_1 I|} \int_{\lambda_1 I} f_N(x) e^{-2\pi ikx} dx - \frac{1}{|\lambda_2 I|} \int_{\lambda_2 I} f_N(x) e^{-2\pi ikx} dx \Big| \\
&\quad \leq 2\|f - f_N\|_{L^\infty(\mathbb{R}^n)} + \varepsilon/2 < \varepsilon.
\end{aligned}$$

This shows that $(\frac{1}{|\lambda_n I|} \int_{\lambda_n I} f(x) e^{-2\pi ikx} dx)_m$ is Cauchy sequence for any sequence $\lambda_n \to \infty$, and hence $\lim_{\lambda\to\infty} \frac{1}{|\lambda I|} \int_{\lambda I} f(x) e^{-2\pi ikx} dx$ exists.

Part (ii) follows the same argument as (i) using only $I = \mathbb{T}^n$ and replacing the $L^\infty(\mathbb{R}^n)$ by the $B^1(\mathbb{R}^n)$ norm. ∎

We also remark that not only do almost periodic functions (and in particular continuous and periodic functions) but also the periodic weights (which are just assumed to be locally integrable) belong to $B^1(\mathbb{R}^n)$.

Remark 4.2 Given $w \in L^1(\mathbb{T}^n)$ and $(\alpha_k)_{k\in\mathbb{Z}^n}$ one can define

$$W(x) = \alpha_{[x]} w\big(x - [x]\big),$$

where $[x]$ stands for the unique value $k \in \mathbb{Z}^n$ such that $x - k \in \mathbb{T}^n$. The argument above shows that $W \in \mathcal{RW}(\mathbb{R}^n)$ whenever there exist the limit

$$\lim_{m\to\infty} \frac{1}{m^n} \sum_{|j|\le m} \alpha_j = A \quad \text{and} \quad \sup_m \frac{1}{m^n} \sum_{|k|\le m} |\alpha_k| < \infty.$$

In such a case,

$$\lim_{\lambda\to\infty} \frac{1}{|\lambda I|} \int_{\lambda I} W(x) e^{-2\pi i k x} dx = A\widehat{w}(k),$$

where $\widehat{w}(k)$ denotes the k-th Fourier coefficient of w.

In particular, selecting $\alpha_k = 1$ for $k \in \mathbb{Z}^n$ one gets that if W is a periodic weight then $W \in \mathcal{RW}(\mathbb{R}^n)$. Moreover, $d\mu_W(x) = W(x)\chi_{\mathbb{T}^n}(x)dx$, because for every $m \in \mathbb{N}$,

$$\lim_{m\to\infty} \frac{1}{(2m+1)^n} \int_{(2m+1)\mathbb{T}^n} e^{-2\pi i k x} W(x) dx = \int_{\mathbb{T}^n} e^{-2\pi i k x} w(x) dx = \widehat{w}(k).$$

We now prove now the main property of this new class of functions to be used in the sequel.

Lemma 4.3 *Let $W \in \mathcal{RW}(\mathbb{R}^n)$ with restriction to $\mathbb{T}^n$, μ_W, as defined in (2.5). Then for any $g \in C(\mathbb{T}^n)$ we have*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} g(x) W(x) dx = \int_{\mathbb{T}^n} g(x) d\mu_W(x).$$

In particular, $gW \in \mathcal{RW}(\mathbb{R}^n)$ for any g nonnegative continuous and periodic function. Moreover, $d\mu_{gW} = g d\mu_W$.

Proof Assume first that g is a trigonometric polynomial; that is,

$$g(x) = \sum_{|j|\le N} \alpha_j e^{2\pi i j x}.$$

Then

$$\frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} g(x) W(x) dx = \sum_{|j|\le N} \alpha_j \frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} W(x) e^{2\pi i j x} dx$$

and, by definition,

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} g(x) W(x) dx = \sum_{|j|\le N} \alpha_j A_{-j} = \int_{\mathbb{T}^n} g(x) d\mu(x).$$

For the general case, we take, for any $\epsilon > 0$, a trigonometric polynomial P such that $\|g - P\|_{L^\infty(\mathbb{R}^n)} < \epsilon B^{-1}$, where B is the constant appearing in (2.3). Then

$$\left| \frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} g(x)W(x)dx - \int_{\mathbb{T}^n} g(x)d\mu_W(x) \right| \leq \left| \frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} (g(x) - P(x))W(x)dx \right|$$
$$+ \left| \frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} P(x)W(x)dx - \int_{\mathbb{T}^n} P(x)d\mu_W(x) \right| + \left| \int_{\mathbb{T}^n} (P(x) - g(x))d\mu_W(x) \right|.$$

The second term tends to zero when λ tends to infinity by the previous case. So, it only remains to prove that the limit of the first and third terms when λ tends to infinity can be made as small as needed. But this is trivial, since

$$\frac{1}{\lambda^n} \left| \int_{\lambda\mathbb{T}^n} (g(x) - P(x))W(x)e^{-2\pi ikx}dx \right| \leq \|g - P\|_{L^\infty(\mathbb{T}^n)} \frac{1}{\lambda^n} \int_{\lambda\mathbb{T}^n} W(x)dx < \epsilon,$$

and the proof is finished. ■

Remark 4.4 It follows from Lemma 4.3 that μ_W is a nonnegative measure.

4.2 Lemmata

Now we develop a lemma to replace the use of $\chi_{\mathbb{T}^n}$ by a more general approximation of the identity. For this, we first prove the following property of functions in $\mathcal{RW}(\mathbb{R}^n)$.

Lemma 4.5 *Let $W \in \mathcal{RW}(\mathbb{R}^n)$ with the associated sequence $(A_k)_{k\in\mathbb{Z}^n}$. Then, for every interval $I = [a, b] \subset \mathbb{R}$ with $a < b$, we have that*

$$\lim_{\lambda\to\infty} \frac{1}{|\lambda J|} \int_{\lambda J} W(x)e^{-2\pi ikx}dx = A_k,$$

where $J = \prod_{j=1}^n (I \cup (-I))$.

Proof We first prove the same property when $J = \prod_{j=1}^n [-a, a]$. In such a case, the result is trivial as we see:

$$\frac{1}{|\lambda J|} \int_{\lambda J} W(x)e^{-2\pi ikx}dx - A_k = \frac{1}{|\lambda J|} \int_{\lambda J} (W(x)e^{-2\pi ikx} - A_k)dx.$$

Since $J = 2a\mathbb{T}^n$, we have that previous expression can be rewritten as

$$\frac{1}{|2a\lambda\mathbb{T}^n|} \int_{2a\lambda\mathbb{T}^n} (W(x)e^{-2\pi ikx} - A_k)dx$$

which tends to zero by the definition of A_k.

Now, for every $I = [a, b] \subset \mathbb{R}$, let $r_1 = \max(|a|, |b|)$ and $r_2 = \min(|a|, |b|)$ Then we have that $I \cup (-I)$ is either $[-r_1, r_1] \backslash [-r_2, r_2]$ if a and b have the same sign or $[-r_1, r_1]$ otherwise. Therefore, $J = 2r_1\mathbb{T}^n \backslash 2r_2\mathbb{T}^n$. This implies

$$\frac{1}{|\lambda J|}\int_{\lambda J}(W(x)e^{-2\pi ikx} - A_k)dx = \frac{|2r_1\mathbb{T}^n|}{|J|}\frac{1}{|2r_1\lambda\mathbb{T}^n|}\int_{2r_1\lambda\mathbb{T}^n}(W(x)e^{-2\pi ikx} - A_k)dx$$
$$- \frac{|2r_2\mathbb{T}^n|}{|J|}\frac{1}{|2r_2\lambda\mathbb{T}^n|}\int_{2r_2\lambda\mathbb{T}^n}(W(x)e^{-2\pi ikx} - A_k)dx,$$

and both terms tend to zero when m tends to infinity.

