

1

Parallel Folds

We start with *parallel folds*, the simplest and easiest pop-up technique to understand. Despite the simplicity of parallel folds, one can reach beautiful and intricate designs with just this one construction technique, repeated. But we'll keep it simple, making our goal pop-up letters, perhaps the most common pop-up card design.

The mathematics behind parallel folds is also simple, which will provide an opportunity to set conventions and notation that will be used throughout. We will aim toward describing the 3D motion of card opening by tracking various key points on the card front and back traveling along circles in space.

1.1 Card Notation

Here we establish the basic notation used throughout the book, which is important because the rigor of mathematics relies on clear definitions and symbolic notation.¹

A card is composed of two identical rectangles, the back B and the front F , joined and hinged along the card centerline L . (The centerline is also called the gutter or the spine.) The front and back are both rigid, usually made of stiff cardstock. We view the back as fixed to a tabletop; only the front moves. See Fig. 1.1. In general we will use uppercase letters for “big” things and lowercase letters for “small” things, for example, edges or specific points. And we will follow the mathematics tradition of using (lowercase) Greek letters for angles: α, β, θ , etc. In particular, we will reserve θ (theta) to represent the card angle, the angle along the centerline between the front and the back. At $\theta = 0^\circ$ the card is closed, at $\theta = 90^\circ$ it is half-open, and at $\theta = 180^\circ$ the card is fully opened flat. Often, the card is intended to only open to 90° , when F becomes a backdrop to the popped-up structure. This will be the case for most of the constructions in this chapter.

¹Symbols are gathered in a table on p. 121.

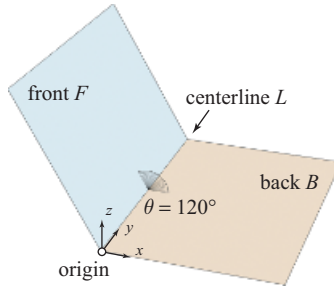


Figure 1.1 Basic card notation.

In geometry, the angle in 3D between two planes, such as θ just described, is known as a *dihedral angle*; see Box 1.1, which relies on vectors, described in Box 1.2.

Our overall goal is to describe the geometry of the motions of the pop-up structures as the card is opened or closed. Often we reach the geometry through algebraic equations. The equations will employ variables representing points in a 3D Cartesian coordinate system, called *Cartesian* because it was introduced by René Descartes. We will consistently use x and y to represent placement in the xy -plane of the card back B , and use z as the third coordinate representing the perpendicular height above B . We place the origin of the coordinate system somewhere along the centerline L , either at the bottom end or the midpoint of L , with x increasing horizontally along the width of the back and y increasing vertically along the centerline.

Box 1.1 Dihedral Angle

In geometry, the angle in 3D between two planes is known as a *dihedral angle*, from the Greek *di-* (two) and *hedra* (faces). A *face* is a region of a plane. Ultimately angles are measured in two dimensions, not three, between what are known as *vectors*: see Box 1.2.

In Fig. 1.2, F_1 and F_2 are faces sharing an edge e . The dihedral angle δ at e can be measured by a protractor perpendicular to edge e ; so

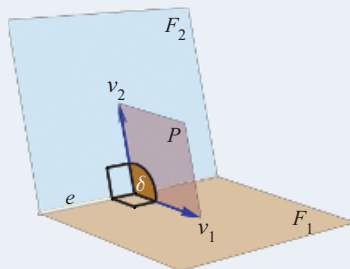


Figure 1.2 The dihedral angle δ between faces F_1 and F_2 .

the protractor lies in a plane P orthogonal to e . (The term *orthogonal* is a synonym for “perpendicular.”) The vectors v_1 and v_2 lie along the intersection of plane P with the planes containing the faces F_1 and F_2 , respectively. The angle δ measures how much v_2 needs to be rotated about e to align with v_1 . In this chapter, e is the card centerline L and δ is the card angle θ .

