The same value of x being a root of both equations we have the following theorem :—A root of the equation

$$-\sqrt{\{(x-a)(x-b)\}}+\sqrt{\{(x-c)(x-d)\}}=e$$

is also a root of the equation

$$\sqrt{\{(x-a)(x-b)\}}+\sqrt{\{(x-c)(x-d)\}}=\{(a+b-c-d)x-ab+cd\}/e \quad (28)$$

or, elimitating x by means of (4) from the right-hand side of this last equation, it is a root of the equation

$$\begin{aligned}&\sqrt{\{(x-a)(x-b)\}}+\sqrt{\{(x-c)(x-d)\}}\\ &\qquad =e\{\mathrm{P}(\mathrm{P}+\mathrm{Q}+\mathrm{R})-4a\}/\Pi \pm 2\mathrm{P}\sqrt{(\mathrm{LM})}/\Pi,\end{aligned} \quad (29)$$

the ambiguity of sign being properly chosen.

Example.—We have seen that $x=9{\cdot}4295\ldots$is the root of the equation

$$-\sqrt{\{x(x-3)\}}+\sqrt{\{(x-2)(x-9)\}}=-6\,;$$

whence we find that the same value of x is the root of the equations

$$\sqrt{\{x(x-3)\}}+\sqrt{\{(x-2)(x-9)\}}=\tfrac{4}{3}x-3,$$

and

$$\sqrt{\{x(x-3)\}}+\sqrt{\{(x-2)(x-9)\}}=\tfrac{2}{3}(5+2\sqrt{30}).$$

Fourth Meeting, February 13, 1891.

R. E. ALLARDICE, Esq., M.A., F.R.S.E., President, in the Chair.

A problem in the theory of numbers.

By T. HUGH MILLER, M.A.

Let it be required to find the square integral numbers which added to a given integer shall produce a square integer, and the smallest such number.

Let N be the given number, and let the sum of N and x^2 be y^2, where x and y are integers.

Then $$\mathrm{N}=y^2-x^2=(y+x)(y-x).$$

Now resolve N into its prime factors, let these $l^a\, m^b\, n^c\ldots$. Then if $y+x$ is put equal to the product of any number of these factors, and $y-x$ equal to the product of all the others, a series of values of y is obtained.

If the factor 2 is one of the factors of N, and of the first degree, then 2 must be a factor of $y+x$ or of $y-x$ only. Therefore y and x cannot be integers. Thus if N is double of an odd number it is impossible to find a solution. If 2 is one factor it must be of at least the second degree.

If N is a square number, $x=0$ is obviously one solution, and the least.

Excluding these values, we may put $x+y$ equal to a series of values the number of which is

$$1+a+b+c\ldots \quad +ab+ac+bc+\ldots \quad +abc+\ldots .$$

Now one half of these values are greater than $\sqrt{N}$, and since $y+x$ is always greater than $y-x$, the number of solutions is

$$\tfrac{1}{2}(1+a)(1+b)(1+c)\ldots$$

Since a, b, c are by hypothesis not all even numbers, this number is of course an integer.

If the least value of x is required, $y+x$ and $y-x$ must differ by as small a number as possible. Then $y+x$ must be taken equal to the product of those prime factors of N, which differs from $\sqrt{N}$ by as small a number as possible, and is greater than $\sqrt{N}$.

For example. It is impossible to add a square number to 6 so that the sum shall be a square.

Let the given number be 525.

Now $525=3\,5^2.7$, thus putting $y+x=5^2$, and $y-x=21$, we get $x=2$, $y=23$.

$$\therefore \qquad 525+2^2=23^2.$$

There are in all $\frac{1}{2}\times 2\times 3\times 2$, that is 6 solutions; the other numbers to be added being

$$10^2,\ 34^2,\ 50^2,\ 86^2,\ 262^2.$$

If 2^a is one of the factors of N, since 2 must be a factor of $y+x$, and of $y-x$, the number of solutions is

$$\tfrac{1}{2}(1+a-2)(1+b)(1+c)\ldots\ldots .$$

For example, let $N=1000$, that is $2^3.5^3$. There are $\frac{1}{2}(3-1)(1+3)$ that is 4 solutions. It is sufficient to consider the factors 2.5^3, as 2 must be a factor of $y+x$ and $y-x$. Then putting $y+x=5^2$, and $y-x=2\times 5$, we get 15^2 as the smallest square required.

In a similar way the least value of N can be found which will make $N^2 + aN$ a square integer, where a is a given constant. For putting the expression equal to y^2

$$N = \{ -a + \sqrt{(a^2 + 4y^2)} \}/2,$$

and the problem is reduced to that of finding the least square of an even number which added to a^2 will make it an integral square.

If a is an even number, N is obviously an integer. If a is odd, $\sqrt{(a^2 + 4y^2)}$ is also an odd number, and therefore N is again an integer.

Sur un Lieu Géométrique.

Par M. PAUL AUBERT.

Par un point fixe A *d'une circonférence donnée on mène deux cordes* AB *et* AC *dont le produit a une valeur constante* m², *puis on joint* BC. *Trouver* 1° *le lieu du pied* D *de la bissectrice de l'angle* A *du triangle* ABC ; 2° *le lieu des centres des cercles inscrits et ex-inscrits à ce triangle.*

FIGURE 14.

1°. Soit H le point où la bissectrice AD rencontre la circonférence circonscrite au triangle ABC.

On sait que $\quad AD \times AH = AB \times AC$;
donc on a $\quad AD = m^2/AH.$

Le lieu du point D est la figure inverse de la circonférence lieu de H, le pôle d'inversion étant en A, la puissance d'inversion m^2. C'est donc la perpendiculaire XY au diamètre AO ; elle coupe la circonférence aux points F et G tels que

$$AF = AG = m.$$

2°. Si on mène BK parallèle à XY, les arcs AFB et ACK sont égaux, d'où $\angle ACB = \angle AB'C'$.
Les droites BC et B′C′ sont alors anti-parallèles, et AD est bissectrice de l'angle B′DC. Par suite le cercle inscrit au triangle ABC est aussi inscrit au triangle AB′C′, et il en est de même du cercle ex-inscrit dans l'angle A. D'ailleurs on a manifestement

$$AB' = AC, \; AC' = AB,$$

d'où $\quad AB' \times AC' = m^2.$

Le problème revient donc à considérer les triangles tels que

https://doi.org/10.1017/S0013091500030741 Published online by Cambridge University Press