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Residue: A Geometric Construction

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Abstract. A new construction of the ordinary residue of differential forms is given. This construction is intrinsic, *i.e.*, it is defined without local coordinates, and it is geometric: it is constructed out of the geometric structure of the local and global cohomology groups of the differentials. The Residue Theorem and the local calculation then follow from geometric reasons.

Introduction

Residue theory is very old, and has largely been developed—usually in relation to duality theory—from different points of view. Here we take the point of view of cohomological residues, developed by J. Lipman, E. Kunz, R. Hubl and others (see [Li1], [Li2], [HK], [Hu]). Moreover, we deal with the most basic case: the residue theory for smooth schemes. Whenever one desires to establish basic residue theory one faces certain problems: if one defines the local residue taking a system of parameters, one then has two problems: first, to prove the independence of the choice of parameters (which is non-trivial, even in the case of curves), and second, to prove the Residue Theorem (see for example [Se] for the case of curves). By contrast, if one gives an intrinsic definition of the local residue (for example, the one of [Ta] in the case of curves), then it is usually easier to prove the Residue Theorem but the local computation, in terms of a local system of parameters, becomes difficult (see [Ha1, pp. 247–8], for some comments about these problems in the case of curves).

The aim of this paper is to show that in conceptual terms the residue map and its main properties are a consequence of the geometry of the local and global cohomology groups of the differentials (the Hodge intersection ring). That is, assuming that one has defined and constructed the cohomology classes and the cup product in this ring and has proved their basic properties, then one can give an intrinsic construction of the residue map and prove its main properties (Residue Theorem and local computation).

Let X be a smooth scheme over an algebraically closed field k. For each closed point p, we shall intrinsically define a morphism of $\mathcal{O}_p \underset{k}{\otimes} \mathcal{O}_p$-modules

$$\phi_p \colon H^n_p(X, \Omega^n_X) \to H^n_p(X, \Omega^n_X) \underset{k}{\otimes} H^n_p(X, \Omega^n_X),$$

which is essentially the cup multiplication with the cohomology class of the diagonal in $X \times X$. We shall construct the residue at p as the only k-linear map $\mathrm{Res}_p \colon H^n_p(X, \Omega^n_X) \to k$ such that:

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1. $\operatorname{Res}_p\bigl(c_X(p)\bigr) \neq 0$, where $c_X(p)$ denotes the cohomology class of p in X.
2. $\operatorname{Res}_p = (\operatorname{Res}_p \otimes \operatorname{Res}_p) \circ \phi_p$. We can state this by saying that the residue map is "compatible with products".

In other words, the Hodge intersection ring allows us to construct a co-product structure in $H^n_p(X, \Omega^n_X)$ (that is, a product in $H^n_p(X, \Omega^n_X)^*$), and the residue map is characterized as the co-unity. This can be translated from the local to the global case: we construct a morphism of $\Gamma(X \times X, \mathcal{O}_{X\times X})$-modules

$$\phi\colon H^n(X, \Omega^n_X) \to H^n(X, \Omega^n_X) \underset{k}{\otimes} H^n(X, \Omega^n_X),$$

which is compatible with ϕ_p for any $p \in X$; that is, $(h_p \otimes h_p) \circ \phi_p = \phi \circ h_p$, $h_p\colon H^n_p(X, \Omega^n_X) \to H^n(X, \Omega^n_X)$ being the natural morphism. The global residue (X proper) is the only (non-zero) k-linear map $H^n(X, \Omega^n_X) \to k$ which is "compatible with products": $\operatorname{Res} = (\operatorname{Res} \otimes \operatorname{Res}) \circ \phi$. The compatibility of ϕ_p with ϕ gives the compatibility of the local and global residue; that is, $\operatorname{Res} \circ h_p = \operatorname{Res}_p$. This compatibility yields the Residue Theorem.

Usually, if one defines the residue in an intrinsic way one encounters difficulties in the explicit computation. We shall see that in our case the explicit computation follows easily from the given construction for geometric reasons. More concretely: let $t_1, \dots, t_n$ be local parameters at p.