Box 1.2 Vectors

Informally, a *vector* can be viewed as a directed line segment. Thus it has both a length and a direction. The two endpoints are called the *tail* and the *head*, with the head marked by an arrowhead. A vector differs from a directed line segment in that its tail is not fixed to one point of the plane or space. Rather it should be imagined to be placed anywhere, as in Fig. 1.3.

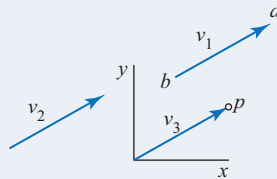


Figure 1.3 Vectors v_1, v_2, v_3 are the same vector $a - b$.

Often it is convenient to view a point p , which has specific coordinates with respect to an origin, as a vector with its tail at the origin and its head at p . Given any two points a and b , the difference in their coordinates, $a - b$, is a vector, placed with the head at a and the tail at b . Here if $a = (a_x, a_y)$ and $b = (b_x, b_y)$, then

$$a - b = (a_x - b_x, a_y - b_y) .$$

We will make cuts and creases in the card (and in attachments to the card). Creases in the card can be either *valley* or *mountain* folds, terminology from origami with the obvious interpretation. All creases can be viewed as hinges between the flat, planar pieces to either side. Those flat pieces are often called *faces*, which are rigid except where hinged along the edges.

1.2 Rhombus Card

Our first pop-up is a *rhombus*, a four-sided figure (a quadrilateral) all of whose edge lengths are equal. Fig. 1.4 shows the construction from a rectangle of cardstock. After folding the cardstock in half along centerline L , the two

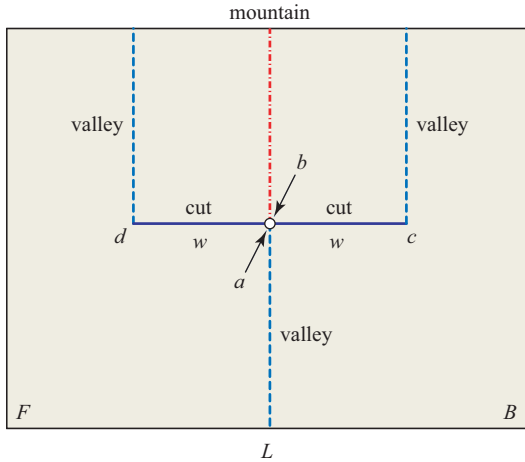


Figure 1.4 Rhombus cuts and folds, producing Fig. 1.5.

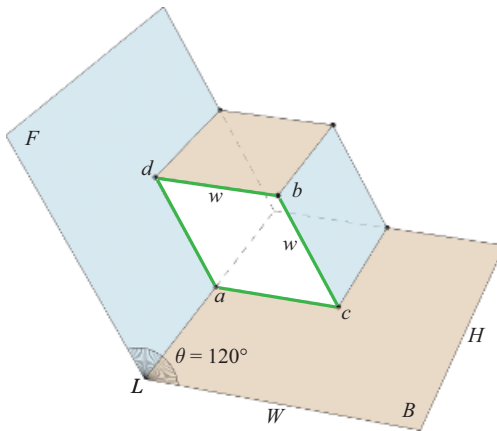


Figure 1.5 Rhombus card from the template in Fig. 1.4.

length- w cuts can be achieved by one scissors cut of length w perpendicular to L . Then the mountain and valley creases shown pop out the rhombus, as illustrated in Fig. 1.5. With several points labeled, the *rim* (green in Fig. 1.5) is a 2D rhombus $abcd$ lying in a plane orthogonal to the centerline L . The sides of the rhombus have length w . We'll use the notation ac to represent the line segment from point a to point c , and $|ac| = w$ to indicate its length (Box 1.3.)