Remark 4.6 Assume that f satisfies (2.3) for some $1 \le p < \infty$. For any rectangle $I \subset \mathbb{R}^n$ with $|I| \neq 0$ there is $k \in \mathbb{N}$ such that $I \subset k\mathbb{T}^n$, and so

$$\limsup_{\lambda\to\infty}\frac{1}{|\lambda I|}\int_{\lambda I}|f(x)|^p dx \le \limsup_{\lambda\to\infty}\frac{|\lambda k\mathbb{T}^n|}{|\lambda I|}\frac{1}{|\lambda k\mathbb{T}^n|}\int_{\lambda k\mathbb{T}^n}|f(x)|^p dx$$
$$= \frac{|k\mathbb{T}^n|}{|I|}\limsup_{\lambda\to\infty}\frac{1}{\lambda^n}\int_{\lambda\mathbb{T}^n}|f(x)|^p dx < \frac{|k\mathbb{T}^n|}{|I|}B^p.$$

Lemma 4.7 *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfying (2.1), (2.3) for some $1 \le p < \infty$. Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ belonging to $L^{p'}(\mathbb{R}^n)$ be ∞-radial with compact support (and continuous in the case $p = 1$). Then,*

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}^n} f(x)D^1_\lambda\varphi(x)dx = M(f)\int_{\mathbb{R}^n}\varphi(x)dx.$$

Proof We first assume that φ is a ∞-radial step function, $\varphi = \sum_{|k|<N}\varphi_k\chi_{\Delta_k}$ with $\Delta_k = \prod_{j=1}^n(I_k \cup (-I_k))$. In this case the proof is trivial, since

$$\int_{\mathbb{R}^n} f(x)D^1_\lambda\varphi(x)dx = \sum_{|k|<N}\varphi_k\frac{1}{\lambda^n}\int_{\mathbb{R}^n} f(x)\chi_{\lambda\Delta_k}(x)dx$$
$$= \sum_{|k|<N}\varphi_k|\Delta_k|\frac{1}{|\lambda\Delta_k|}\int_{\lambda\Delta_k} f(x)dx,$$

and so by the same arguments as in the previous lemma

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}^n} f(x)D^1_\lambda\varphi(x)dx = \sum_{|k|<N}\varphi_k|\Delta_k|\lim_{\lambda\to\infty}\frac{1}{|\lambda\Delta_k|}\int_{\lambda\Delta_k} f(x)dx$$
$$= \left(\int_{\mathbb{R}^n}\varphi(x)dx\right)M(f).$$

For the general case, we argue as follows. Let $K = [-a, a]^n \subset \mathbb{R}^n$ rectangle such that $\text{supp}(\varphi) \subset K$. For any $\epsilon > 0$ let φ_ϵ be a ∞-radial step function as previously described with support in K such that $\|\varphi - \varphi_\epsilon\|_{L^{p'}(\mathbb{R}^n)} < (|K|^{1/p}B)^{-1}\epsilon$. Notice that the step functions φ_ϵ have domain defined by a finite union of bounded intervals. Therefore, when $p = 1$, the approximation is only possible if φ is assumed to be continuous. Then

$$\int_{\mathbb{R}^n} f(x)D^1_\lambda\varphi(x)dx = \int_{\mathbb{R}^n} f(x)D^1_\lambda\varphi_\epsilon(x)dx + \int_{\mathbb{R}^n} f(x)\big(D^1_\lambda\varphi(x) - D^1_\lambda\varphi_\epsilon(x)\big)\,dx.$$

The first term tends to the expected limit because φ_ϵ is a step function. So we just need to prove that the limit of the second term when λ tends to infinity can be done as small as needed. This second term can be rewritten and bounded in the following way:

$$\begin{aligned}
&\Big|\int_{\lambda K} f(x)(D^1_\lambda\varphi(x) - D^1_\lambda\varphi_\epsilon(x))dx\Big| \\
&\quad \leq \|D^1_\lambda(\varphi - \varphi_\epsilon)\|_{L^{p'}(\mathbb{R}^n)}\left(\int_{\lambda K}|f(x)|^p dx\right)^{1/p} \\
&\quad = \|\varphi - \varphi_\epsilon\|_{L^{p'}(\mathbb{R}^n)}\left(\frac{1}{\lambda^n}\int_{\lambda K}|f(x)|^p dx\right)^{1/p} \\
&\quad = \|\varphi - \varphi_\epsilon\|_{L^{p'}(\mathbb{R}^n)}|K|^{1/p}\left(\frac{1}{|\lambda K|}\int_{\lambda K}|f(x)|^p dx\right)^{1/p}
\end{aligned}$$

Taking limit in λ we have

$$\lim_{\lambda\to\infty}\Big|\int_{\lambda K} f(x)(D^1_\lambda\varphi(x) - D^1_\lambda\varphi_\epsilon(x))dx\Big| \leq \|\varphi - \varphi_\epsilon\|_{L^{p'}(\mathbb{R}^n)}|K|^{1/p}B < \epsilon. \qquad \blacksquare$$

Finally, we prove the following corollary, which will be the usual way in which we will use the previous result:

Corollary 4.8 *Let $W \in \mathcal{RW}(\mathbb{R}^n)$ and let μ be its restriction to $\mathbb{T}^n$. Let $f\colon \mathbb{R}^n \to E$ be a vector-valued trigonometric polynomial and let φ be a ∞-radial continuous function with compact support. Then*

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}^n}\|f(x)D^p_\lambda\varphi(x)\|^p W(x)dx = \left(\int_{\mathbb{T}^n}\|f(x)\|^p d\mu(x)\right)\left(\int_{\mathbb{R}^n}|\varphi(x)|^p dx\right).$$

Proof Since $\|f\|^p$ is a periodic continuous function, we have by Lemma 4.3 that $\|f\|^p W \in \mathcal{RW}(\mathbb{R}^n)$, and we then obtain the result by Lemma 4.7 applied to $\|f\|^p W$ and $|\varphi|^p$. $\blacksquare$

4.3 Proof of Theorem 2.4

Proof in the case of symbols continuous at $\mathbb{Z}^n$ Now we give two basic lemmata that describe the way we can compare T_ϕ and $T_{\phi|_{\mathbb{Z}^n}}$ for continuous symbols.

Lemma 4.9 *Let $\phi\colon \mathbb{R}^n \to E$ be a bounded function continuous in $\mathbb{Z}^n$ and let $\Phi \in L^1(\mathbb{R}^n)$ such that $\widehat{\Phi} \in L^1(\mathbb{R}^n)$. Then*

$$\lim_{\lambda\to\infty} \big\| T_\phi(PD_\lambda\Phi) - T_{\phi|_{\mathbb{Z}^n}}(P)D_\lambda\Phi \big\|_{L^\infty(\mathbb{R}^n,E)} = 0$$

for any $P \in \mathcal{P}(\mathbb{T}^n)$.