1. $\operatorname{Res}_p\bigl(\frac{d t_1 \wedge \cdots \wedge d t_n}{t_1 \cdot \cdots \cdot t_n}\bigr) = 1$. This follows from the following facts:
   a) $\frac{d t_1 \wedge \cdots \wedge d t_n}{t_1 \cdot \cdots \cdot t_n}$ is the cohomology class of p,
   b) $\phi_p\bigl(c_X(p)\bigr) = c_X(p) \otimes c_X(p)$, as follows from the construction of ϕ_p and the basic properties of cohomology classes, and
   c) Res_p is "compatible with products".
2. $\operatorname{Res}_p\bigl(\frac{d t_1 \wedge \cdots \wedge d t_n}{t_1^{k_1} \cdot \cdots \cdot t_n^{k_n}}\bigr) = 0$, for k_i positive integers, some of them greater than 1. This follows from the invariance of Res_p under automorphisms (the group of automorphisms of the completion $\widehat{\mathcal{O}}$ at p acts naturally on $H^n_p(X, \Omega^n_X)$).

Let us denote by G the group of automorphisms of the completion $\widehat{\mathcal{O}}$ at p. The action of G at $H^n_p(X, \Omega^n_X)$ is essential. For example, $H^n_p(X, \Omega^n_X)$ and $H^n_p(X, \mathcal{O}_X)$ are isomorphic as $\mathcal{O}_p$-modules, but *not* as G-modules. This is why $H^n_p(X, \Omega^n_X)$ has a canonical residue map but $H^n_p(X, \mathcal{O}_X)$ does not.

In Section 0 we give the elementary results of local cohomology that we require for the rest of the work.

0 Basic Results of Local Cohomology $H^d_Y(X, \Omega^d_X)$

Let X be a k-scheme and $j\colon Y \hookrightarrow X$ a locally complete intersection closed subscheme of codimension d of ideal $\mathcal{I}$. There are canonical morphisms

$$(1) \qquad \operatorname{Hom}_{\mathcal{O}_X}\bigl(\Lambda^d(\mathcal{I}/\mathcal{I}^2), j^*\Omega^d_{X/k}\bigr) = \operatorname{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_Y, \Omega^d_{X/k}) \hookrightarrow H^d_Y(X, \Omega^d_{X/k}).$$

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The first map is obtained via the Koszul complex (see [Gro1]). For the second one, see [Gro2].

The natural map $\mathcal{I}/\mathcal{I}^2 \to j^*\Omega_{X/k}$ induces a map $\Lambda^d(\mathcal{I}/\mathcal{I}^2) \to j^*\Omega^d_{X/k}$ that defines, via (1), an element $c_X(Y) \in H^d_Y(X, \Omega^d_{X/k})$, which is called the *cohomology class* of Y in X.

The main facts that we need are (see [Gro1] for a proof):

Theorem 0.1

(a) *Let Y, Z be two closed subschemes of X, locally complete intersections of codimension d, d' respectively. If Y and Z meet transversally, then:*

$$c_X(Y) \cdot c_X(Z) = c_X(Y \cap Z),$$

where $c_X(Y) \cdot c_X(Z)$ denotes the cup product:

$$H^d_Y(X, \Omega^d_{X/k}) \times H^{d'}_Z(X, \Omega^{d'}_{X/k}) \to H^{d+d'}_{Y\cap Z}(X, \Omega^{d+d'}_{X/k}).$$

(b) *If $\pi\colon X \times T \to X$ is the natural projection, then*

$$\pi^* c_X(Y) = c_{X\times T}(Y \times T)$$

where $\pi^*\colon H^d_Y(X, \Omega^d_{X/k}) \to H^d_{Y\times T}(X \times T, \Omega^d_{X\times T/k})$ is the inverse image.

Local computations Assume X is affine, $X = \operatorname{Spec} \mathcal{O}$. Set $Y = \operatorname{Spec} \mathcal{O}/I$, with $I = (f_1, \dots, f_d)$ and $f_1, \dots, f_d$ a regular sequence.