Box 1.3 Segment Length $|ab|$

Throughout this book, we use lowercase letters to represent points, in either 2D or 3D. We often need to refer to the length of a segment

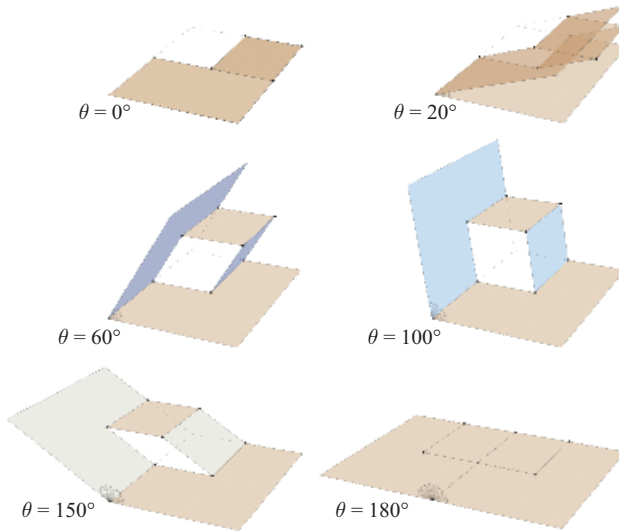


Figure 1.6 Snapshots of the rhombus card. Animation available (O'Rourke, 2021).

between two points. If a and b are two points, we use $|ab|$ to mean the length of the line segment from a to b (which of course is the same as $|ba|$, the length from b to a). In terms of 3D coordinates, with $a = (a_x, a_y, a_z)$ and $b = (b_x, b_y, b_z)$,

$$|ab| = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2}.$$

Often this is expressed as $|a - b|$, because the difference of the point coordinates is a vector from b to a , and $|ab|$ is the length of this vector, as explained in Box 1.2. Other sources notate this as $\|a - b\|$, where $\|$ indicates the *norm* of the vector.

The cut segment creates two sides, which then become the four sides of the rhombus rim. Note that in the flattened template, points a and b have the same location—they are *co-located*—and become opposite corners of the rhombus. Fig. 1.6 shows snapshots of the card at increasing values of θ from 0° —completely closed—to 180° —fully opened, when it matches the template in Fig. 1.4.

This construction is called a *parallel fold* because the two valley creases and the one mountain crease are parallel (and parallel to the card centerline). For aesthetic reasons, often an extra layer of cardstock is pasted behind the front and back faces of the card, so that the gap formed by popping up the rhombus has a backdrop, rather than the hole shown in Fig. 1.6.

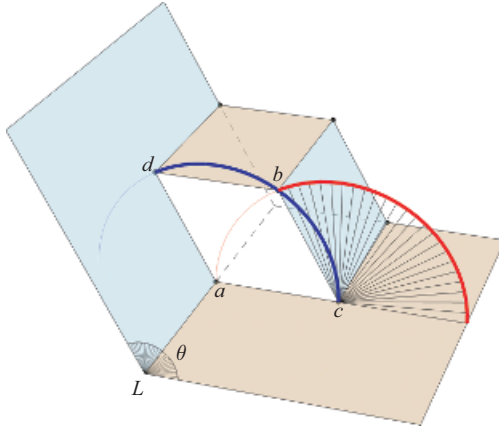


Figure 1.7 Points b and d follow circular arcs centered on c and a , respectively.

1.3 Circular Arc Equations

As is evident in Fig. 1.7, point b at the corner of the rhombus must remain the fixed distance $|cb| = w$ from point c throughout the opening motion, and therefore must ride on a circle of radius w centered on c . Similarly, point d remains distance $|ad| = w$ from a , and so follows a circle centered on a . In preparation of more complicated motions in later chapters, we now describe these motions via algebraic equations.

