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INSTANTANEOUS POINT SOURCE SOLUTIONS IN NONLINEAR DIFFUSION WITH NONLINEAR LOSS OR GAIN

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Abstract

Exact solutions are developed for instantaneous point sources subject to nonlinear diffusion and loss or gain proportional to nth power of concentration, with $n > 1$. The solutions for the loss give, at large times, power-law decrease to zero of slug central concentration and logarithmic increase of slug semi-width. Those for gain give concentration decreasing initially, going through a minimum, and then increasing, with blow-up to infinite concentration in finite time. Slug semi-width increases with time to a finite maximum in finite time at a blow-up. Taken in conjunction with previous studies, these new results provide an overall schema for instantaneous nonlinear diffusion point sources with nonlinear loss or gain for the total range $n \geq 0$. Six distinct regimes of behaviour of slug semi-width and concentration are identified, depending on the range of n, $0 \leq n < 1$, $n = 1$, or $n > 1$. Three of them are for loss, and three for gain. The classical Barenblatt-Pattle nonlinear instantaneous point-source solutions with material concentration occupy a central place in the total schema.

1. Introduction

Nonlinear diffusion with linear or nonlinear loss may involve chemical reaction, irreversible absorption on a porous substrate, radioactive decay, solution or evaporation. The relevant equation is

$$\frac{\partial \theta}{\partial t_*} = r_*^{1-s} \frac{\partial}{\partial r_*} \left(r_*^{1-s} D(\theta) \frac{\partial \theta}{\partial r_*} \right) - k(\theta). \tag{1}$$

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John Philip died last year so this paper is part of his mathematical last testament. John was a very distinguished Australian mathematician and was elected a Fellow of the Royal Society and a Fellow of the Australian Academy. He had very lively interest in a variety of mathematical topics and was a staunch supporter of this Journal. He will be missed.—Ed.

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Here θ (≥ 0) is concentration, normalized with respect to some standard concentration, t_* is time, r_* is the radial space coordinate ($0 \leq r_* \leq \infty$) and s is the number of space dimensions (1, 2 or 3). The diffusivity may take the power-law form

$$D(\theta) = D_1\theta^m, \qquad 0 < m \leq 1 \tag{2}$$

and the time rate of material loss may have the form

$$k(\theta) = k_1\theta^n. \tag{3}$$

Putting (2) and (3) in (1) and using the substitutions

$$r = \left(\frac{k_1}{D_1}\right)^{1/2} r_*, \qquad t = k_1 t_*, \tag{4}$$

we obtain the dimensionless equation

$$\frac{\partial\theta}{\partial t} = r^{1-s}\frac{\partial}{\partial r}\left(r^{1-s}\theta^m\frac{\partial\theta}{\partial r}\right) - \theta^n. \tag{5}$$

Instantaneous point-source solutions of (5) satisfy the initial condition

$$t = 0, \quad 0 \leq r \leq \infty, \quad \theta = Q\delta(r), \tag{6}$$

with Q the dimensionless instantaneous source strength ($0 \leq Q \leq \infty$) and with $\delta(\,)$ defined by an appropriate limiting process. Note that the physical instantaneous source strength with dimensions $[\text{length}]^s$, $Q_* = (D_1/k_1)^{s/2}Q$.

Philip gave exact solutions of (5), (6) for linear loss with $m \geq 0$, $n = 1$ [9] and for nonlinear loss with $0 < m \geq 1$, $n = 1 - m$ (that is, $0 \leq n < 1$) [10]. The solutions applied for all $s > 0$, with s normally 1, 2 or 3 in physical applications. Solutions for $n = 1$ were given previously for $(s, m) = (1, 1)$ by Kersner [6] and for $(s, m) = (2, 1)$ by Miller and van Duijn [7]. The latter authors also gave the solution for $n = 1 - m$ in the special case $(s, m) = (2, 1)$ [7].

For $n = 1$ the slug approaches a finite maximum radius and vanishes in the limit as $t \to \infty$ [7, 9]. On the other hand, for $0 \leq n < 1$, the slug radius increases to a finite maximum and then decreases to zero in a finite time [7, 10]. The difference in behaviour is explained physically in terms of the different modes of approach of $k(\theta)$ to zero as $\theta \to 0$ [9, 10].

Those studies covered only the range $0 \leq n \leq 1$ and left open the question of slug behaviour for $n > 1$. The solutions of [9, 10] were similarity solutions. This paper extends the range of n by establishing relevant similarity solutions of (5), (6) for $1 < n < \infty$. A limitation is that the results are solely for the case $s = 1$, so (5) is replaced by the specialized

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\theta^m \frac{\partial \theta}{\partial x}\right) - \theta^n. \tag{7}$$

Certain thermal and biological problems involve diffusion with gain, not loss. The minus on the right of (1) and (5) is replaced by a plus. As Philip [9, 10] showed, analyses for the cases $m \geq 0, n = 1$ and $0 < m \leq 1, n = 1 - m$ (that is, $0 \leq n < 1$) go similarly to those for loss, with appropriate modifications. In both cases, the solutions for gain exhibit an initial decrease of central concentration, followed by indefinite increase as $t \to \infty$. For $0 \leq n < 1$ [10] the influence of the initial conditions on the solution tends to be lost in the limit as $t \to \infty$, whereas for $n = 1$ [9] it persists for all $t < 0$.

In this paper we extend the range of n for diffusion with gain also to $1 < n < \infty$. The results for gain are also solely for $s = 1$. The relevant equation is (7) with the minus on the right replaced by plus.

2. Exact solutions for instantaneous point sources with loss

Bertsch *et al.* [4] gave an exact solution for the case

$$0 < m < \infty, \qquad n = 1 + m. \tag{8}$$

They gave no attention to the physical significance of their solution. Their formulation contained two excess parameters and, more importantly, two additional constants which were left arbitrary and undetermined. Here we remove these extraneous elements by reference to the physics of the process and exploit the essentials of their ingenious method of solution.