Proof Let us first assume that $P(x) = e^{2\pi ikx}$ for $k \in \mathbb{Z}^n$. Notice that $PD_\lambda\Phi \in L^1(\mathbb{R}^n)$, so $T_\phi(PD_\lambda\Phi)$ is well defined. Moreover, since $\widehat{D_\lambda\Phi} = D^1_{1/\lambda}\widehat{\Phi}$,

$$\begin{aligned}
&T_\phi(PD_\lambda\Phi)(x) - T_{\phi|_{\mathbb{Z}^n}}(P)(x)D_\lambda\Phi(x)\\
&\quad= \int_{\mathbb{R}^n} \phi(y)D^1_{1/\lambda}\widehat{\Phi}(y-k)e^{2\pi ixy}dy - \phi(k)e^{2\pi ikx}\int_{\mathbb{R}^n} D^1_{1/\lambda}\widehat{\Phi}(y)e^{2\pi ixy}dy\\
&\quad= e^{2\pi ikx}\int_{\mathbb{R}^n} (\phi(y+k)-\phi(k))D^1_{1/\lambda}\widehat{\Phi}(y)e^{2\pi ixy}dy.
\end{aligned}$$

Therefore, by linearity, for a polynomial $P(x) = \sum_{|k|\le N} \alpha_k e^{2\pi ikx}$,

$$\begin{aligned}
&T_\phi(PD_\lambda\Phi)(x) - T_{\phi|_{\mathbb{Z}^n}}(P)(x)D_\lambda\Phi(x) =\\
&\qquad\qquad \sum_{|k|\le N} \alpha_k\Big(\int_{\mathbb{R}^n}\big(\phi(y+k)-\phi(k)\big)\,D^1_{1/\lambda}\widehat{\Phi}(y)e^{2\pi ixy}dy\Big)\,e^{2\pi ikx}.
\end{aligned}$$

This shows that

$$\begin{aligned}
&\big\| T_\phi(PD_\lambda\Phi)(x) - T_{\phi|_{\mathbb{Z}^n}}(P)(x)D_\lambda\Phi(x)\big\|_{L^\infty(\mathbb{R}^n,E)} \le\\
&\qquad\qquad \sum_{|k|\le N} |\alpha_k| \int_{\mathbb{R}^n} \big\|\phi(y+k)-\phi(k)\big\|_{L^\infty(\mathbb{R}^n,E)} |D^1_{1/\lambda}\widehat{\Phi}(y)|dy.
\end{aligned}$$

Therefore we conclude the result by using Lebesgue's Dominated Convergence Theorem and the continuity of ϕ at $k \in \mathbb{Z}^n$. ■

Lemma 4.10 *Let $0 < p < \infty$, $W \in \mathcal{RW}(\mathbb{R}^n)$ with restriction measure μ_W. Let $\Phi \in C_c(\mathbb{R}^n)$, ∞-radial and such that $\|\Phi\|_{L^p(\mathbb{R}^n)} = 1$. Let $\phi\colon \mathbb{R}^n \to E$ be continuous in $\mathbb{Z}^n$. Then*

$$\Big(\int_{\mathbb{T}^n} \|T_{\phi|_{Z^n}}(P)(x)\|^p d\mu_W(x)\Big)^{1/p} = \lim_{\lambda\to\infty}\Big(\int_{\lambda K} \|T_\phi(PD^p_\lambda\Phi)(x)\|^p W(x)dx\Big)^{1/p}$$

for any $P \in \mathcal{P}(\mathbb{T}^n)$, where $K = \prod_{j=1}^n [-a,a]$ such that $\operatorname{supp}\Phi \subset K$.

Proof Let P be a polynomial and $\Phi \in C_c(\mathbb{R}^n)$ ∞-radial such that $\operatorname{supp}(\Phi) \subset K = \prod_{j=1}^{n}[-a,a]$ with $\|\Phi\|_{L^p(\mathbb{R}^n)} = 1$.

Since $f = T_{\phi|_{Z^n}}(P)$ is a trigonometric polynomial and $\varphi = |\Phi|^p$ is continuous, ∞-radial and compactly supported, we have from Corollary 4.8

$$\int_{\mathbb{T}^n} \|T_{\phi|_{Z^n}}(P)(x)\|^p d\mu(x) = \lim_{\lambda\to\infty} \int_{\lambda K} \|T_{\phi|_{Z^n}}(P)(x) D_\lambda^p \Phi(x)\|^p W(x)dx.$$

Now, when $1 \le p < \infty$ we use Minkowski's inequalities

$$\|f\|_{L^p(\mathbb{R}^n,W)} - \|g\|_{L^p(\mathbb{R}^n,W)} \le \|f+g\|_{L^p(\mathbb{R}^n,W)} \le \|f\|_{L^p(\mathbb{R}^n,W)} + \|g\|_{L^p(\mathbb{R}^n,W)},$$

while when $0 < p < 1$, we use the pointwise estimates

$$\|f(x)\|^p - \|g(x)\|^p \le \|f(x)+g(x)\|^p \le \|f(x)\|^p + \|g(x)\|^p$$

to obtain for $s = \max(p,1)$,

$$\Big(\int_{\lambda K} \|T_{\phi|_{Z^n}}(P)(x) D_\lambda^p \Phi(x)\|^p W(x)dx\Big)^{1/s} \le \Big(\int_{\lambda K} \|T_\phi(PD_\lambda^p \Phi)(x)\|^p W(x)dx\Big)^{1/s}$$
$$+ \Big(\lambda^{-n} \int_{\lambda K} \|T_{\phi|_{Z^n}}(P)(x) D_\lambda \Phi(x) - T_\phi(PD_\lambda \Phi)(x)\|^p W(x)dx\Big)^{1/s}$$

and

$$\Big(\int_{\lambda K} \|T_{\phi|_{Z^n}}(P)(x) D_\lambda^p \Phi(x)\|^p W(x)dx\Big)^{1/s} \ge \Big(\int_{\lambda K} \|T_\phi(PD_\lambda^p \Phi)(x)\|^p W(x)dx\Big)^{1/s}$$
$$- \Big(\lambda^{-n} \int_{\lambda K} \|T_{\phi|_{Z^n}}(P)(x) D_\lambda \Phi(x) - T_\phi(PD_\lambda \Phi)(x)\|^p W(x)dx\Big)^{1/s}$$

In both cases we just need to prove that the second term tends to zero when λ tends to infinity. This second term can be bounded by

$$\|T_{\phi|_{Z^n}}(P)(x) D_\lambda \Phi(x) - T_\phi(PD_\lambda \Phi)(x)\|_{L^\infty(\mathbb{R}^n,E)}^{p/s} \Big(\lambda^{-n} \int_{\lambda K} W(x)dx\Big)^{1/s},$$

so the result follows by taking lim as $\lambda \to \infty$ and invoking Lemma 4.9 and that $\limsup_{\lambda\to\infty} |\lambda^{-n} \int_{\lambda K} W(x)dx| \le C|K|M(W) < \infty$. ■

Proof of Theorem 2.4 for symbols continuous at $\mathbb{Z}^n$

Select $\Phi \in C_c(\mathbb{R}^n)$ ∞-radial with $\|\Phi\|_p = 1$. From Lemma 4.10 one has

$$\Big(\int_{\mathbb{T}^n} \|T_{\phi|_{Z^n}}(P)(x)\|^p d\mu_2(x)\Big)^{1/p}$$
$$= \lim_{\lambda\to\infty} \Big(\int_{\mathbb{R}^n} \|T_\phi(PD_\lambda^p \Phi)(x)\|^p W_2(x)dx\Big)^{1/p}$$
$$\le \|\phi\|_{M_{\overline{p},\overline{U}}(\mathbb{R}^n,E)} \lim_{\lambda\to\infty} \Big(\int_{\mathbb{R}^n} |P(x)|^p |D_\lambda^p \Phi(x)|^p W_1(x)dx\Big)^{1/p}.$$