The equation of a radius- w circle in the xy -plane, centered on the origin, is

$$x^2 + y^2 = w^2 . \tag{1.1}$$

The circles followed by points b and d lie instead in the xz -plane, a plane orthogonal to the y -axis along the centerline L . Let's focus on point d revolving around a . One might think that the equation for d 's position is

$$x^2 + z^2 = w^2 , \tag{1.2}$$

simply substituting z for y and otherwise not mentioning y . But because that equation does not constrain y , it describes circles for all values of y . So Eq. 1.2 describes an infinite cylinder centered on the y -axis (and in a 3D coordinate system, Eq. 1.1 denotes an infinite cylinder centered on the z -axis). To describe the circle that d follows, we need two equations, including one pinning down the precise y -coordinate for point a .

Let us say that the card back B has width W and height H (uppercase distinguishing from lowercase w and h). Then point a has coordinates $a = (0, \frac{H}{2}, 0)$, and d is constrained by these two equations “simultaneously”:

$$\begin{aligned} x^2 + z^2 &= w^2 , \\ y &= \frac{H}{2} . \end{aligned}$$

We will frequently need to “solve” equations simultaneously, but here there is nothing to solve: we are just specifying that both equations constrain x, y, z at the same time.

Similarly, point b follows the circle arc centered on $c = (w, \frac{H}{2}, 0)$ defined by these two equations, where x is shifted by w :

$$\begin{aligned}(x - w)^2 + z^2 &= w^2, \\ y &= \frac{H}{2}.\end{aligned}$$

Lastly we presage *parametric equations*, which were used to draw the spokes in Fig. 1.7 and which will play a significant role in later chapters (Box 2.2). Parametric equations rely on a *parameter*, which in our situation is the card angle θ . The equations we just derived describe the full circle for point b “all at once.” Parametric equations pinpoint the location of b for a specific value of θ :

$$\begin{aligned}x &= w + w \cos \theta, \\ y &= \frac{H}{2}, \\ z &= w \sin \theta.\end{aligned}\tag{1.3}$$

For example, if $\theta = 90^\circ$, then $\cos \theta$ and $\sin \theta$ are 0 and 1, respectively, so $(x, y, z) = (w, \frac{H}{2}, w)$, which places b directly above $c = (w, \frac{H}{2}, 0)$. (For more on cosine and sine, see Box 1.4.)

Box 1.4 Trigonometry

We will use trigonometry sparingly and never need more than the basic relationships illustrated in Fig. 1.8, always measuring angles counterclockwise. Note that not only is $\tan \theta$ the ratio of the altitude to the base of the triangle, but it is the length of the tangent from p to the x -axis. This latter relationship helps to understand why, as θ approaches 90° , $\tan \theta$ grows without bound—approaches infinity.

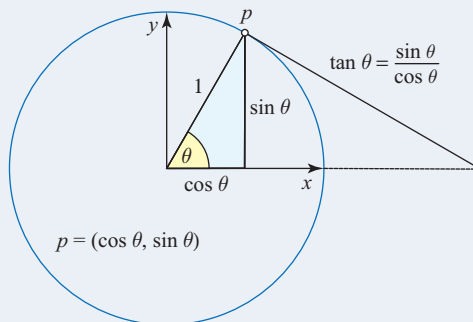


Figure 1.8 Basic trigonometry: unit-radius circle.

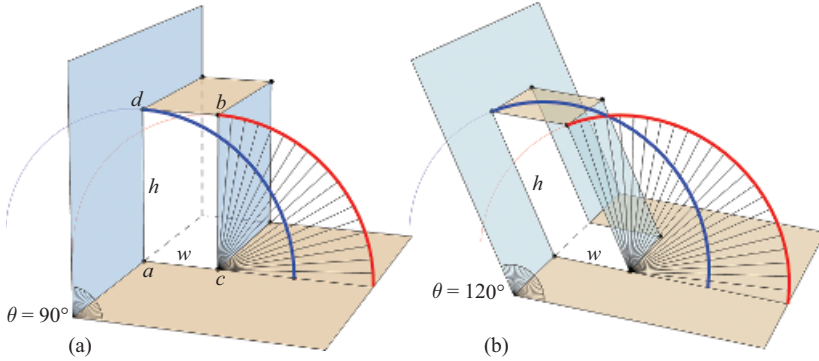


Figure 1.9 (a) Asymmetric cut: $h > w$. (b) Two parallel asymmetric cuts.