We seek instantaneous source solutions of

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\theta^m \frac{\partial \theta}{\partial x}\right) - \theta^{m+1} \tag{9}$$

subject to initial condition (6), which we rewrite as

$$t = 0, \quad 0 \leq |x| \leq \infty, \quad \theta = Q\delta(r). \tag{10}$$

In terms of $v = \theta^m$, (9) becomes

$$\frac{\partial v}{\partial t} = v\frac{\partial^2 v}{\partial x^2} + \frac{1}{m}\left(\frac{\partial v}{\partial x}\right)^2 - mv^2. \tag{11}$$

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Following [3], we seek solutions of the form

$$
\begin{aligned}
v(x,t) &= \frac{1}{f(t)}\left[1 - \frac{\cosh(\alpha x) - 1}{g(t)}\right], \quad 0 \le |x| \le X(t);\\
&= 0, \quad |x| > X(t);\\
X(t) &= \alpha^{-1}\cosh^{-1}(1+g).
\end{aligned}
\tag{12}
$$

Here f and g are positive functions and α is a positive constant. These are determined in the course of the analysis. Bertsch *et al.* included the unnecessary extra unknown constant ρ in their formulation. This is absorbed into f.

Substituting (12) in (11) and equating coefficients of $\cosh^0(\alpha, x)$, $\cosh(\alpha, x)$ and $\cosh^2(\alpha, x)$, we obtain

$$g^2 f' + gf' + fg' = \frac{\alpha^2}{m} + m(1 + 2g + g^2), \tag{13}$$

$$gf' + fg' = -\alpha^2 g - \alpha^2 + 2m(1+g), \tag{14}$$

$$\alpha^2 = m^2/(m+1). \tag{15}$$

These are the conditions under which (12) satisfies (11). Primes signify differentiation with respect to t.

Subtracting (14) from (13) and using (15), we find

$$f' = m + \frac{m^2}{(m+1)g} \tag{16}$$

and putting (15) in (14) gives

$$f'g + fg' = \frac{m(m+2)}{(m+1)}(1+g). \tag{17}$$

Combining this with (16) and rearranging, we get

$$f = \frac{m(2+g)}{(m+1)g'}. \tag{18}$$

Eliminating f' between (17) and the result of differentiating (18), we obtain

$$(2+g)g'' + m\left(1 + \frac{1}{g}\right)(g')^2 = 0. \tag{19}$$

Solving (19) involves two constants of integration. One is fixed by the physical requirement for an instantaneous point source that $g \to 0$ as $t \to 0$. The second (positive) constant, c, is evaluated later. The relevant solution is thus

$$ct = \int_0^g [g_1(2+g_1)]^{m/2}\, dg_1 \tag{20}$$

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and it follows from (18) that

$$f = \frac{mg^{m/2}(2+g)^{(m+2)/2}}{(m+1)c}. \tag{21}$$

We see that (20) gives $g(t)$ implicitly and that putting $g(t)$ in (21) gives $f(t)$. The required solution for θ is then

$$\begin{aligned}\theta(x,t) &= \left\{\frac{1}{f}\left[1 - \frac{\cosh(\alpha x) - 1}{g}\right]\right\}^{1/m}, \quad 0 \le |x| \le X(t);\\ &= 0, \quad |x| > X(t).\end{aligned} \tag{22}$$

To complete the solution we evaluate c in terms of the dimensionless instantaneous source strength Q. This requires us to examine the properties of the solution in the limit as $t \to 0$. In this limit $g \ll 2$, so that, to a good approximation, the solution reduces to

$$g = 1/2[(m+2)ct]^{2/(m+2)}, \tag{23}$$

$$f = \frac{2m}{(m+1)c}[(m+2)ct]^{m/(m+2)}. \tag{24}$$

In the small t approximation then,

$$\begin{aligned}\theta &= \left[\frac{(m+1)c}{2m}\right]^{1/m}[(m+2)ct]^{-1/(m+2)}\left[1 - \left(\frac{x}{X}\right)^2\right]^{1/m}, \quad 0 \le |x| \le X,\\ &= 0, \quad |x| > X,\end{aligned} \tag{25}$$

with

$$X = \alpha^{-1}[(m+2)ct]^{1/(m+2)}. \tag{26}$$

Now

$$\begin{aligned}Q = 2\lim_{t\to 0}\int_0^{X(t)}\theta(x,t)\,dx &= \frac{2}{\alpha}\left[\frac{(m+1)c}{2m}\right]^{1/m}\int_0^1(1-w^2)^{1/m}\,dw\\ &= \alpha^{-1}\left[\frac{(m+1)c}{2m}\right]^{1/m}\frac{\sqrt{\pi}\,\Gamma\left(\frac{m+1}{m}\right)}{\Gamma\left(\frac{3m+2}{2m}\right)}.\end{aligned} \tag{27}$$

It follows that

$$c = \frac{2m^{m+1}}{(m+1)^{(m+2)/2}}\left(\frac{\Gamma\left(\frac{3m+2}{2m}\right)Q}{\sqrt{\pi}\,\Gamma\left(\frac{m+1}{m}\right)}\right)^m. \tag{28}$$

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This completes the solution.

It is of interest that the foregoing solution (9), (10) in the limit as $t \to 0$ is exactly the classical Barenblatt-Pattle [3, 8] solution of the no-loss equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\theta^m \frac{\partial \theta}{\partial x}\right), \tag{29}$$

subject to (10).

3. Implications of the solution with loss

Numerous mathematical and physical implications follow.