Notation For each $\omega \in \Omega^d_{X/k}$, we shall denote by $\left[{\omega \atop f_1,\dots,f_d} \right]$ the element of $H^d_Y(X, \Omega^d_{X/k})$ which corresponds, via (1), to the morphism:

$$\Lambda^d(\mathcal{I}/\mathcal{I}^2) \to j^*\Omega^d_{X/k}$$
$$\overline{f}_1 \wedge \cdots \wedge \overline{f}_d \mapsto \overline{\omega}$$

The following Proposition is elementary from the definitions (only 2. requires something further: the functoriality of the morphisms of (1) with respect to Y).

Proposition 0.2

(1) $c_X(Y) = \left[{\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_d \atop f_1,\dots,f_d} \right]$.

(2) *Let $(g_1, \dots, g_d) \subset (f_1, \dots, f_d)$ be two ideals with the same radical, and set $g_i = \sum a_{ij} \cdot f_j$. Then:*

$$\begin{bmatrix} \det(a_{ij}) \cdot \omega \\ f_1, \dots, f_d \end{bmatrix} = \begin{bmatrix} \omega \\ g_1, \dots, g_d \end{bmatrix}.$$

(3) *If $\omega \in I \cdot \Omega^d_{X/k}$, then* $\left[{\omega \atop f_1,\dots,f_d} \right] = 0$.

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For each $f \in \mathcal{O}$, let us denote by U_f the affine subset $U_f = X - (f)_0$. Then, $U = X - Y$ is covered by $U_{f_1}, \dots, U_{f_d}$. The exact sequence of local cohomology defines an epimorphism (isomorphism for $d > 0$)

$$H^{d-1}(U, \Omega^d_{X/k}) \to H^d_Y(X, \Omega^d_{X/k}) \tag{2}$$

and the Cech cohomology defines an epimorphism:

$$H^0(U_{f_1} \cap \dots \cap U_{f_d}, \Omega^d_{X/k}) \to H^{d-1}(U, \Omega^d_{X/k}). \tag{3}$$

We shall denote by $\left[\frac{\omega}{f_1 \cdots f_d}\right]$ the image of $(-1)^d \frac{\omega}{f_1 \cdots f_d} \in H^0(U_{f_1} \cap \dots \cap U_{f_d}, \Omega^d_{X/k})$ in $H^d_Y(X, \Omega^d_{X/k})$ via the composition of (2) and (3). We then have (see, for example, [Li1, p. 61, (7.2.1)], although the sign $(-1)^d$ was missing there):

Proposition 0.3 *One has* $\left[{\omega \atop f_1, \dots, f_d}\right] = \left[\frac{\omega}{f_1 \cdots f_d}\right]$, *and hence the properties of Proposition 0.2 may be translated to the symbol* $\left[\text{—}\right]$.

Dualizing Modules Let $\mathcal{O}$ be a local and rational k-algebra with maximal ideal $\mathfrak{m}$ (it would be enough that $k \to \mathcal{O}/\mathfrak{m}$ were finite and separable, if k is not algebraically closed). The closed point will be denoted by p. For each $\mathcal{O}$-module M supported at p we define:

$$F(M) = \operatorname{Hom}_k(M, k)$$

which is a functor in M in the natural way.

Definition We say that E is a *dualizing* module if it is supported at p and there exists an isomorphism of functors:

$$F \simeq \operatorname{Hom}_{\mathcal{O}}(\quad, E).$$

It is well known (Yoneda's Lemma) that a morphism of functors $\operatorname{Hom}_{\mathcal{O}}(\quad, E) \to F$ is equivalent to an element of $F(E)$, a linear form on E. In fact, given a linear form, $w \colon E \to k$, the corresponding morphism of functors is:

$$w_* \colon \operatorname{Hom}_{\mathcal{O}}(M, E) \to \operatorname{Hom}_k(M, k)$$
$$f \mapsto w \circ f.$$

We wish to know what conditions w must satisfy for w_* to be an isomorphism; that is, when the pair (E, w) represents the functor.