By using Corollary 4.8 again for $f = P$ and $\varphi = |\Phi|^p$, we get

$$\lim_{\lambda\to\infty}\left(\int_{\mathbb{R}^n}|P(x)|^p|D^1_\lambda|\Phi|^p(x)W_1(x)dx\right)^{1/p} = \left(\int_{\mathbb{T}^n}|P(x)|^p d\mu_1(x)\right)^{1/p}.$$

Therefore,

$$\|T_{\phi|_{Z^n}}(P)\|_{L^p_E(\mathbb{T}^n,\mu_2)} \leq \|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}\|P\|_{L^p(\mathbb{T}^n,\mu_1)},$$

and, by the density of polynomials in $L^p(\mathbb{T}^n,\mu_1)$, the proof is complete. ■

Remark 4.11 The argument above shows that if ϕ is continuous at $\mathbb{Z}^n$ and $\phi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)$ for $\overline{p} = (p_1,p_2)$ for $p_2 < p_1$, then $\phi(k) = 0$ for any $k \in \mathbb{Z}^n$.

Proof in the case of regulated symbols We now prove the same result in the setting of regulated symbols. The main idea in the following lemma appears in [5, 13] when handling weak-type inequalities. Despite the fact that the restriction theorem proven so far is truly meaningful only when both integration exponents coincide, we state and prove the result about restriction of regulated symbols for two different exponents. We do so because the same method can be applied for other type of restriction theorems in the nonhomogeneous setting.

Lemma 4.12 *Let $0 < p_1, p_2 < \infty$. Let W_1, W_2 be weights in $\mathbb{R}^n$ and let $\varphi \in L^1(\mathbb{R}^n)$. Let $\phi\colon \mathbb{R}^n \to E$ be a bounded measurable function. If $\phi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)$, then $\phi * \varphi \in M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)$. Moreover, there exist $K_{\overline{p}} > 0$ such that*

$$\|\phi * \varphi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} \leq K_{\overline{p}}\|\varphi\|_{L^1(\mathbb{R}^n)}\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)},$$

where $K_{\overline{p}} = 1$ for $p_2 \geq 1$.

Proof Denoting $M_y f(x) = e^{2\pi yxi}f(x)$ we have that

$$T_{\phi*\varphi}(f)(x) = \int_{\mathbb{R}^n} M_y T_\phi M_{-y}(f)(x)\varphi(y)dy$$

for functions $f \in C^\infty_c(\mathbb{R}^n)$.

Then, for $p_2 \geq 1$, Minkowski's inequality shows that

$$\begin{aligned}\|T_{\phi*\varphi}(f)\|_{L^{p_2}_E(\mathbb{R}^n,W_2)} &\leq \int_{\mathbb{R}^n}\|T_\phi M_{-y}(f)\|_{L^{p_2}_E(\mathbb{R}^n,W_2)}|\varphi(y)|dy\\ &\leq \|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}\int_{\mathbb{R}^n}\|M_{-y}(f)\|_{L^{p_1}(\mathbb{R}^n,W_1)}|\varphi(y)|dy\\ &= \|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}\|\varphi\|_{L^1(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n,W_1)}.\end{aligned}$$

For $0 < p_2 < 1$ we write

$$T_{\phi*\varphi}(f)(x) = \int_{\mathbb{R}^n} T_\phi M_{-y}(f)(x)M_x\varphi(y)dy.$$

Now for each $x \in \mathbb{R}^n$, $f \in C_c^\infty(\mathbb{R}^n)$ and $e^* \in E'$, where E' is the topological dual of E, we have

$$|\langle T_{\phi*\varphi}(f)(x), e^*\rangle| \leq \int_{\mathbb{R}^n} \big|\langle T_\phi M_{-y}(f)(x), e^*\rangle\big|\, |\varphi(y)|dy.$$

On the other hand, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\|T_\phi M_{-y}(f)(x) - T_\phi M_{-y'}(f)(x)\| &= \left\| \int_{\mathbb{R}^n} \phi(\xi)(\tau_{-y}\widehat{f}(\xi) - \tau_{-y'}\widehat{f}(\xi))e^{2\pi i\xi x}d\xi \right\| \\ &\leq \|\phi\|_{L^\infty_E(\mathbb{R}^n)}\|\tau_{-y}\widehat{f} - \tau_{-y'}\widehat{f}\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Hence, given $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \|T_\phi M_{-y}(f)(x) - T_\phi M_{-y'}(f)(x)\|_E < \varepsilon \text{ whenever } |y - y'| < \delta.$$

Hence,

$$\begin{aligned}&|\langle T_{\phi*\varphi}(f)(x), e^*\rangle| \\ &\quad\leq \sum_{k\in\mathbb{Z}^n} \int_{\delta k+\delta\mathbb{T}^n} \big|\langle T_\phi M_{-y}(f)(x), e^*\rangle\big|\, |\varphi(y)|dy \\ &\quad\leq \sum_{k\in\mathbb{Z}^n} \big|\langle T_\phi M_{-\delta k}(f)(x), e^*\rangle\big| \int_{\delta k+\delta\mathbb{T}^n} |\varphi(y)|dy \\ &\qquad + \sum_{k\in\mathbb{Z}^n} \int_{\delta k+\delta\mathbb{T}^n} \big|\langle T_\phi M_{-\delta k}(f)(x), e^*\rangle - \langle T_\phi M_{-\delta k}(f)(x), e^*\rangle\big|\, |\varphi(y)|dy\end{aligned}$$

Therefore, denoting $y_k = y_k(\varepsilon) = \delta k$ and $\lambda_k^2 = \int_{\delta k+\delta\mathbb{T}^n} |\varphi(y)|dy$, one gets

$$\begin{aligned}&\big|\langle T_{\phi*\varphi}(f)(x), e^*\rangle\big| \\ &\quad\leq \sum_{k\in\mathbb{Z}^n} \big|\big\langle \lambda_k T_\phi M_{-y_k}(f)(x), e^*\big\rangle\big|\, \lambda_k + \varepsilon \sum_{k\in\mathbb{Z}^n} \lambda_k^2 \\ &\quad\leq \left(\sum_{k\in\mathbb{Z}^n} \big|\langle T_\phi(\lambda_k M_{-y_k}(f))(x), e^*\rangle\big|^2 \right)^{1/2} \left(\sum_{k\in\mathbb{Z}^n} |\lambda_k|^2 \right)^{1/2} + \varepsilon\|\varphi\|_{L^1(\mathbb{R}^n)} \\ &\quad\leq C\left(\int_0^1 \Big| \sum_{k\in\mathbb{Z}^n} \langle T_\phi(\lambda_k M_{-y_k}(f))(x), e^*\rangle r_k(t) \Big|^{p_2} dt \right)^{1/p_2} \|\varphi\|_{L^1(\mathbb{R}^n)}^{1/2} + \varepsilon\|\varphi\|_{L^1(\mathbb{R}^n)} \\ &\quad= C\left(\int_0^1 \Big|\Big\langle T_\phi\Big(\sum_{k\in\mathbb{Z}^n} \lambda_k M_{-y_k}(f) r_k(t) \Big)(x), e^* \Big\rangle\Big|^{p_2} dt \right)^{1/p_2} \|\varphi\|_{L^1(\mathbb{R}^n)}^{1/2} + \varepsilon\|\varphi\|_{L^1(\mathbb{R}^n)}\end{aligned}$$

where $(r_k)_{k\in\mathbb{Z}}$ denotes the Rademacher system and the last estimate follows from Khinchine's inequalites.