1.4 Parallelograms

Two simple modifications of the rhombus construction turn it into a useful pop-up structure. First, rather than cut a length w to each side of the centerline L , an asymmetric cut of w on the back B and h on the front F leads to a pop-up parallelogram, $w \times h$ at card angle $\theta = 90^\circ$. See Fig. 1.9(a).

Second, two identical parallel cuts, symmetric or asymmetric about L , result in a centered pop-up, rather than the parallelogram extending to the card’s top edge. See Fig. 1.9(b). With $h > w$ as in the figure, the pop-up can serve as a face into which letters can be carved. When $h < w$, the pop-up often serves as a platform on which other structures are built.

Exercise 1.1 Practice: Within Card Profile

With the card width W and height H , which combinations of the pop-up parallelogram dimensions w and h ensure that no portion of the pop-up “sticks out” when the card is fully closed at $\theta = 0^\circ$? (See Fig. 1.6.)

Exercise 1.2 Understanding: Arcs Intersection

In Fig. 1.9(b), what are the coordinates of the point p at which the red and blue arcs cross? Here red marks the trajectory of point b , and blue that of point d .

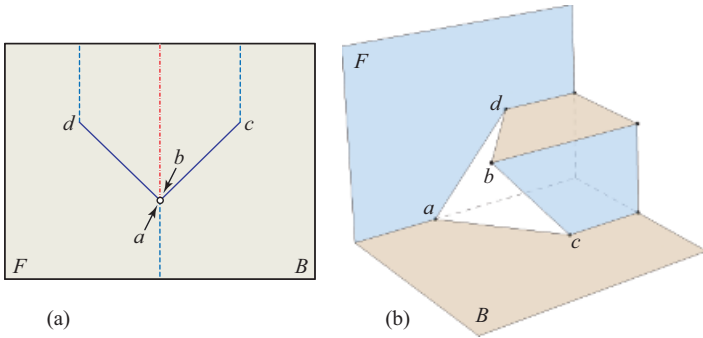


Figure 1.10 (a) Template for angled cuts. (b) The pop-up.

1.5 Cut Variations

We've examined symmetric and asymmetric cuts, but many other variations are possible. For example, angled but symmetric cuts (and still creases parallel to L), as in Fig. 1.10(a), lead to a very similar pop-up with parallel creases, as in Fig. 1.10(b).

Before we turn to popping up letters, the reader might enjoy exploring several other variants in the following exercises, perhaps verifying understanding with scissors and paper (cardstock is not essential for quick experiments).

Exercise 1.3 Understanding: Parallel Unequal

Describe the behavior of the pop-up whose template is shown in Fig. 1.11.

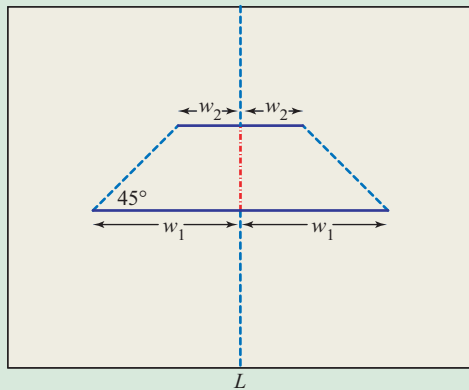


Figure 1.11 Cuts are centered on L , parallel, but of unequal lengths.

Exercise 1.4 Understanding: Three Parallel Cuts

Describe the behavior of the pop-up whose template is shown in Fig. 1.12.