From the Matlis theory it is known that $\operatorname{End}_{\mathcal{O}}(E) = \widehat{\mathcal{O}}$; that is, any $\mathcal{O}$-endomorphism of E is the multiplication by a unique element of $\widehat{\mathcal{O}}$. Consequently, if $w \colon E \to k$ is a linear form representing the functor F (that is, w_* is an isomorphism), then for each linear form $w' \colon E \to k$, there exists a unique $f \in \widehat{\mathcal{O}}$ such that $w' = w \cdot f$. Moreover w' represents the functor F if and only if f is invertible.

Definition The $\mathfrak{m}$-torsion of E is the set of elements of E annihilated by $\mathfrak{m}$:

$$E^{\mathfrak{m}} = \{ j \in E \text{ such that } a \cdot j = 0 \text{ for any } a \in \mathfrak{m} \}.$$

One has that $E^{\mathfrak{m}} = \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, E) \simeq \operatorname{Hom}_k(\mathcal{O}/\mathfrak{m}, k)$, and hence $E^{\mathfrak{m}}$ is a 1-dimensional vectorial subspace of E.

Proposition 0.4 *A linear form $w \colon E \to k$ represents the functor F (that is, w_* is an isomorphism) if and only if w is not zero on the $\mathfrak{m}$-torsion of E.*

Proof Assume that $w \colon E \to k$ represents the functor. Then, the composition

$$k = \operatorname{Hom}_k(\mathcal{O}/\mathfrak{m}, k) \overset{w_*}{=\!=} \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, E) \hookrightarrow E \xrightarrow{w} k$$

is the identity and therefore w is not null on the $\mathfrak{m}$-torsion. Conversely, let $w' \colon E \to k$ be a linear form that is non-zero on the $\mathfrak{m}$-torsion. There exists a unique $f \in \widehat{\mathcal{O}}$ such that $w' = w \cdot f$. Then, if j belongs to the $\mathfrak{m}$-torsion, we have that:

$$w'(j) = (w \cdot f)(j) = w(f \cdot j) = w(\overline{f} \cdot j) = \overline{f} \cdot w(j)$$

$\overline{f}$ being the class of f in $\mathcal{O}/\mathfrak{m}$. Since w' is not null on the $\mathfrak{m}$-torsion, it follows that $\overline{f} \neq 0$; that is, f is invertible and hence w' represents the functor. ∎

A basic result of local duality theory is the following:

Proposition 0.5 *If $\mathcal{O}$ is regular, then $H^n_p(\Omega^n)$ is a dualizing module and $c_X(p)$ is a basis of the $\mathfrak{m}$-torsion.*

1 The Residue Map

Let X be a smooth scheme over an algebraically closed field k (a perfect field would be sufficient). Let $\overline{X} = X \times X$, $\pi_2 \colon \overline{X} \to X$ be the second projection and Δ the diagonal of $X \times X$.

Let p be a closed point of X. Consider the inverse image:

$$\pi^* \colon H^n_p(X, \Omega^n_X) \to H^n_{X \times p}(\overline{X}, \Omega^n_{\overline{X}}), \tag{1}$$

and the cup multiplication with $c_{\overline{X}}(\Delta)$

$$\begin{aligned} H^n_{X \times p}(\overline{X}, \Omega^n_{\overline{X}}) &\longrightarrow H^{2n}_{p \times p}(\overline{X}, \Omega^{2n}_{\overline{X}}) \\ a &\longmapsto a \cdot c_{\overline{X}}(\Delta). \end{aligned} \tag{2}$$

Moreover, there is a natural isomorphism:

$$\begin{aligned} H^n_p(X, \Omega^n_X) \underset{k}{\otimes} H^n_p(X, \Omega^n_X) &\overset{\pi_1^* \otimes \pi_2^*}{=\!=\!=} H^{2n}_{p \times p}(\overline{X}, \Omega^{2n}_{\overline{X}}) \\ a \otimes b &\longmapsto \pi_1^*(a) \cdot \pi_2^*(b) \end{aligned} \tag{3}$$

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with $\pi_1, \pi_2 \colon \overline{X} \to X$ as the natural projections.