Hence,

$$\|T_{\phi*\varphi}(f)(x)\| \leq C\|\varphi\|^{1/2}_{L^1(\mathbb{R}^n)} \liminf_{\varepsilon\to 0}\Big(\int_0^1 \Big\| T_\phi\Big(\sum_{k\in\mathbb{Z}^n}\lambda_k M_{-y_k}(f)r_k(t)\Big)(x)\Big\|^{p_2} dt\Big)^{1/p_2},$$

and by Fatou's lemma one concludes that $\|T_{\phi*\varphi}(f)\|_{L^{p_2}_E(\mathbb{R}^n,W_2)}$ is bounded by

$$\begin{aligned}
&C\|\varphi\|^{1/2}_{L^1(\mathbb{R}^n)} \liminf_{\varepsilon\to 0}\Big(\int_0^1\int_{\mathbb{R}^n} \Big\| T_\phi\Big(\sum_{k\in\mathbb{Z}^n}\lambda_k M_{-y_k}(f)r_k(t)\Big)(x)\Big\|^{p_2} W_2(x)dxdt\Big)^{1/p_2}\\
&= C\|\varphi\|^{1/2}_{L^1(\mathbb{R}^n)} \liminf_{\varepsilon\to 0}\Big(\int_0^1 \Big\| T_\phi\Big(\sum_{k\in\mathbb{Z}^n}\lambda_k M_{-y_k}(f)r_k(t)\Big)\Big\|^{p_2}_{L^{p_2}(\mathbb{R}^n,W_2)} dt\Big)^{1/2}\\
&\leq C\|\varphi\|^{1/2}_{L^1(\mathbb{R}^n)}\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}\Big(\int_0^1 \Big\| \sum_{k\in\mathbb{Z}^n}\lambda_k M_{-y_k}(f)r_k(t)\Big\|^{p_2}_{L^{p_1}(\mathbb{R}^n,W_1)} dt\Big)^{1/p_2}
\end{aligned}$$

for any function $f \in C_c^\infty(\mathbb{R}^n)$.

Now Kahane's inequalities (see [7, page 211]) allow us to write

$$\begin{aligned}
&\Big(\int_0^1 \Big\| \sum_{k\in\mathbb{Z}^n}\lambda_k M_{-y_k}(f)r_k(t)\Big\|^{p_2}_{L^{p_1}(\mathbb{R}^n,W_1)} dt\Big)^{1/p_2}\\
&\leq C\Big(\int_0^1 \Big\| \sum_{k\in\mathbb{Z}^n}\lambda_k M_{-y_k}(f)r_k(t)\Big\|^{p_1}_{L^{p_1}(\mathbb{R}^n,W_1)} dt\Big)^{1/p_1}\\
&\leq C\Big\|\Big(\sum_{k\in\mathbb{Z}^n}|\lambda_k M_{-y_k}(f)|^2\Big)^{1/2}\Big\|_{L^{p_1}(\mathbb{R}^n,W_1)}\\
&= C\Big(\sum_{k\in\mathbb{Z}^n}|\lambda_k|^2\Big)^{1/2}\|f\|_{L^{p_1}(\mathbb{R}^n,W_1)}\\
&= C\|\varphi\|^{1/2}_{L^1(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n,W_1)}.
\end{aligned}$$

This, combined with the above estimate, implies the result using the density of $C_c^\infty(\mathbb{R}^n)$ in $L^{p_1}(\mathbb{R}^n, W_1)$. ∎

Proof of Theorem 2.4 for regulated symbols Since ϕ is regulated, we take $\varphi \in L^1(\mathbb{R}^n)$ with compact support such that $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$ and $\lim_{\epsilon\to 0}\phi*\varphi_\epsilon(x) = \phi(x)$ for all $x \in \mathbb{R}^n$, where $\varphi_\epsilon = D^1_\epsilon\varphi$.

From Lemma 4.12 we conclude that $\phi*\varphi_\epsilon \in M_{\overline{p},\overline{W}}(\mathbb{R}^n, E)$ with

$$\|\phi*\varphi_\epsilon\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} \leq K_{\overline{p}}\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}.$$

Now, since $\phi*\varphi_\epsilon$ is continuous in $\mathbb{R}^n$, we can apply the previous case and obtain that $(\phi*\varphi_\epsilon)|_{\mathbb{Z}^n} \in M_{\overline{p},\overline{w}}(\mathbb{T}^n, E)$ with

$$\|(\phi*\varphi_\epsilon)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}}(\mathbb{T}^n,E)} \leq \|\phi*\varphi_\epsilon\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} \leq K_{\overline{p}}\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}.$$

Finally, for any $P \in \mathcal{P}(\mathbb{T}^n)$ we have $T_{\phi|_{\mathbb{Z}^n}}(P)(x) = \lim_{\epsilon\to 0} T_{(\phi*\varphi_\epsilon)|_{\mathbb{Z}^n}}(P)(x)$ and so by Fatou's lemma

$$\begin{aligned}\|T_{\phi|_{\mathbb{Z}^n}}(P)\|_{L_E^{p_2}(\mathbb{T}^n,\mu_2)} &\le \liminf_{\epsilon\to 0} \|T_{(\phi*\varphi_\epsilon)|_{\mathbb{Z}^n}}(P)\|_{L_E^{p_2}(\mathbb{T}^n,\mu_2)}\\ &\le K_{\overline{p}}\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)}\|P\|_{L^{p_1}(\mathbb{T}^n,\mu_1)},\end{aligned}$$

which ends the proof in this case. ∎

5 Extension of Multipliers

5.1 Notation and Lemmata

Recall that for $f \in L^1(\mathbb{R}^n)$ the periodization of f, denoted by $\widetilde{f}$ (see [14, pp. 250–253]), is the integrable function in $\mathbb{T}^n$ defined by

$$\widetilde{f}(x) = \sum_{k\in\mathbb{Z}^n} f(x+k),$$

where the convergence of the series is understood in $L^1(\mathbb{T}^n)$.

In particular, for a continuous integrable function f defined in $\mathbb{R}^n$ satisfying

$$|f(x)| + |\widehat{f}(x)| \le A/(1+|x|^2)^{\alpha/2}$$

for some $A > 0$ and $\alpha > n$ (which imply that f and $\widehat{f}$ are uniformly continuous in $\mathbb{R}^n$) it turns out that $\widetilde{f}$ is the well-defined periodic function given by any of the two following formulae

$$\widetilde{f}(x) = \sum_{k\in\mathbb{Z}^n} f(x+k) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)e^{2\pi ik\cdot x}.$$

The second equality is the expression of Poisson's summation formula.