3.1. The overall process: new dimensionless quantities The foregoing solutions describe the spread and loss of a finite slug of the material with dimensionless semi-width $X(t)$ increasing indefinitely and concentration $\theta(x, t)$ decreasing to zero, as $t \to \infty$. The central concentration of the slug

$$\theta(0, t) = f^{-1/m}. \tag{30}$$

Superposed on the mnemonic decrease of $\theta(0, t)$ is systematic change of the concentration profile as t increases. This is conveniently measured by the shape function

$$\vartheta(\xi, t) = \frac{\theta(|x|, t)}{\theta(0, t)}, \quad \xi = \frac{|x|}{X}, \quad 0 \le \xi \le 1. \tag{31}$$

Expressing $X(t)$, $\theta(0, t)$, and $\vartheta(\xi, t)$ in terms of new dimensionless variables, we may remove the dependence of each of these on the dimensionless source strength Q. Accordingly, we introduce the new variables

$$\tau = ct = \int_0^g [g_1(2 + g_1)]^{m/2}\, dg_1, \tag{32}$$

$$\Theta_0(\tau) = \left[\frac{m}{(m+1)c}\right]^{1/m} \theta(0, t) = g^{-1/2}(2 + g)^{-(m+2)/(2m)}. \tag{33}$$

The shape function then becomes

$$\vartheta(\xi, \tau) = \left\{1 - \frac{\cosh[\xi \cosh^{-1}(1 + g)] - 1}{g}\right\}^{1/m}. \tag{34}$$

We shall use also the new dimensionless slug semi-width

$$\Xi = \alpha X = \cosh^{-1}(1 + g). \tag{35}$$

3.2. Small-time behaviour We have seen that the limiting small-time behaviour, (25), (26), is that of the analogous Barenblatt-Pattle no-loss solutions. We thus have, for small t,

$$\Theta_0 \approx 2^{-1/m}[(m+2)\tau]^{-1/(m+2)}, \tag{36}$$

$$\Xi \approx [(m+2)\tau]^{1/(m+2)}, \tag{37}$$

$$\vartheta \approx (1-\xi^2)^{1/m}. \tag{38}$$

It is of some interest to estimate the time when the influence on the solutions of the loss process becomes significant. We may do this by replacing $(2+g_1)^{m/2}$ in (32) by the two leading terms of its Taylor expansion, so obtaining

$$\tau \approx \frac{(2g)^{(m+2)/2}}{m+2}\left[1+\frac{m(m+2)g}{4(m+4)}\right]. \tag{39}$$

The fractional error in (23) of τ for fixed g is therefore about $m(m+2)g/(4m+16)$. For $m=1$, this is about 0.05 with $g \approx 0.33$, $\tau \approx 0.19$.

3.3. Large-time behaviour For large τ, $g \gg 2$ and (32) give

$$\tau \approx g^{m+1}/(m+1),$$

that is,

$$g \approx [(m+1)\tau]^{1/(m+1)}. \tag{40}$$

It follows from (33) that, in this approximation,

$$\Theta_0 \approx [(m+1)\tau]^{-1/m} \tag{41}$$

and from (35) that

$$\Xi \approx \frac{1}{m+1}\ln[(m+1)\tau] + \ln 2. \tag{42}$$

At large times the central concentration decreases like $\tau^{-1/m}$ and slug semi-width increases like $\ln\tau$.

It is a consequence of (42) that, for large τ,

$$\vartheta \approx \left\{1-[(m+1)\tau]^{(\xi-1)/(m+1)}\right\}^{1/m}. \tag{43}$$

3.4. Solutions in closed form For arbitrary $m > 0$ the integral of (32) must be evaluated numerically so that $g(\tau)$, $\Theta_0(\tau)$, $\Xi(\tau)$ and other aspects of the solutions are necessarily expressed numerically. Some special values of m, however, yield solutions in closed forms. They include the following.

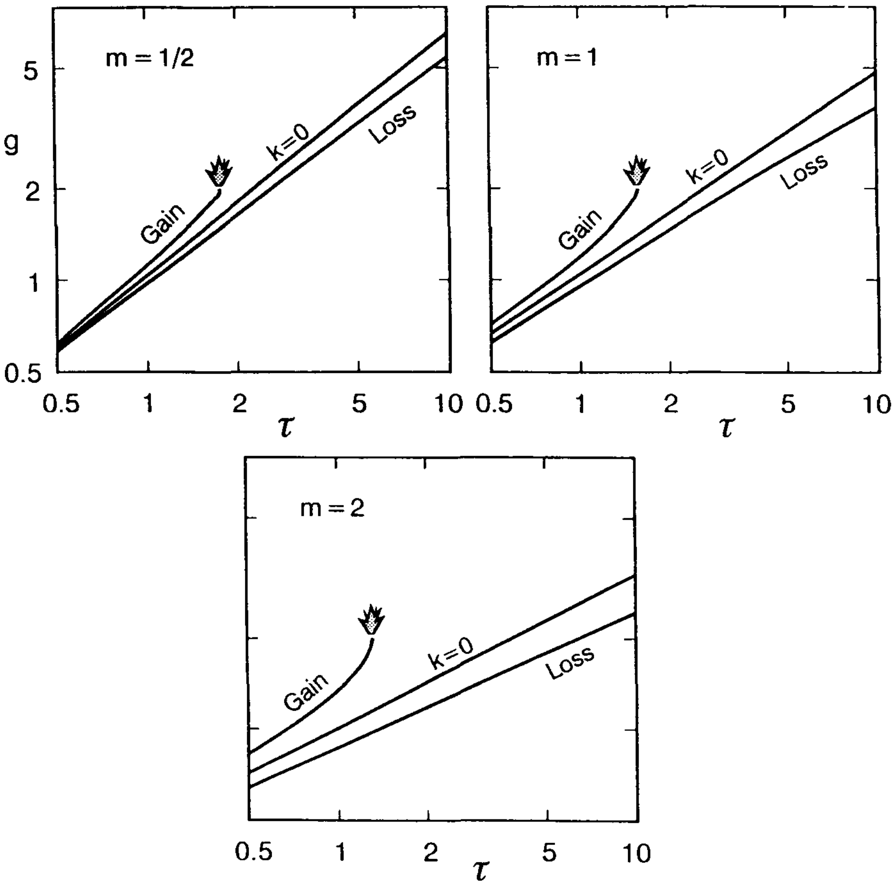

FIGURE 1. The function $g(\tau)$ for $m = 1/2$, 1 and 2. Comparison for loss, material conservation ($k = 0$) and gain. The flame symbol signifies blow-up of the gain solution at $g = 2$.