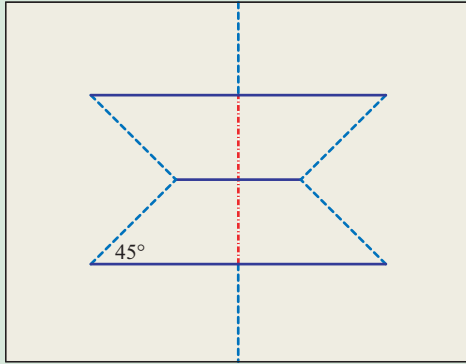


Figure 1.12 Three centered, parallel cuts.

Exercise 1.5 Understanding: Slanted Parallel

Describe the behavior of the pop-up whose template is shown in Fig. 1.13, with valley creases as shown.

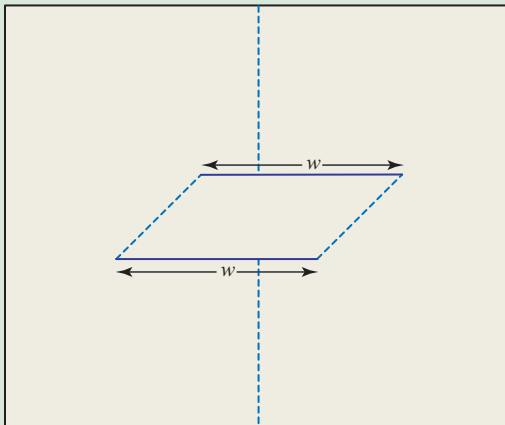


Figure 1.13 Parallel but slanted cuts.

1.6 Pop-Up Letters

It is now a short step from popping up a parallelogram box to popping up letters “carved” into the front of such a box. Fig. 1.14 illustrates the idea for the letter

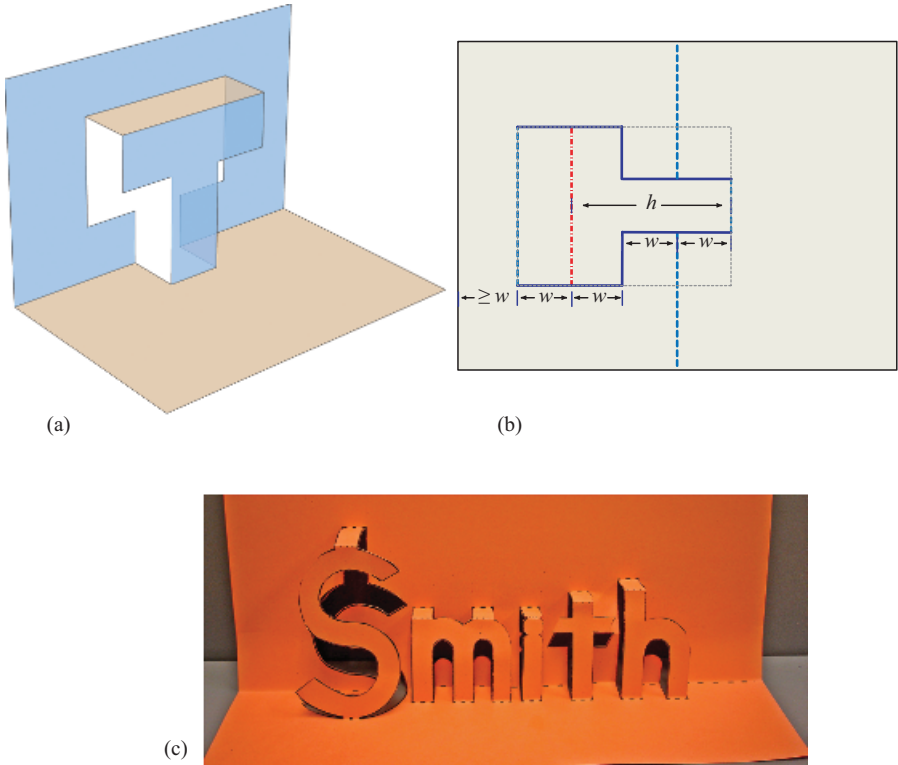


Figure 1.14 (a) Pop-up letter T and (b) template. (c) Several letters. Animation available for (a) (O'Rourke, 2021). [Pattern from Card3D software.]