5.2 Proof of Theorem 2.5

Let $f \in C_c^\infty(\mathbb{R}^n)$. Then, by using vector-valued Riemann integration,

$$\begin{aligned}T_\phi(f)(x) &= \int_{\mathbb{R}^n} \phi(y)\widehat{f}(y)e^{2\pi ixy}\,dy\\ &= \lim_{m\to\infty} m^{-n}\sum_{k\in\mathbb{Z}^n} \phi(k/m)\widehat{f}(k/m)e^{2\pi ixk/m}.\end{aligned}$$

By the Poisson summation formula, we have

$$\widetilde{D_{1/m}f}(x) = m^{-n}\sum_{k\in\mathbb{Z}^n} \widehat{f}(k/m)e^{2\pi ikx},$$

and so

$$m^{-n}\sum_{k\in\mathbb{Z}^n}\phi(k/m)\widehat{f}(k/m)e^{2\pi ikx/m}=D_mT_{(D_m\phi)|_{\mathbb{Z}^n}}(\widetilde{D_{1/m}f})(x).$$

We select $0\le\nu_2\le 1$ supported in $\mathbb{T}^n$ continuous at the origin with $\nu_2(0)=1$ and $\nu_1=\chi_{\mathbb{T}^n}$. Then we have for every $x\in\mathbb{R}^n$,

$$T_\phi(f)(x)=\lim_{m\to\infty}D_m\Big(T_{(D_m\phi)|_{\mathbb{Z}^n}}(\widetilde{D_{1/m}f})\nu_2^{1/p_2}\Big)(x). \tag{5.1}$$

Using that $D_{1/m}f$ is supported in $\mathbb{T}^n$ for m big enough, we obtain

$$D_{1/m}f(x)=(\widetilde{D_{1/m}f})(x),x\in\mathbb{T}^n.$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^n}\|D_m(T_{(D_m\phi)|_{\mathbb{Z}^n}}(\widetilde{D_{1/m}f})\nu_2^{1/p_2})(x)\|^{p_2}W_2(x)dx\\ &\quad\le\int_{\mathbb{R}^n}\|T_{(D_m\phi)|_{\mathbb{Z}^n}}(\widetilde{D_{1/m}f})(x)\|^{p_2}\nu_2(x)D^1_{1/m}W_2(x)dx\\ &\quad=\int_{\mathbb{T}^n}\|T_{(D_m\phi)|_{\mathbb{Z}^n}}(\widetilde{D_{1/m}f})(x)\|^{p_2}w_{m,2}(x)dx\\ &\quad\le\|(D_m\phi)|_{\mathbb{Z}^n}\|^{p_2}_{M_{\overline{p},\overline{w}_m}(\mathbb{T}^n,E)}\Big(\int_{\mathbb{T}^n}|\widetilde{D_{1/m}f}(x)|^{p_1}w_{m,1}(x)dx\Big)^{p_2/p_1}\\ &\quad=\|(D_m\phi)|_{\mathbb{Z}^n}\|^{p_2}_{M_{\overline{p},\overline{w}_m}(\mathbb{T}^n,E)}\Big(\int_{\mathbb{T}^n}|D_{1/m}f(x)|^{p_1}\nu_1(x)D^1_{1/m}W_1(x)dx\Big)^{p_2/p_1}\\ &\quad\le\|(D_m\phi)|_{\mathbb{Z}^n}\|^{p_2}_{M_{\overline{p},\overline{w}_m}(\mathbb{T}^n,E)}\Big(\int_{\mathbb{R}^n}|f(x)|^{p_1}\nu_1\Big(\frac{x}{m}\Big)W_1(x)dx\Big)^{p_2/p_1}\end{aligned}$$

Now, by Fatou's lemma in (5.1) we obtain

$$\begin{aligned}&\|T_\phi(f)\|_{L^{p_2}_E(\mathbb{R}^n,w_2)}\\ &\quad\le\sup_{m\in\mathbb{N}}\|(D_m\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}_m}(\mathbb{T}^n,E)}\lim_{m\to\infty}\Big(\int_{\mathbb{R}^n}|f(x)|^{p_1}\nu_1\Big(\frac{x}{m}\Big)W_1(x)dx\Big)^{1/p_1}\\ &\quad=\sup_{m\in\mathbb{N}}\|(D_m\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}_m}(\mathbb{T}^n,E)}\ \|f\|_{L^{p_1}(\mathbb{R}^n,W_1)}.\end{aligned}$$

The result now follows from the density of $C_c^\infty(\mathbb{R}^n)$ in $L^{p_1}(\mathbb{R}^n,W_1)$. ∎

6 Other Transference Results

As previously stated, the restriction methods presented so far do not work for all types of weights. This is the case in particular of power weights. In this section, we will proceed in a different way to obtain new restriction theorems that are valid for power weights.

The basic idea is to adapt the restriction procedure we used before to weights for which the dilation operator is bounded (which can be of power type).

6.1 Proof of Theorem 2.10

We write the proof only when ϕ is continuous at the integers. The extension to the case when ϕ is regulated can be done following the same ideas as in Theorem 2.4.

Let us show that $\phi|_{\mathbb{Z}^n} \in M_{\overline{p},\overline{\mu}}(\mathbb{Z}^n, E)$. We take $\gamma > 0$ and Φ continuous such that $\Phi \geq 0$, ∞-radial and with support in $\{\delta \leq |x| \leq \rho\}$, $\|\Phi\|_{L^{p_1}(\mathbb{R}^n)} = 1 + \gamma$, and $\|\Phi\|_{L^{p_2}(\mathbb{R}^n)} = 1$. Also let $P \in \mathcal{P}(\mathbb{T}^n)$.

We first apply Lemma 4.10 to get

$$\Big(\int_{\mathbb{T}^n} \|T_{\phi|_{\mathbb{Z}^n}} P(x)\|^{p_2} d\mu_2(x)\Big)^{1/p_2} = \lim_{\lambda\to\infty} \Big(\lambda^{-n} \int_{\{\delta\lambda \leq |x| \leq \rho\lambda\}} \|T_\phi(PD_\lambda\Phi)(x)\|^{p_2} W_2(x)dx\Big)^{1/p_2}.$$

Now, we denote $C_2^{-1} = \inf_{\{\delta\leq|x|\leq\rho\}} U_2(x), C_1 = \sup_{\{\delta\leq|x|\leq\rho\}} U_1(x)$, and use (2.6) to obtain the following estimates:

$$\begin{aligned}
&\Big(\lambda^{-n} \int_{\{\delta\lambda \leq |x| \leq \rho\lambda\}} \|T_\phi(PD_\lambda\Phi)(x)\|^{p_2} W_2(x)dx\Big)^{1/p_2}\\
&\leq \lambda^{-n/p_2} C_2 \Big(\int_{\{\delta\lambda \leq |x| \leq \rho\lambda\}} \|T_\phi(PD_\lambda\Phi)(x)\|^{p_2} D_\lambda U_2(x) W_2(x)dx\Big)^{1/p_2}\\
&\leq \lambda^{-n/p_2} C_2 \|D_\lambda\|_{p_2,W_2} \Big(\int_{\mathbb{R}^n} \|D_{1/\lambda} T_\phi(PD_\lambda\Phi)(x)\|^{p_2} U_2(x) W_2(x)dx\Big)^{1/p_2}\\
&= \lambda^{-n/p_2} C_2 \|D_\lambda\|_{p_2,W_2} \Big(\int_{\mathbb{R}^n} \|T_{D_\lambda\phi}(D_{1/\lambda}(PD_\lambda\Phi))(x)\|^{p_2} V_2(x)dx\Big)^{1/p_2}\\
&\leq \lambda^{-n/p_2} C_2 \|D_\lambda\|_{p_2,W_2} \|D_\lambda\phi\|_{M_{\overline{p},\overline{V}}(\mathbb{R}^n,E)} \Big(\int_{\mathbb{R}^n} |D_{1/\lambda}(PD_\lambda\Phi)(x)|^{p_1} V_1(x)dx\Big)^{1/p_1}\\
&\leq \lambda^{-n/p_2} C_2 \|D_\lambda\|_{p_2,W_2} \|D_\lambda\phi\|_{M_{\overline{p},\overline{V}}(\mathbb{R}^n,E)} \|D_{1/\lambda}\|_{p_1,W_1}\\
&\qquad \Big(\int_{\{\delta\lambda \leq |x| \leq \rho\lambda\}} |P(x)D_\lambda\Phi(x)|^{p_1} D_\lambda U_1(x) W_1(x)dx\Big)^{1/p_1}\\
&\leq \lambda^{-n(1/p_2 - 1/p_1)} C_2 C_1 D(\lambda, \overline{p}, \overline{W}) \|D_\lambda\phi\|_{M_{\overline{p},\overline{V}}(\mathbb{R}^n,E)}\\
&\qquad \Big(\lambda^{-n} \int_{\mathbb{R}^n} |P(x)|^{p_1} |D_\lambda\Phi(x)|^{p_1} W_1(x)dx\Big)^{1/p_1}\\
&\leq C_2 C_1 D^1(\lambda, \overline{p}, \overline{W}) \|\phi\|_{M_{\overline{p},D^1_\lambda\overline{V}}(\mathbb{R}^n,E)}\\
&\qquad \Big(\lambda^{-n} \int_{\mathbb{R}^n} |P(x)|^{p_1} |D_\lambda\Phi(x)|^{p_1} W_1(x)dx\Big)^{1/p_1} 0
\end{aligned}$$

Applying Corollary 4.8 for $f = |P|^{p_1} W_1$ and $\varphi = |\Phi|^{p_1}$, one gets

$$\Big(\int_{\mathbb{T}^n} \|T_{\phi|_{\mathbb{Z}^n}} P(x)\|^{p_2} d\mu_2(x)\Big)^{1/p_2} \leq C\|\Phi\|_{L^{p_1}(\mathbb{R}^n)} \Big(\int_{\mathbb{T}^n} |P(x)|^{q_1} d\mu_1(x)\Big)^{1/p_1},$$

where the constant is given by

$$C_1C_2 \sup_{\lambda>1} D^1(\lambda, \overline{p}, \overline{W})\|\phi\|_{M_{\overline{p},D^1_\lambda \overline{V}}(\mathbb{R}^n,E)}.$$

Since $\|\Phi\|_{L^{p_1}(\mathbb{R}^n)} = 1+\gamma$ making limits as $\gamma \to 0$, one gets the desired estimate for $\|\phi|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{w}}(\mathbb{T}^n,E)}$.

Part (ii) follows combining (i) with Theorem 2.5. Consider the weights $D^1_{1/m}V_i = (D_{1/m}U_i)(D^1_{1/m}W_i)$ and the multipliers $D_m\phi$ for a given $m \in \mathbb{N}$. Note first that the assumption (2.10) gives that the weights $D_{1/m}U_i$ satisfy (2.9) with uniform bound. On the other hand, it is elementary to show that

$$D^1(\lambda, \overline{p}, D^1_{1/\delta}\overline{W}) = D^1(\lambda, \overline{p}, \overline{W}),\ \lambda > 1, \delta > 0$$

and

$$\|D_\delta\phi\|_{M_{\overline{p},D^1_\lambda D^1_{1/\delta}\overline{V}}(\mathbb{R}^n,E)} = \|\phi\|_{M_{\overline{p},D^1_\lambda \overline{V}}(\mathbb{R}^n,E)},\ \lambda > 1, \delta > 0.$$

Hence $A(D_m\phi) = A(\phi)$. Using that W_i are periodic and also that

$$d\mu_i = (D^1_{1/m}W_i)\chi_{\mathbb{T}^n}(x)dx,$$

and applying Theorem 2.5 and part (i), one gets

$$\begin{aligned}\|\phi\|_{M_{\overline{p},\overline{W}}(\mathbb{R}^n,E)} &\le \sup_{m\in\mathbb{N}} \|(D_m\phi)|_{\mathbb{Z}^n}\|_{M_{\overline{p},\overline{((D^1_{1/m}W)\chi_{\mathbb{T}^n})}}(\mathbb{T}^n,E)} \\ &\le \sup_{m\in\mathbb{N}} C_2(m)C_1(m)A(\phi) \le CA(\phi).\end{aligned}$$

■

To obtain the proof of Theorem 2.12 we considered the class $\mathcal{RW}^q(\mathbb{R}^n)$ in the introduction. We can now use Lemma 4.7 to obtain the following analogue of Corollary 4.8 whose proof is left to the interested reader.

Proposition 6.1 *Let $1 < q < \infty$ and $W \in \mathcal{RW}^q(\mathbb{R}^n)$ and let μ be its restriction to $\mathbb{T}^n$.*

Let $f : \mathbb{R}^n \to E$ be a vector-valued trigonometric polynomial and let φ be a ∞-radial function in $L^{pq'}(\mathbb{R}^n)$ with compact support.

Then

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}^n} \|f(x)D^p_\lambda\varphi(x)\|^p W(x)dx = \left(\int_{\mathbb{T}^n} \|f(x)\|^p d\mu(x)\right)\left(\int_{\mathbb{R}^n} |\varphi(x)|^p dx\right)$$

6.2 Proof of Theorem 2.12

(i) As in the previous case, we will write the proof only in the case when ϕ is assumed to be continuous at the integers.

We recall that for $1 < p < \infty$ and $F \in L^p_E(\mathbb{T}^n, \mu_2)$, $\|F\|_{L^p_E(\mathbb{T}^n,\mu_2)}$ coincides with

$$\sup\left\{\left|\int_{\mathbb{T}^n} \langle F(x), Q(x)\rangle d\mu_2(x)\right| : Q \in \mathcal{P}(\mathbb{T}^n, E'), \|Q\|_{L^{p'}_{E'}(\mathbb{T}^n,\mu_2)} = 1\right\},$$

where E' denotes the topological dual of E. Hence, it suffices to show that

$$\int_{\mathbb{T}^n} \|T_{\phi|_{\mathbb{Z}^n}}(P)(x)\| \, \|Q(x)\| d\mu_2 \leq C \|P\|_{L^p(\mathbb{T}^n, w_1)} \|Q\|_{L^{p'}_{E'}(\mathbb{T}^n, \mu_2)}$$

for all trigonometric polynomials $P \in \mathcal{P}(\mathbb{T}^n)$ and $Q \in \mathcal{P}(\mathbb{T}^n, E')$.

We fix $\delta > 0$ and choose $\Phi \in \mathcal{S}(\mathbb{R}^n)$, $\Phi \geq 0$, ∞-radial with support in $\{|x| \leq 1\}$, $\|\Phi\|_{L^\infty(\mathbb{R}^n)} = 1 + \delta$ and $\|\Phi\|_{L^1(\mathbb{R}^n)} = 1$.