3.5. Closed-form solutions for $m = 2N - 1$ When $m = 2N - 1$, with N being a positive integer, $\tau(g)$ is expressible in terms of $\cosh^{-1}(1+g)$. The solutions for $N = 1$ and 2 are:

for $m = 1$,

$$\tau = \frac{1}{2}\left[(1+g)[g(2+g)]^{1/2} - \cosh^{-1}(1+g)\right]; \tag{44}$$

for $m = 3$,

$$\tau = \frac{1}{8}\left[[2g(2+g)-3](1+g)[g(2+g)]^{1/2} + 3\cosh^{-1}(1+g)\right]. \tag{45}$$

3.6. Closed-form solutions for $m = 2N$ In these cases $\tau(g)$ is polynomial. The solution for $N = 1$ and 2 are:
for $m = 2$,

$$\tau = g^2 + g^3/3; \tag{46}$$

for $m = 4$,

$$\tau = 4g^3/3 + g^4 + g^5/5. \tag{47}$$

3.7. The explicit solution for $m = 2$ In general the foregoing closed-form solutions require various aspects of the solution to be expressed parametrically in terms of g, with the dependence on τ implicit. However (46) is a cubic, so the special case $m = 2$ gives $g(\tau)$ and hence the τ-dependence of all aspects of the solution explicitly. We find for $m = 2$,

$$\begin{aligned} g(\tau) &= 2\cos\left[\frac{1}{3}\cos^{-1}\frac{3\tau-2}{2}\right] - 1, \quad 0 \le \tau < 4/3; \\ &= 1, \quad \tau = 4/3; \\ &= \sqrt[3]{\frac{3\tau-2}{2} + \sqrt{\frac{9\tau^2-12\tau}{4}}} + \sqrt[3]{\frac{3\tau-2}{2} - \sqrt{\frac{9\tau^2-12\tau}{4}}} - 1, \quad \tau > 4/3. \end{aligned} \tag{48}$$

Here, and in what follows, $\cos^{-1}$ denotes the principal value such that $0 \le \cos^{-1} \le \pi$.

4. Illustrative examples of diffusion with loss

We illustrate the preceding results in Figures 1–4. Each figure shows separately the cases (a) $m = 1/2$, (b) $m = 1$, and (c) $m = 2$. On each figure we present, in addition to the results for diffusion with loss, those for diffusion with gain (Section 5), and those for no loss ($k = 0$).

4.1. Solutions for $g(\tau)$ These basic solutions are shown in Figure 1. The loss solutions decrease gradually relative to the $k = 0$ solutions as τ decreases. The growth of g with τ is like $\tau^{1/(m+1)}$ as $\tau \to \infty$.

4.2. Slug semi-width Figure 2 shows $\Xi(\tau)$, the evolution of slug semi-width in dimensionless form. Here also, Ξ for loss gradually decreases relative to that for $k = 0$. Its increase with τ becomes logarithmically slow as $\tau \to \infty$.

4.3. Slug central concentration Figure 3 depicts $\Theta_0(\tau)$, the evolution of the central concentration expressed in dimensionless form. Here also, Θ_0 for loss gradually decreases relative to that for $k = 0$ as τ increases. The decrease of Θ_0 is like $\tau^{-1/m}$ as $\tau \to \infty$.

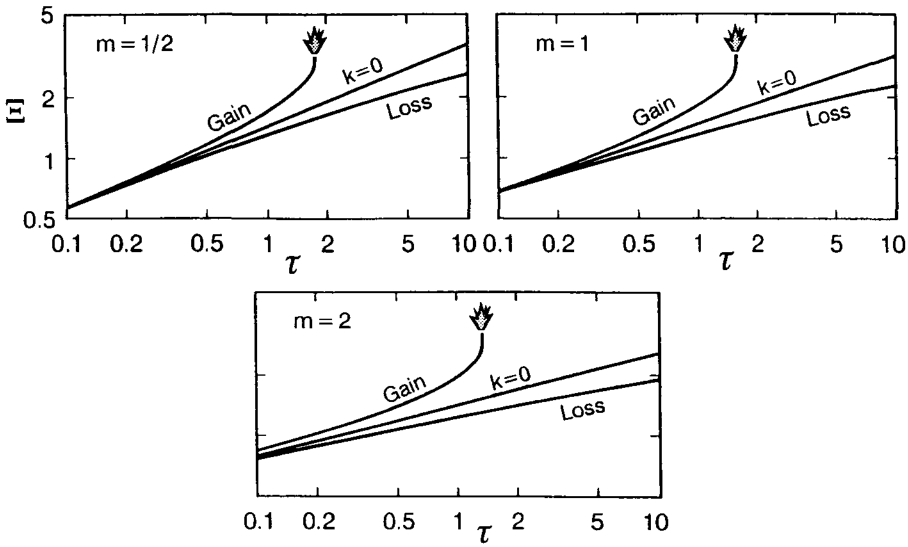

FIGURE 2. As Figure 1 for the function $\Xi(\tau)$. Ξ is the dimensionless slug semi-width. For the gain solution $\Xi = \pi$ at blow-up.

4.4. Concentration profile shape Figure 4 shows $\vartheta(\tau)$ for the indicated values of τ and so presents in dimensionless form the evolution of slug concentration profile shape. We see that the shapes for loss deviate from the initial shape ($\tau = 0$), becoming increasingly convex upward (decreasingly concave upward) as τ increases. In the limit as $\tau \to \infty$, $\vartheta = 1$ for $0 \le \xi < 1$, with $\vartheta = 0$ for $\xi = 1$.

Comparison of these results with those for gain is developed in later sections.