T, which can be generalized for any sequence of letters. There is software all over the Web that creates a template for a given sequence of letters, and indeed we used software for Fig. 1.14(c).

Exercise 1.6 Practice: Tallest Letter

If a card has dimensions $W \times H$, what is the tallest letter one can pop-up, without having some of the letter protrude when the card is fully closed? (See also Exercise 1.1.)

1.7 Tents

So far we have explored constructions that pop-up cut portions of the front and back of the card. This constrains the structure so that the top (the w -side) of the popped-up parallelogram box (in Figs. 1.6 and 1.9) remains at all times parallel to the card back B and the h -side of the parallelogram remains parallel to the card front F . One can add structures not cut from the card, but instead

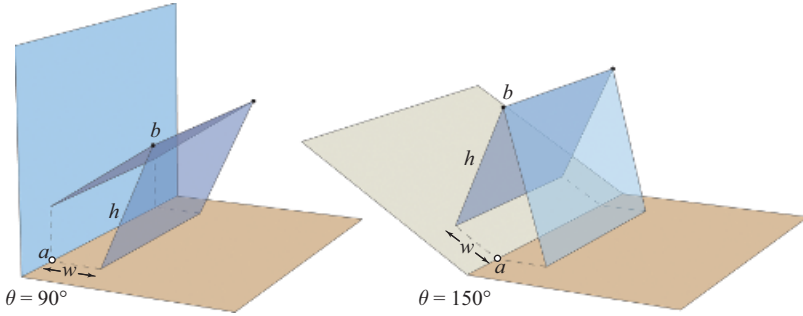


Figure 1.15 Tent, with $h = 3w$.

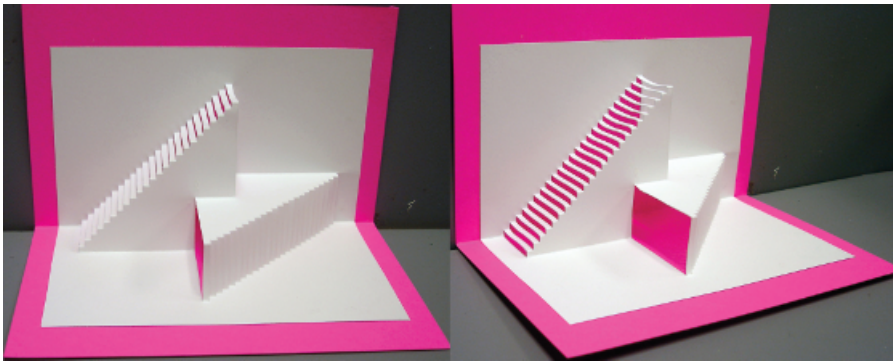


Figure 1.16 Two staircases. [Design and construction by Gail Parsloe. Based on construction in (Jackson, 1993, p. 63).]

separately creased cardstock pasted onto the card back and front. Fig. 1.15 shows a simple *tent* that does not flatten at $\theta = 180^\circ$ as have our previous examples. When the card is fully opened at $\theta = 180^\circ$, the tent has height

$$|ab| = \sqrt{h^2 - w^2},$$

which is positive whenever $h > w$. We will examine similar pasted-in structures in the next chapter and especially in Chapters 5, 6, and 7.

1.8 Artistic Designs

Even with just the simple parallel fold, artistic designs can be made. Fig. 1.16 shows two intersecting staircases, one vertical and one horizontal. Each is constructed from parallel cuts staggered by the stair height. The extra layer of cardstock as a backdrop highlights the design.

Fig. 1.17 shows a beautiful design, again constructed solely of parallel cuts and a few “windows.” One can view the central circular region as akin to the platform into which the T is cut in Fig. 1.14. We will see an analogous artistic example in Chapter 6, Fig. 6.1.