Definition We shall denote by ϕ_p the morphism:

$$\phi_p \colon H^n_p(X, \Omega^n_X) \to H^n_p(X, \Omega^n_X) \underset{k}{\otimes} H^n_p(X, \Omega^n_X),$$

which is the composition of (1), (2) and (3). Notice that ϕ_p is a morphism of $\mathcal{O} \underset{k}{\otimes} \mathcal{O}$-modules, where $H^n_p(X, \Omega^n_X)$ is an $\mathcal{O} \underset{k}{\otimes} \mathcal{O}$-module via the diagonal morphism $\mathcal{O} \underset{k}{\otimes} \mathcal{O} \to \mathcal{O}$, $a \otimes b \mapsto ab$.

Lemma 1.1 **(Fundamental)** *Let p be a closed point of X. Then:*

$$\phi_p\big(c_X(p)\big) = c_X(p) \otimes c_X(p).$$

Proof This follows from the equalities (Theorem 0.1):

$$c_{\overline{X}}(X \times p) \cdot c_{\overline{X}}(\Delta) = c_{\overline{X}}(p \times p) = c_{\overline{X}}(X \times p) \cdot c_{\overline{X}}(p \times X) = c_X(p) \otimes c_X(p),$$

where the last equality is via $\pi_1^* \otimes \pi_2^*$. ■

Local and Global Residue: Compatibility Let $\mathcal{O}$ be a local, regular and rational k-algebra of dimension n. Set $X = \operatorname{Spec} \mathcal{O}$ and let us take p as the closed point. Let us denote $\Omega^n = \Omega^n_{X/k}$.

Theorem 1.2 **(Local Residue)** *There exists a unique non-zero k-linear map:*

$$\operatorname{Res}_p \colon H^n_p(\Omega^n) \to k,$$

representing the functor F and compatible with products (i.e., $(\operatorname{Res}_p \otimes \operatorname{Res}_p) \circ \phi_p = \operatorname{Res}_p$*).*

Proof Let $w \colon H^n_p(\Omega^n) \to k$ be a linear map representing the functor F. Then, w is not null on $c_X(p)$ and hence $(w \otimes w) \circ \psi$ is not null either, since $\phi_p\big(c_X(p)\big) = c_X(p) \otimes c_X(p)$. Therefore, $(w \otimes w) \circ \phi_p$ represents the functor and hence there exists a unique invertible $f \in \widehat{\mathcal{O}}$ such that $(w \otimes w) \circ \psi = w \cdot f$. Taking $w \cdot \frac{1}{f}$, one concludes.

Unicity: Given a linear map w, there exists a unique $f \in \widehat{\mathcal{O}}$ such that $w = \operatorname{Res}_p \cdot f$. Now, if $w = (w \otimes w) \circ \phi_p$, then:

$$\operatorname{Res}_p \cdot f = (\operatorname{Res}_p \cdot f \otimes \operatorname{Res}_p \cdot f) \circ \phi_p = \operatorname{Res}_p \cdot f^2,$$

and hence $f = 0$ or $f = 1$. If w represents the functor, then $f \neq 0$ and one concludes. ■

Lemma 1.3 $\operatorname{Res}_p\big(c_X(p)\big) = 1.$

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Proof Since $\phi_p\big(c_X(p)\big) = c_X(p)\otimes c_X(p)$, one has:

$$\operatorname{Res}_p\big(c_X(p)\big) = \big((\operatorname{Res}_p\otimes\operatorname{Res}_p)\circ\phi_p\big)\big(c_X(p)\big) = \operatorname{Res}_p\big(c_X(p)\big)^2,$$

and hence $\operatorname{Res}_p\big(c_X(p)\big) = 0$ or 1. Since Res_p is not null on the $\mathfrak{m}$-torsion, one concludes. ∎