Using Lemma 4.10 for $W = \|Q\| W_2$ and $\varphi = \Phi$ and $p = 1$, we have

$$\int_{\mathbb{T}^n} \|T_{\phi|_{\mathbb{Z}^n}} P(x)\| \|Q(x)\| d\mu_2(x) = \lim_{\lambda \to \infty} \int_{\{|x| \leq \lambda\}} \|T_\phi(PD^1_\lambda \Phi)(x)\| \|Q(x)\| W_2(x) dx.$$

Now, using Hölder inequality one obtains that

$$\lambda^{-n} \int_{\{|x| \leq \lambda\}} \|T_\phi(PD_\lambda \Phi)(x)\| \|Q(x)\| W_2(x) dx$$

is bounded by $I_1 I_2$, where

$$I_1(\lambda) = \Big(\lambda^{-n} \int_{\{|x| \leq \lambda\}} \|T_\phi(PD_\lambda \Phi)(x)\|^{p_2} D_\lambda U_2(x) W_2(x) dx \Big)^{1/p_2}$$

and

$$I_2(\lambda) = \Big(\lambda^{-n} \int_{\{|x| \leq \lambda\}} \|Q(x)\|^{p_2'} D_\lambda U_2(x)^{-p_2'/p_2} W_2(x) dx \Big)^{1/p_2'}.$$

To estimate I_1, we use the fact that $D_{1/\lambda} T_\phi(f) = T_{D_\lambda \phi}(D_{1/\lambda} f)$ to get

$$\begin{aligned}
I_1 &\leq \Big(\lambda^{-n} \int_{\mathbb{R}^n} \|T_\phi(PD_\lambda \Phi)(x)\|^{p_2} D_\lambda U_2(x) W_2(x) dx \Big)^{1/p_2} \\
&\leq \lambda^{-n/p_2} \|D_\lambda\|_{p_2, W_2} \Big(\int_{\mathbb{R}^n} \|D_{1/\lambda} T_\phi(PD_\lambda \Phi)(x)\|^{p_2} U_2(x) W_2(x) dx \Big)^{1/p_2} \\
&= \lambda^{-n/p_2} \|D_\lambda\|_{p_2, W_2} \Big(\int_{\mathbb{R}^n} \|T_{D_\lambda \phi}(D_{1/\lambda}(PD_\lambda \Phi))(x)\|^{p_2} V_2(x) dx \Big)^{1/p_2} \\
&\leq \lambda^{-n/p_2} \|D_\lambda\|_{p_2, W_2} \|D_\lambda \phi\|_{M_{\overline{p}, \overline{V}}(\mathbb{R}^n, E)} \Big(\int_{\mathbb{R}^n} |D_{1/\lambda}(PD_\lambda \Phi)(x)|^{p_1} V_1(x) dx \Big)^{1/p_1} \\
&= \lambda^{-n/p_2} \|D_\lambda\|_{p_2, W_2} \|D_\lambda \phi\|_{M_{\overline{p}, \overline{V}}(\mathbb{R}^n, E)} \Big(\int_{\mathbb{R}^n} |D_{1/\lambda}(PD_\lambda \Phi)(x)|^{p_1} U_1(x) W_1(x) dx \Big)^{1/p_1} \\
&\leq \lambda^{-n/p_2} \|D_\lambda\|_{p_2, W_2} \|D_\lambda \phi\|_{M_{\overline{p}, \overline{V}}(\mathbb{R}^n, E)} \|D_{1/\lambda}\|_{p_1, W_1} \\
&\qquad \Big(\int_{\mathbb{R}^n} |P(x) D_\lambda \Phi(x)|^{p_1} D_\lambda U_1(x) W_1(x) dx \Big)^{1/p_1}
\end{aligned}$$

$$\leq \lambda^{-n(1/p_2-1/p_1)} D(\lambda,\overline{p},\overline{W}) \|D_\lambda \phi\|_{M_{\overline{p},\overline{V}}(\mathbb{R}^n,E)}$$
$$\Big(\lambda^{-n}\int_{\mathbb{R}^n} |P(x)D_\lambda\Phi(x)|^{p_1} D_\lambda U_1(x)W_1(x)dx\Big)^{1/p_1}$$
$$= D^1(\lambda,\overline{p},\overline{W})\|\phi\|_{M_{\overline{p},D^1_\lambda\overline{V}}(\mathbb{R}^n,E)}\Big(\lambda^{-n}\int_{\mathbb{R}^n} |P(x)D_\lambda\Phi(x)|^{p_1} D_\lambda U_1(x)W_1(x)dx\Big)^{1/p_1}.$$

Now, we apply Proposition 6.1 for W_1, p_1, P, and $\varphi_1 = |\Phi| U_1^{1/p_1}$ (note that $\varphi_1 \in L^{q_1'p_1}(\mathbb{R}^n)$ is ∞-radial with compact support) to conclude

$$\limsup_{\lambda\to\infty} I_1 \leq \sup_{\lambda\geq 1} D^1(\lambda,\overline{p},\overline{W})\|\phi\|_{M_{\overline{p},D^1_\lambda\overline{V}}(\mathbb{R}^n,E)}$$
$$\lim_{\lambda\to\infty}\Big(\lambda^{-n}\int_{\mathbb{R}^n} |P(x)D_\lambda\Phi(x)|^{p_1} D_\lambda U_1(x)W_1(x)dx\Big)^{1/p_1}$$
$$\leq A(\phi)\Big(\int_{\mathbb{R}^n}|\Phi(x)|^{p_1}U_1(x)dx\Big)^{1/p_1}\Big(\int_{\mathbb{T}^n}|P(x)|^{p_1}d\mu_1(x)\Big)^{1/p_1}$$
$$\leq A(\phi)(1+\delta)\Big(\int_{\{|x|<1\}} U_1(x)dx\Big)^{1/p_1}\|P\|_{L^{p_1}(\mathbb{T}^n,\mu_1)}.$$

On the other hand, to bound I_2 we use Proposition 6.1 again for W_2, p_2', Q, and $\varphi_2 = U_2^{-1/p_2}\chi_{\{|x|<1\}}$ (note note that $\varphi_2 \in L^{q_2'p_2'}(\mathbb{R}^n)$ is ∞-radial with compact support) to get

$$\lim_{\lambda\to\infty} I_2^{p_2'} = \lim_{\lambda\to\infty}\int_{\mathbb{R}^n}\|Q(x)\|^{p_2'} D^1_\lambda(U_2^{-p_2'/p_2})(x)W_2(x)dx$$
$$= \Big(\int_{\{|x|<1\}} U_2(x)^{-p_2'/p_2}dx\Big)\Big(\int_{\mathbb{T}^n}\|Q(x)\|^{p_2'}d\mu_2(x)\Big).$$

Finally applying the two previous estimates we get

$$\int_{\mathbb{T}^n}\|T_{\phi|_{\mathbb{Z}^n}}(P)(x)\|\,\|Q(x)\|d\mu_2(x) \leq C\|P\|_{L^{p_1}(\mathbb{T}^n,w_1)}\|Q\|_{L^{p_2}_{E'}(\mathbb{T}^n,\mu_2)},$$

where the constant equals

$$A(\phi)\|U_1^{1/p_1}\chi_{\{|x|<1\}}\|_{L^{p_1}(\mathbb{R}^n)}\|U_2^{-1/p_2}\chi_{\{|x|<1\}}\|_{L^{p_2'}(\mathbb{R}^n)}(1+\delta).$$

Now taking limits as $\delta\to 0$ the proof of (i) is complete.

Part (ii) follows the same argument as the proof of Theorem 2.10(ii) and is left to the interested reader. ■

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