5. Exact solution for instantaneous point sources with gain

We extend the search to solution of

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(\theta^m \frac{\partial\theta}{\partial x}\right) + \theta^{m+1}. \tag{49}$$

Ames *et al.* [1] cite a solution of (49) due to Zmitrenko *et al.* [11], and state that "it describes a blow-up regime". It is readily shown, however, that the cited "solution" fails to satisfy (49). All is not lost, however, since our analysis of Section 3 suggests the means of finding the instantaneous point-source solutions for (49), again subject to initial conditions (10). In this case (11) is replaced by

$$\frac{\partial v}{\partial t} = v\frac{\partial^2 v}{\partial x^2} + \frac{1}{m}\left(\frac{\partial v}{\partial x}\right)^2 + mv^2, \tag{50}$$

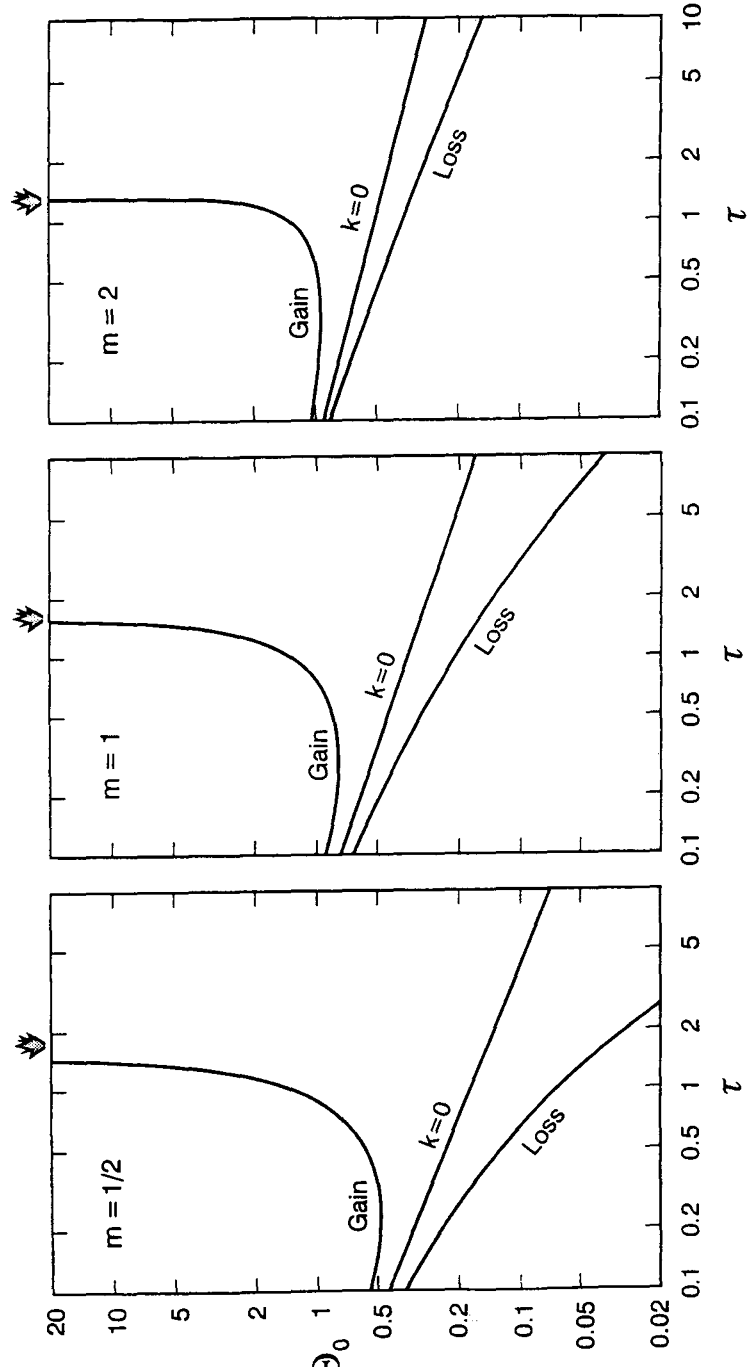

FIGURE 3. As Figure 1 for the function $\Theta_0(\tau)$. Θ_0 is the dimensionless central concentration. The gain solution goes to infinity at blow-up.

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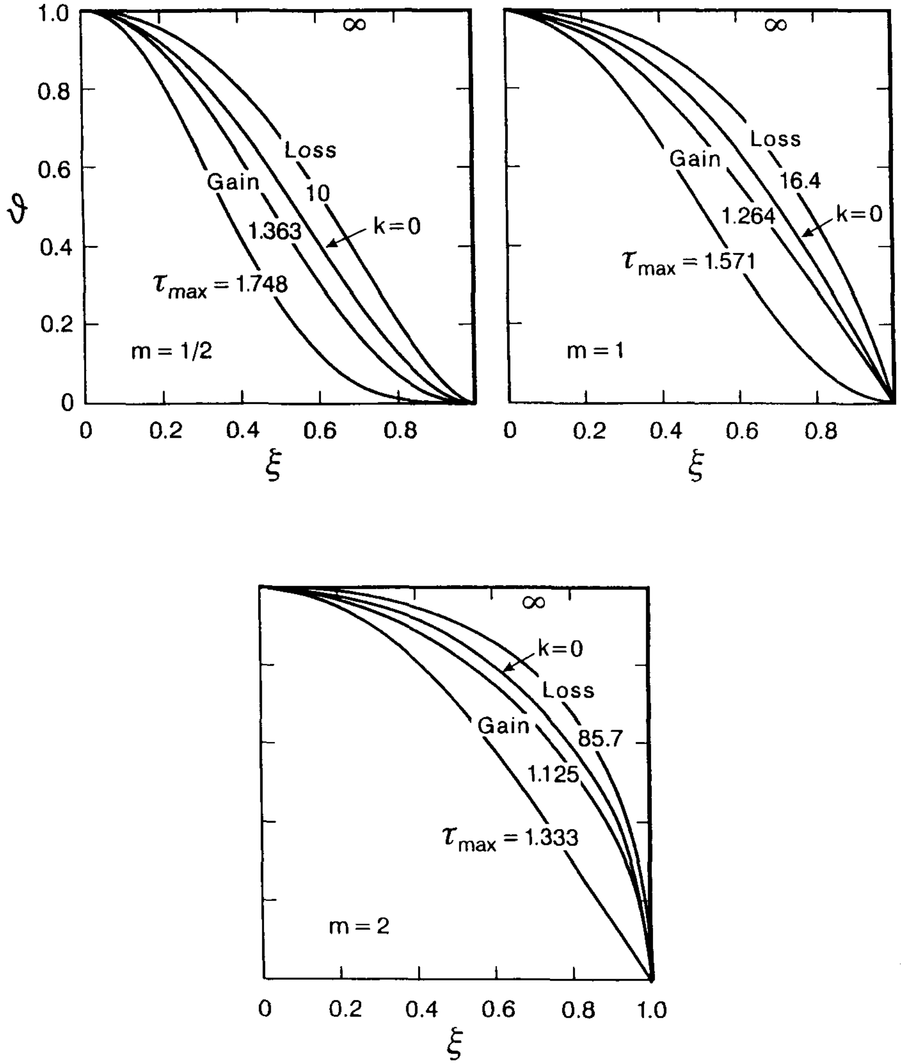

