MODULAR REPRESENTATIONS OF $C_2 \times C_2$

S. B. CONLON

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The representations of V_4 $(= C_2 \times C_2)$ over characteristic 2 are put down in matrix form in sect. 2 of [1]. As such representations are of particular interest to finite group theorists, we present the following "geometric" descriptions of them which give immediate insight into their structures. Indeed, without such pictures it is difficult to see how they can be handled. Finally the relative Grothendieck algebra (relative to a copy of C_2 in V_4) falls out immediately from these diagrams. These diagrams have already helped towards the more general calculation of such algebras [2].

The notation of [1] is followed as far as convenience allows. Thus

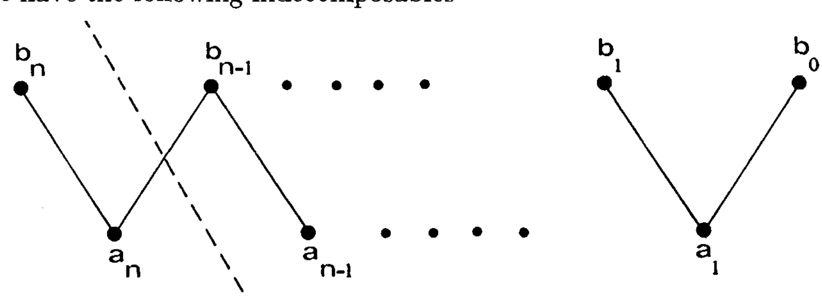
$$V_4 = \{X, Y | X^2 = Y^2 = E, XY = YX\},$$
$$P = X+E, \; Q = Y+E.$$

Thus $P^2 = Q^2 = 0$ and $PQ = QP$ if k is a field of characteristic 2, and $k(V_4) \approx k[P, Q]/(P^2, Q^2)$.

However the following is a complete description of all indecomposable finite dimensional (over k) R-modules, where k is a field of any characteristic and $R = k[P, Q]/(P^2, Q^2)$, P, Q being regarded as indeterminates over k.

Each indecomposable is represented by a diagram composed of dots and diagonal lines. The dots represent the elements of a k-basis. A diagonal line $\diagdown(/)$ represents the action of the linear transformation corresponding to $P(Q)$, the element above being carried into the element below. If a line does not emanate downwards in a given direction from a dot the appropriate action of P (or Q) on such an element gives the zero element.

We have the following indecomposables

A_n: 

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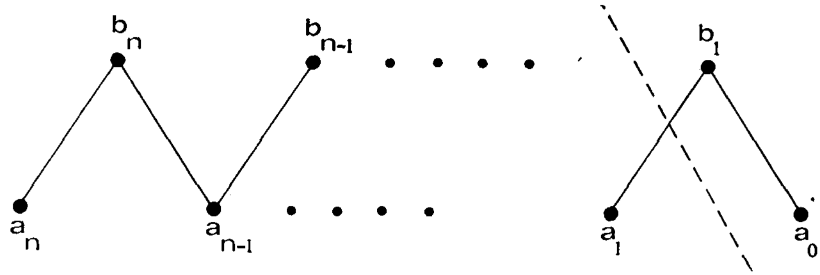
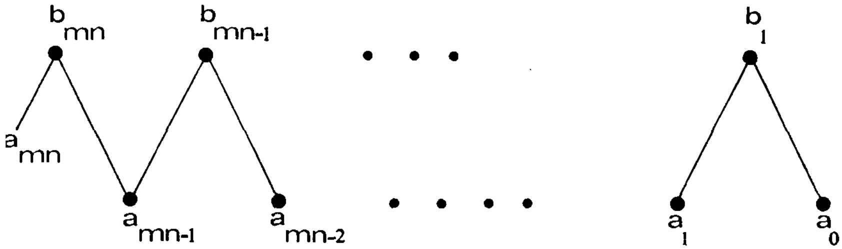
B_n:

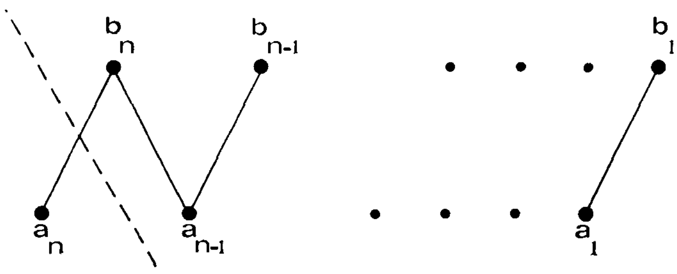


$C_n(\pi)$:

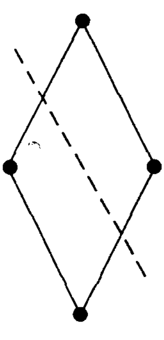


where π is an irreducible polynomial of degree m over k in the indeterminate T, $\pi^n = T^{mn} - u_{mn-1}T^{mn-1} - \ldots - u_0$ and $a_{mn} = u_{mn-1}a_{mn-1} + \ldots + u_0 a_0$.

$C_n(\infty)$:



D:



If G is a finite group, write $a(G)$ for the representation algebra formed from integral combinations of G-module isomorphism classes $\{M\}$. Let H be normal in G and let I denote the ideal of $a(G)$ spanned by elements of the

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form $\{M\}-\{M'\}-\{M''\}$, where there exists an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of G-modules such that its restriction to H splits. Then the quotient

$$g(G, H) = a(G)/I$$

is the *relative Grothendieck algebra*. A G-module M is called *H-simple* if it cannot be the centre term in any non-trivial H-split G-exact sequence as above.

Returning to the case where k has characteristic 2, $G = V_4$, we obtain the structure of the module restricted to the subgroup $H = (X) \approx C_2$ by simply ignoring lines / and considering the action of P given by $\diagdown$.

Clearly

$A_0 = B_0$: 

and

$C_1(T)$: 

being indecomposable upon restriction to H, must be H-simple.

In A_n it is clear that b_n, a_n span a V_4-submodule which is a H-direct summand. This part is isolated on the diagram for A_n by the diagonal dotted line $\ddots$. Thus there exists a H-split V_4-exact sequence:

$$0 \to C_1(T) \to A_n \to A_{n-1} \to 0. \tag{1}$$

Similarly we have the following H-split V_4-exact sequences:

$$0 \to B_{n-1} \to B_n \to C_1(T) \to 0, \tag{2}$$

$$0 \to A_0 \to C_n(\infty) \to A_{n-1} \to 0, \tag{3}$$

$$0 \to C_1(T) \to D \to C_1(T) \to 0. \tag{4}$$

In B_{mn} consider the submodule generated by the element

$$a_{mn} + u_{mn-1}a_{mn-1} + \dots + u_0 a_0$$

which is isomorphic to A_0, where the $u_{mn-1}, \dots, u_0$ are the coefficients associated to $C_n(\pi)$. This gives another H-split V_4-exact sequence:

$$0 \to A_0 \to B_{mn} \to C_n(\pi) \to 0. \tag{5}$$

Modulo the H-split V_4-exact sequences 1, 2, 3, 5 and 5, it is clear that every indecomposable can be reduced to integral multiples of A_0 and $C_1(T)$. On the other hand, using the restriction function $r_{V_4 H}$ the H-split sequences disappear and we have an epimorphism $g(V_4, H) \to a(H)$. But $(A_0)_H$ and $(C_1(T))_H$ form a free $\boldsymbol{Z}$-basis of $a(H)$ and so they are free in $g(V_4, H)$ and $g(V_4, H) \approx a(H)$.

References

[1] Conlon, S. B., 'Certain representation algebras', *J. of Austral. Math. Soc.* 5 (1965), 83–99.
[2] Lam, T. Y. and Reiner, I., 'Relative Grothendieck groups', *J. of Algebra*, 11 (1969), 213–242.

Mathematics Department
University of Sydney
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