Invariance under Automorphisms Let $G = \operatorname{Aut}_{k-\mathrm{alg}}\widehat{\mathcal{O}}$ be the group of automorphisms of the completion $\widehat{\mathcal{O}}$. Note that there is a natural action of G in $H^n_p(\Omega^n)$. In fact, let $\widehat{\Omega}^n$ be the completion of Ω^n. The natural morphism $H^n_p(\Omega^n)\to H^n_p(\widehat{\Omega}^n)$ is an isomorphism, and it therefore suffices to see that G acts on $\widehat{\Omega}^n$. This follows from the following equalities. Set $\mathcal{O}_i = \mathcal{O}/\mathfrak{m}^i$, then:

$$G = \varprojlim_i \operatorname{Aut}_{k-\mathrm{alg}}\mathcal{O}_i, \quad \widehat{\Omega}^n = \varprojlim_i \Omega^n_{\mathcal{O}_i/k}.$$

The natural action of $\operatorname{Aut}_{k-\mathrm{alg}}\mathcal{O}_i$ in $\Omega^n_{\mathcal{O}_i/k}$ induces the action of G in $\widehat{\Omega}^n$.

For any $g\in G$, we shall denote by $g^*\colon H^n_p(\Omega^n)\to H^n_p(\Omega^n)$ the action of g in $H^n_p(\Omega^n)$. From the very construction of ϕ_p, it follows that it is invariant under automorphisms; that is, for any $g\in G$ the diagram:

$$\begin{array}{ccc}
H^n_p(\Omega^n) & \xrightarrow{\phi_p} & H^n_p(\Omega^n)\underset{k}{\otimes} H^n_p(\Omega^n)\\
{\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*\otimes g^*}\\
H^n_p(\Omega^n) & \xrightarrow{\phi_p} & H^n_p(\Omega^n)\underset{k}{\otimes} H^n_p(\Omega^n)
\end{array}$$

is commutative. Then, by the unicity of the residue, it is also invariant: for any $g\in G$, one has $\operatorname{Res}_p\circ g^* = \operatorname{Res}_p$.

Global Residue Let X be a proper and smooth scheme over k. We shall construct a canonical isomorphism:

$$\operatorname{Res}\colon H^n(X,\Omega^n_X)\to k$$

with the same idea as for the local residue; that is, imposing the condition of compatibility with products.

Theorem 1.4 *There is a canonical morphism*

$$\phi\colon H^n(X,\Omega^n_X)\to H^n(X,\Omega^n_X)\underset{k}{\otimes} H^n(X,\Omega^n_X)$$

such that for any closed point $p\in X$ the diagram:

$$\begin{array}{ccc}
H^n_p(\Omega^n) & \xrightarrow{\phi_p} & H^n_p(\Omega^n)\underset{k}{\otimes} H^n_p(\Omega^n)\\
\downarrow & & \downarrow\\
H^n(X,\Omega^n_X) & \xrightarrow{\phi} & H^n(X,\Omega^n_X)\underset{k}{\otimes} H^n(X,\Omega^n_X)
\end{array}$$

is commutative (compatibility of ϕ_p and ϕ).

Proof ϕ is constructed as in the local case. Set $\overline{X} = X \times X$, $\pi_2 \colon \overline{X} \to X$ the second projection and $c_{\overline{X}}(\Delta)$ the cohomology class of the diagonal. One has canonical morphisms:

$$H^n(X, \Omega_X^n) \xrightarrow{\pi_2^*} H^n(\overline{X}, \Omega_{\overline{X}}^n) \xrightarrow{\cdot c_{\overline{X}}(\Delta)} H^{2n}(\overline{X}, \Omega_{\overline{X}}^{2n}) = H^n(X, \Omega_X^n) \otimes H^n(X, \Omega_X^n),$$

whose composition is ϕ. The commutativity of the diagram follows from the constructions. ■