FIGURE 4. Dependence of concentration profile shape function $\vartheta(\xi)$ on dimensionless time τ for $m = 1/2, 1$ and 2. Comparison for loss, material conservation ($k = 0$) and gain. ϑ : concentration normalized with respect to central concentration. ξ : modulus of space coordinate normalized with respect to slug semi-width. The $k = 0$ curves hold for all $\tau \geq 0$ for material conservation and are also the $\tau \to 0$ limits of the curves for both gain and loss. Numerals on other curves denote values of τ. Curves labelled τ_{max} are limiting shapes at blow-up.

and we seek solutions of the form

$$\begin{aligned} v(x,t) &= \frac{1}{f(t)}\left[1 - \frac{1-\cosh(\alpha x)}{g(t)}\right], \quad 0 \le |x| \le X(t); \\ &= 0, \quad |x| > X(t); \\ X(t) &= \alpha^{-1}\cosh^{-1}(1-g). \end{aligned} \tag{51}$$

Once again we determine the positive functions f and g and the positive constant α, in the course of the analysis. By (51) and (50) and equating coefficients of $\cosh^0(\alpha, x)$, $\cosh(\alpha, x)$ and $\cosh^2(\alpha, x)$, we secure the three conditions that (51) satisfies (49). Equations (13) and (14) are replaced by

$$-g^2 f' + gf' + fg' = \frac{\alpha^2}{m} + m(1 - 2g + g^2) \tag{52}$$

and

$$gf' + fg' = \alpha^2 g - \alpha^2 + 2m(1-g). \tag{53}$$

Equation (15) for α^2 is unchanged.

We see that the transforms $f \to -f$, $g \to -g$, change (13) and (14) into (52) and (53). Applying these transformations to the previous analysis therefore gives the required solution. Equations (16)–(19) are transformed thus and the solution is then

$$ct = \int_0^g [g_1(2-g_1)]^{m/2}\, dg_1; \tag{54}$$

$$f = \frac{mg^{m/2}(2-g)^{(m+2)/2}}{(m+1)c}; \tag{55}$$

$$\begin{aligned} \theta(x,t) &= \left\{\frac{1}{f}\left[1 - \frac{1-\cosh(\alpha x)}{g}\right]\right\}^{1/m}, \quad 0 \le |x| \le X(t); \\ &= 0, \quad |x| > X(t). \end{aligned} \tag{56}$$

Evaluation of c goes precisely as before, with (23)–(26) still applicable. In consequence, c is again given by (29) and just one matter remains. It is evident from (54) and (55) that the solution holds only for $0 \le g \le 2$. This translates into the bounds $0 \le t \le t_{\max}$ and it follows from (54) that

$$ct_{\max} = \int_0^2 [g(2-g)]^{m/2}\, dg = 2^{m+1}\left[\Gamma\left(\frac{2+m}{2}\right)\right]^2 \Big/ \Gamma(2+m). \tag{57}$$

This completes the solution.

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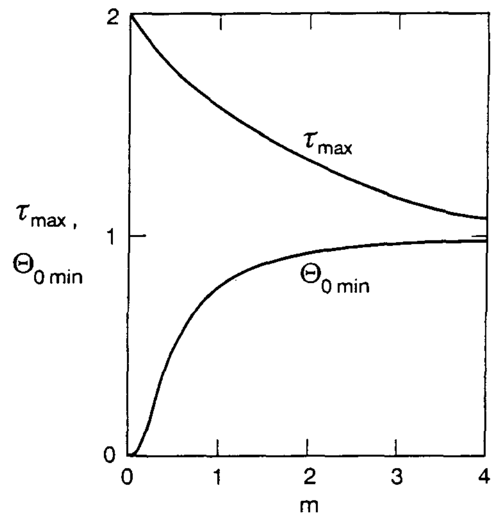

FIGURE 5. Diffusion with gain: dependence on m of τ_{max} and $\Theta_{0\,min}$. τ_{max} : dimensionless time of blow-up. $\Theta_{0\,min}$: dimensionless minimum central concentration.

6. Implications of the solution with gain

6.1. The overall process The solutions of Section 5 describe the spread and gain of an initially finite slug of material. The process is completed within the finite reduced time interval $0 \le \tau \le \tau_{max}$, where

$$\tau_{max} = ct_{max} = 2^{m+1}\left[\Gamma\left(\frac{2+m}{2}\right)\right]^2 \Big/ \Gamma(2+m). \tag{58}$$

Figure 5 shows the dependence of τ_{max} on m.

The dimensionless slug-semi-width Ξ increases with τ and attains the finite maximum value Ξ_{max} at $\tau = \tau_{max}$. The dimensionless concentration Θ_0 initially decreases, passes through a minimum and increases with τ until the slug blows up, the concentration becoming infinite everywhere in $0 \le \xi < 1$ at the instant $\tau = \tau_{max}$. The following equations (59)–(62) here replace equations (32)–(35) for loss:

$$\tau = \int_0^g [g_1(2-g_1)]^{m/2}\, dg_1, \tag{59}$$

$$\Theta_0(\tau) = g^{-1/2}(2-g)^{-(m+2)/(2m)}, \tag{60}$$

$$\vartheta(\xi,\tau) = \left\{1 - \frac{1-\cos[\xi\cos^{-1}(1-g)]}{g}\right\}^{1/m}, \tag{61}$$

$$\Xi = \cos^{-1}(1-g). \tag{62}$$

These results apply only for $0 \leq \tau \leq \tau_{\max}$.

6.2. Small-time behaviour Here also, the limiting small-time behaviour is described by (25), (26) and is that of the Barenblatt-Pattle no-loss solutions. Equations (36)–(38) apply here also. In this case the fractional error of the small-time approximation is equal in magnitude, but opposite in sign, to that indicated in Section 3.2.