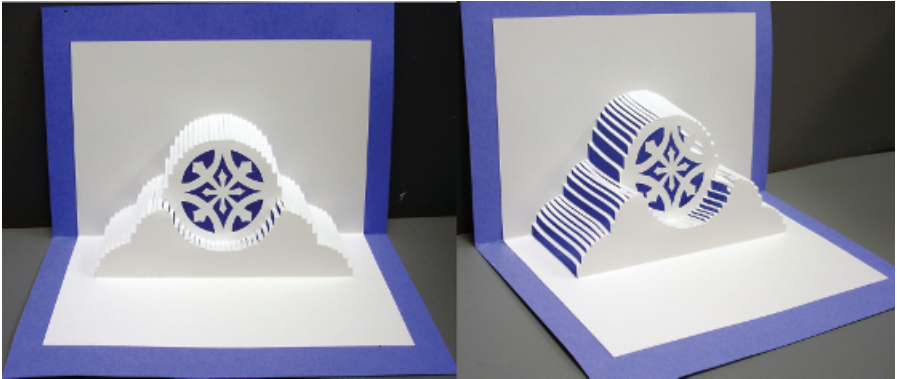


Figure 1.17 Logo for Evermore Origamic Architecture. [Construction by Gail Parsloe. Pattern from origamicarchitecture.com.]

Exercise 1.7 Challenge: Half-Cylinder

Design a pop-up that approximates half of a cylinder when opened to $\theta = 90^\circ$, as in Fig. 1.18.

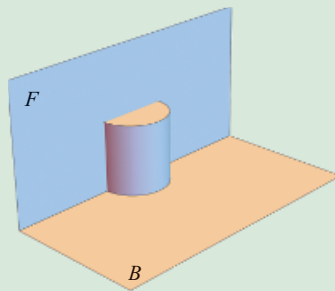


Figure 1.18 A pop-up half-cylinder.

1.9 Creases as Centerlines

A theme throughout this book will be identifying substructures in a pop-up design that have a simple relationship to the card angle θ during the opening motion. The simplest relationship is identity, and that is what is present in the pop-up parallelogram structure. As mentioned earlier, the top face of the parallelogram “box” always remains parallel to the card back B , and the lateral face of the parallelogram remains parallel to the front face F . So the valley creases where the parallelogram meets B and F (at points c and d in Figs. 1.5 and 1.9) are “copies” of the card centerline L , exhibiting the same dihedral angle θ .

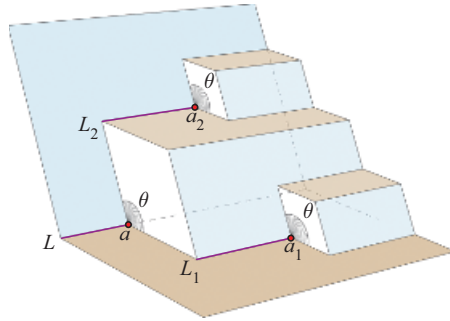


Figure 1.19 The card angle θ at a repeats at a_1 and a_2 . Animation available (O'Rourke, 2021). [Design from (Jackson, 1993, p. 36).]



Figure 1.20 Fractal Christmas tree. [Design and construction by David Richeson; used by permission of author.]

This permits building another pop-up structure on those creases, viewing each crease as a displaced centerline. Fig. 1.19 shows one possibility. Here θ occurs at a along L , as well as at points a_1 and a_2 along L_1 and L_2 , respectively. The pattern can be repeated, leading to the impressive fractal-like recursive pop-up in Fig. 1.20.

Notes

Impressive commercial software is available for designing letter (and more complex) pop-ups, e.g., POP-UP CARD DESIGNER PRO, tamasoft.co.jp/craft/popupcard-pro_en/.

For the Christmas tree design (Fig. 1.20), see Dave Richeson's blog, artfulmaths.com/blog/folding-christmas-fractals.