Proposition 1.5 *Let p be a closed point of X. The cohomology class $c_X(p)$ is a generator of $H^n(X, \Omega_X^n)$. Moreover $\phi\big(c_X(p)\big) = c_X(p) \otimes c_X(p)$ (this property characterizes the cohomology class of a point and it proves, in particular, that the cohomology class does not depend on the point).*

Proof For the first part, it suffices to prove that the morphism:

$$\operatorname{Ext}^n_{\mathcal{O}_X}\big(k(p), \Omega_X^n\big) \to H^n(X, \Omega_X^n) = \operatorname{Ext}^n_{\mathcal{O}_X}(\mathcal{O}_X, \Omega_X^n)$$

is an isomorphism. Now, the cokernel is $\operatorname{Ext}^n_{\mathcal{O}_X}(\mathfrak{m}_p, \Omega_X^n)$, with $\mathfrak{m}_p$ the sheaf of ideals of the point p, and by duality $\operatorname{Ext}^n_{\mathcal{O}_X}(\mathfrak{m}_p, \Omega_X^n) \simeq H^0(X, \mathfrak{m}_p) = 0$.

The second part follows from the compatibility of ϕ_p and ϕ and from the fundamental lemma 1.1. ■

Theorem 1.6 **(Global Residue)** *There exists a unique non-zero morphism (and hence isomorphism):*

$$\operatorname{Res} \colon H^n(X, \Omega_X^n) \to k$$

that is compatible with products: $(\operatorname{Res} \circ \operatorname{Res}) \circ \phi = \operatorname{Res}$.

Proof Existence: Let $w \colon H^n(X, \Omega_X^n) \to k$ be a non-zero linear map. Then, w does not vanish on $c_X(p)$, with p any closed point of X, and hence $(w \otimes w) \circ \phi$ does not vanish on $c_X(p)$ either, since $\phi\big(c_X(p)\big) = c_X(p) \otimes c_X(p)$. There exists a unique $\lambda \in k^*$ such that $(w \otimes w) \circ \phi = \lambda \cdot w$. Taking $\operatorname{Res} = \frac{1}{\lambda} \cdot w$ one concludes.

Unicity: if w were another morphism, then $w = \lambda \cdot \operatorname{Res}$, with $\lambda \in k^*$, and hence:

$$w = (w \otimes w) \circ \phi = \lambda^2 (\operatorname{Res} \otimes \operatorname{Res}) \circ \phi = \lambda^2 \cdot \operatorname{Res} = \lambda \cdot w.$$

Therefore, $\lambda = 1$. ■

Theorem 1.7 **(Compatibility of Local and Global Residue)** *Let p be a closed point of X. If $h_p \colon H^n_p(X, \Omega_X^n) \to H^n(X, \Omega_X^n)$ is the natural map, one has that* $\operatorname{Res} \circ h_p = \operatorname{Res}_p$. *In particular,* $\operatorname{Res}\big(c_X(p)\big) = 1$ *for any closed point p.*

Proof This follows from the compatibility of ϕ_p and ϕ and from the unicity of Res_p. ■

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2 Residue Theorem and Computation

Let $\mathcal{O}$ be the local ring at a closed point p and $\Omega^n = \Omega^n_{\mathcal{O}/k}$. Let $f_1, \dots, f_n \in \mathfrak{m}$ be a regular sequence. For each n-form $\omega \in \Omega^n$, we define $\operatorname{Res}_p\bigl(\frac{\omega}{f_1\cdots f_n}\bigr) = \operatorname{Res}_p\bigl[\frac{\omega}{f_1\cdots f_n}\bigr]$.

Let X be a proper variety, and let $D_1, \dots, D_n$ be effective divisors with intersection $D_1 \cap \cdots \cap D_n$ a finite number of points. Set $D = D_1 + \cdots + D_n$ and let

$$\theta \in H^0\bigl(X, \Omega^n(D)\bigr)$$

be a meromorphic n-form with pole divisor D. For each $p \in D_1 \cap \cdots \cap D_n$, we take the local ring at p and we can define the residue $\operatorname{Res}_p \theta$.