6.3. Behaviour as $\tau \to \tau_{\max}$ It follows from (57)–(59) that, as $\tau \to \tau_{\max}$ and $g \to 2$,

$$g(\tau) \approx 2 - \tfrac{1}{2}\left[(m+2)\left(\tau_{\max} - \tau\right)\right]^{2/(2+m)}, \tag{63}$$

with $dg/d\tau$ tending to infinity as $\tau \to \tau_{\max}$.

Then (60) gives for $\tau \to \tau_{\max}$, $g \to 2$,

$$\Theta_0(\tau) \approx \left[\frac{m+2}{2}\left(\tau_{\max} - \tau\right)\right]^{-1/m}. \tag{64}$$

We see that, as $\tau \to \tau_{\max}$, the central concentration goes to infinity like $(\tau_{\max} - \tau)^{-1/m}$.

In addition (62) gives for this limiting behaviour

$$\Xi(\tau) \approx \pi - \left[(m+2)\left(\tau_{\max} - \tau\right)\right]^{1/(m+2)}. \tag{65}$$

We see that

$$\Xi_{\max} = \pi, \tag{66}$$

with $d\Xi/d\tau$ tending to infinity as $\tau \to \tau_{\max}$.

Finally, combining (61), (62) and (65), we find that, as $\tau \to \tau_{\max}$,

$$\vartheta(\xi, \tau) \approx \left[\frac{1}{2}\left\{1 + \cos\left[\xi\left\{\pi - \left[(m+2)\left(\tau_{\max} - \tau\right)\right]^{1/(m+2)}\right\}\right]\right\}\right]^{1/m} \tag{67}$$

and

$$\vartheta(\xi, \tau_{\max}) = \left\{\tfrac{1}{2}\left[1 + \cos(\xi\pi)\right]\right\}^{1/m}. \tag{68}$$

6.4. Solutions in closed form As before, solutions must be expressed numerically for arbitrary $m > 0$, but special values of m give solutions in closed forms.

6.5. Closed-form solutions for $m = 2N - 1$ In these cases $\tau(g)$ is expressible in terms of $\cos^{-1}(1-g)$. The solutions for $N = 1$ and 2 are:
for $m = 1$,

$$\tau = \frac{1}{2}\left\{\cos^{-1}(1-g) - (1-g)[g(2-g)]^{1/2}\right\}; \tag{69}$$

for $m = 3$,

$$\tau = \frac{1}{8}\left[3\cos^{-1}(1-g) - [2g(2-g)+3](1-g)[g(2-g)]^{1/2}\right]. \tag{70}$$

6.6. Closed-form solutions for $m = 2N$ Here again $\tau(g)$ is polynomial. The solution for $N = 1$ and 2 are:
for $m = 2$,

$$\tau = g^2 - g^3/3; \tag{71}$$

for $m = 4$,

$$\tau = 4g^3/3 - g^4 + g^5/5. \tag{72}$$

6.7. The explicit solution for $m = 2$ Here also we may solve the cubic (71) to obtain $g(\tau)$ explicitly. We thus obtain for $m = 2$,

$$g(\tau) = 2\cos\left[\frac{\pi}{3} + \frac{1}{3}\cos^{-1}\frac{3\tau - 2}{2}\right] + 1, \quad 0 \le \tau \le 4/3 = \tau_{\max}. \tag{73}$$

7. Illustrative examples of diffusion with gain

Figures 1–4 illustrate the results for diffusion with gain, as well as diffusion with loss.

7.1. Solutions for $g(\tau)$ See Figure 1. These basic solutions increase relative to the $k = 0$ solutions as τ increases. All solutions terminate at the blow-up point $\tau = \tau_{\max}$, $g = 2$, where $dg/d\tau$ becomes infinite.

It is evident from (59) that $g(\tau)$ has the symmetry property

$$g(\tau) = 2 - g(\tau_{\max} - \tau), \quad g(\tau_{\max}/2) = 1. \tag{74}$$

Unfortunately the logarithmic plots of Figure 1 obscure this symmetry.

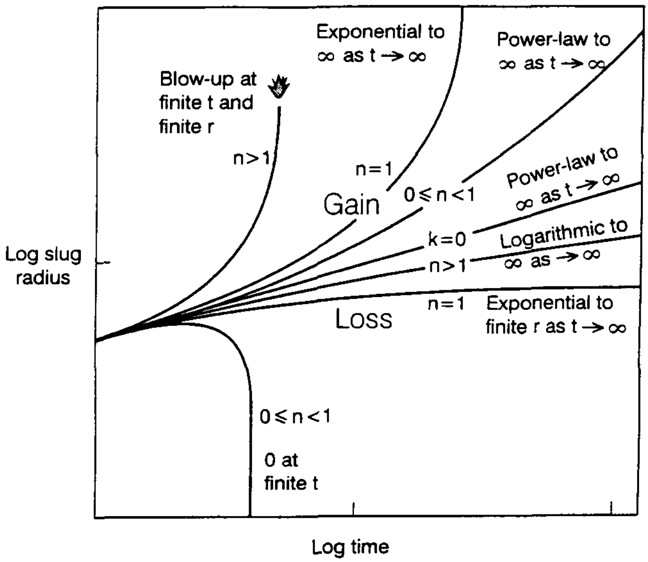

FIGURE 6. Schematic representation of nonlinear diffusion with nonlinear loss or gain. There are different regimes of evolution of slug radius, according to the range of n. The solution for material conservation ($k = 0$) is the separatrix between the solutions for loss and those for gain. All solutions converge in the limit as $\tau \to 0$.

7.2. Slug semi-width See Figure 2. Here also, Ξ for gain gradually increases relative to the no-loss ($k = 0$) solutions. All solutions terminate at the blow-up point $\tau = \tau_{\max}$, $\Xi = \Xi_{\max} = \pi$. At this point $d\Xi/d\tau$ becomes infinite.

7.3. Slug central concentration See Figure 3. Initially, Θ_0 for gain decreases with increasing τ, though it gradually increases relative to Θ_0 for $k = 0$. It has a minimum $\Theta_{0\min}$ at $\tau = \tau_{\min}$ and then increases, approaching infinity as $\tau \to \tau_{\max}$ at the blow-up point.

Differentiating (60) shows that

$$g(\tau_{\min}) = m/(m+1) \tag{75}$$

and it follows that

$$\Theta_{0\min} = \left[\frac{\left[(m+1)^{m+1}\right]^2}{m^m(m+2)^{m+2}}\right]^{1/(2m)}. \tag{76}$$

Figure 5 shows the dependence on m of $\Theta_{0\min}$. Note that $\lim_{m\to\infty}\Theta_{0\min} = 1$.

7.4. Concentration profile shape The shapes for gain deviate from the initial ($k = 0$) shape in the sense opposite to those for loss. For $\tau = \tau_{\max}/2$, $\vartheta = [\cos(\xi\pi/2)]^{1/m}$,

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and for $\tau = \tau_{\max}$, $\vartheta = \{[\cos(\xi\pi/2)]/2\}^{1/m}$.

8. Discussion

It remains to discuss some aspects of this work and to put it in perspective with respect to previous studies, specifically [9, 10]. First, however, we place the present results against a wider context by recognizing that our equations (9) and (49) are particular cases of those established by Arrigo and Hill [2] as admitting classical symmetry reductions. They are special cases of line 3 of their Table 5 with (our) $n = m + 1$. We notice further that Hill *et al.* [5] found that separable solutions of (7) hold only for $n = 1$, as in [9], and for $n = m + 1$, as in [10] and the present work. The solutions of [9, 10] and this study are, however, *not* separable in the sense of Hill *et al.*

8.1. Dependence of the results on source strength We have factored dimensionless source strength Q out of results expressed in terms of τ and Θ_0. It follows from (3), (33) and (34) that

$$t \propto Q^{-m}\tau, \tag{77}$$

and that, at fixed τ

$$\theta(x, 0) \propto Q\Theta_0. \tag{78}$$

Doubling Q shortens the time-scale by the factor 2^{-m} and increases the central concentration (at fixed τ) by the factor 2.

8.2. Persistence of initial conditions in solutions for gain It is of interest that, in the present results for gain, the slug semi-width at blow-up is $\pi\alpha$ when expressed in terms of X. Slug dimensions at blow-up are thus independent of Q. The value of $t_{\max}$ at blow-up is, however, determined by Q. The effect of the initial conditions (that is, the value of Q) thus persists for solutions with gain for $0 \le t \le t_{\max}$.

Contrast this with the results for $n = 1$ [9], where the effect of the initial conditions on the solutions for gain persists for all $t > 0$ and with those for $0 \le n < 1$ [10], where the effect of the initial conditions disappears as $t \to \infty$.

8.3. Importance of the Barenblatt-Pattle solutions The Barenblatt-Pattle [3, 8] no-loss solutions were central to the instantaneous point-source solutions of (5) for $0 < n < 1$ [10] and for $n = 1$ [9]. They are important here also; they give small-time approximations to, and form separatrices between, the solutions for loss and gain. See Figures 1–4.

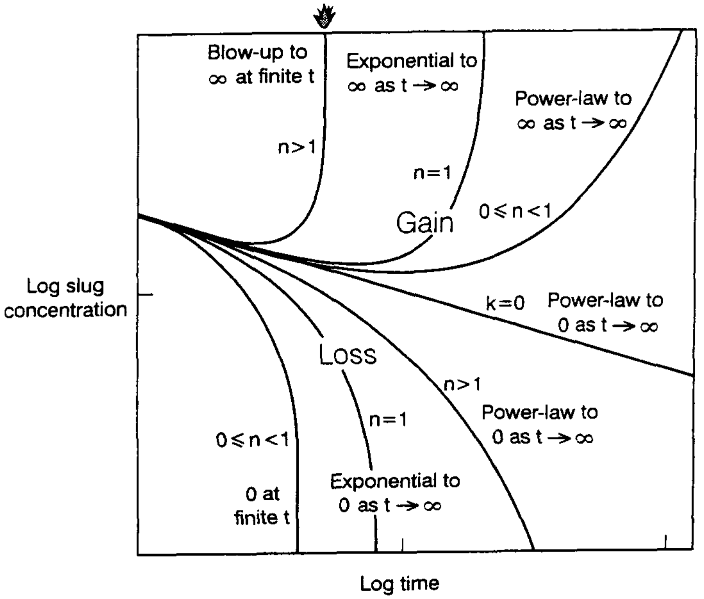

FIGURE 7. As Figure 6, but showing the different regimes of evolution of slug central concentration.

8.4. Dependence on n of solution behaviour It remains to place the present results in the context of earlier studies of the classes $0 \leq n < 1$ [10] and $n = 1$ [9]. Figure 6 represents, schematically, the time-dependence of slug radius (= semi-width in one space dimension) for diffusion with loss and gain for three cases $0 \leq n < 1$, $n = 1$ and the present $n > 1$. Also shown is the Barenblatt-Pattle ($k = 0$) result. Figure 7 gives the corresponding information for the slug central concentration.

Note that, in the limit as time approaches zero, all solutions (for fixed m and s) agree with each other and with the Barenblatt-Pattle ($k = 0$) solution. The latter is a separatrix between solutions for loss and those for gain. It gives a slug radius with power-law decrease (slower than that for loss with $n > 1$) to zero at infinite time.

A limit to the generality of this schema is that the results for $0 < n < 1$ and $n = 1$ are for arbitrary $s > 0$, whereas the present results for $n > 1$ are for $s = 1$ only. The plausible conjecture that the schema holds also for all $s > 0$ when $n > 1$ remains unproven.

8.5. Physical remark The different slug behaviour for diffusion with loss for $0 \leq n < 1$ and for $n = 1$ has been physically explained [9, 10] in terms of the relative magnitudes of the loss function k at small θ in the two cases. The explanation carries over to the present new solutions: for $n > 1$, k is so small as $\theta \to 0$ that, in this case, the rate of loss becomes so slow that the slug semi-width is able to increase indefinitely.

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