Theorem 2.1 **(Residue Theorem)** *With the preceding hypothesis, one has that:*

$$\sum_p \operatorname{Res}_p \theta = 0.$$

Proof Let $U = X - (D_1 \cap \cdots \cap D_n) = U_1 \cup \cdots \cup U_n$, with $U_i = X - D_i$. Then, θ belongs to $H^0(U_1 \cap \cdots \cap U_n, \Omega^n_X)$ and defines, by Cech cohomology, an element of $H^{n-1}(U, \Omega^n_X)$. Now, the local cohomology exact sequence

$$H^{n-1}(U, \Omega^n_X) \to \bigoplus_p H^n_p(\Omega^n_X) \to H^n(X, \Omega^n_X) \xrightarrow[\quad]{\operatorname{Res}} k$$

and the compatibility of the local and global residue allow us to conclude. ∎

Proposition 2.2 **(Explicit Calculation of the Residue)** *Let $t_1, \dots, t_n$ be a system of parameters at p. Then*

(a) *If $k_1, \dots, k_n$ are positive integers, and $k_1 \cdot \cdots \cdot k_n > 1$, then we have that:*

$$\operatorname{Res}_p\left(\frac{\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n}{t_1^{k_1}\cdots t_n^{k_n}}\right) = 0.$$

(b) $\operatorname{Res}_p\bigl(\frac{\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_n}{t_1\cdots t_n}\bigr) = 1.$

(c) *Let $\omega = g \cdot \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$ be an n-form regular at p and $f_1, \dots, f_n$ a regular sequence. For each i there exists k_i such that $t_i^{k_i} \in (f_1, \dots, f_n)$ and hence $t_i^{k_i} = \sum a_{ij} f_j$. Then $\operatorname{Res}_p\bigl(\frac{\omega}{f_1\cdots f_n}\bigr)$ is the coefficient of $t_1^{k_1-1}\cdots t_n^{k_n-1}$ in the Taylor expansion of the function $g \cdot \det(a_{ij})$.*

Proof (a) Let us consider the automorphism of $\widehat{\mathcal{O}}$ given by $t_i' = \lambda \cdot t_i$, with $\lambda \in k$. Since the residue is invariant under automorphisms:

$$\begin{aligned}\operatorname{Res}_p\left(\frac{\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n}{t_1^{k_1}\cdots t_n^{k_n}}\right) &= \operatorname{Res}_p\left(\frac{\mathrm{d}t_1' \wedge \cdots \wedge \mathrm{d}t_n'}{t_1'^{k_1}\cdots t_n'^{k_n}}\right)\\ &= \lambda^{n-(k_1+\cdots+k_n)} \operatorname{Res}_p\left(\frac{\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n}{t_1^{k_1}\cdots t_n^{k_n}}\right)\end{aligned}$$

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and one concludes.

(b) By Proposition 0.3,

$$\operatorname{Res}_p\left(\frac{\mathrm{d}\,t_1 \wedge \cdots \wedge \mathrm{d}\,t_n}{t_1 \cdots t_n}\right) = \operatorname{Res}_p \begin{bmatrix} \mathrm{d}\,t_1 \wedge \cdots \wedge \mathrm{d}\,t_n \\ t_1, \ldots, t_n \end{bmatrix}.$$

But $\left[\begin{smallmatrix} \mathrm{d}\,t_1 \wedge \cdots \wedge \mathrm{d}\,t_n \\ t_1, \ldots, t_n \end{smallmatrix}\right] = c_X(p)$ (Proposition 0.2, part (1)) and hence one concludes by Lemma 1.3.

(c) By Propositions 0.2 and 0.3,

$$\operatorname{Res}_p\left(\frac{\omega}{f_1 \cdots f_n}\right) = \operatorname{Res}_p\left(\frac{g \cdot \det(a_{ij})\,\mathrm{d}\,t_1 \wedge \cdots \wedge \mathrm{d}\,t_n}{t_1^{k_1-1} \cdots t_n^{k_n-1}}\right).$$

One concludes then by (a), (b), and (3) of Proposition 0.2. ■

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