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A NOTE ON THE EQUATION $\lambda * \rho * \mu = \rho$

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Let G be a Hausdorff topological group and μ and λ be probability measures on G. We prove necessary and sufficient conditions for the existence of a probability measure ρ such that $\lambda * \rho * \mu = \rho$ under certain conditions. We prove a similar result for probability measures on semigroups.

In this note we consider the problem of proving necessary and sufficient conditions for the existence of a measure ρ such that the equation

$$(\star) \qquad \lambda * \rho * \mu = \rho$$

holds where λ and μ are two given probability measures. This problem originated from the convergence of concentration functions in the following way: given a probability measure μ either the concentration functions converge to zero or $\check{\mu}^n * \mu^n \to \rho$ (see [1]) and hence $\check{\mu} * \rho * \mu = \rho$.

Let G be a Hausdorff topological group. Let ν be a probability measure on G. Then ν is said to be *adapted* if the closed subgroup generated by the support of ν is G. When ν is adapted we denote by $\mathcal{H}(\nu)$ the smallest closed normal subgroup of G a coset of which contains the support of ν. We say that ν is *concenterated* if there exist a compact subset C and a sequence (g_n) in G such that $\nu^n(g_nC) = 1$ for all n. Let ν_1 and ν_2 be two adapted probability measures on G. Then $\mathcal{H}(\nu_1, \nu_2)$ denotes the smallest closed normal subgroup such that for some $x, y \in G$, $x\mathcal{H}(\nu_1, \nu_2)$ and $y\mathcal{H}(\nu_1, \nu_2)$ contains the support of ν_1 and the support of ν_2 respectively.

Let λ and μ be adapted probability measures on G. Let us now consider the following:

(1) the subgroups $\mathcal{H}(\mu)$, $\mathcal{H}(\lambda)$ and $\mathcal{H}(\lambda, \mu)$ are all compact and the same and the measures μ and $\check{\lambda}$ are supported on $g\mathcal{H}(\mu)$ for all g in the support of μ, in particular, λ and μ are concentrated;

(2) there exist compact sets L_1 and L_2 and a sequence (g_n) in G such that for some $\delta > 0$, $\mu^n(g_n^{-1}L_1) > \delta$ and $\check{\lambda}^n(g_n^{-1}L_2) > \delta$ for all n;

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(3) there exists a probability measure ρ such that $\lambda * \rho * \mu = \rho$.

In [**2**], Bartoszek considered adapted probability measures on a countable group and proved that (1), (2) and (3) are equivalent. In general condition (1) need not be necessary for the existence of ρ satisfying ($\star$), that is (3) implies (1) need not be true (see [**3**]). We prove the equivalence of (1), (2) and (3) for adapted probability measures under certain conditions (see Theorem 1.1 and Theorem 1.2). We also prove a similar result for probability measures on semigroups (see Theorem 2.1). The sufficient condition for the existence of ρ satisfying ($\star$) that is, (2) implies (3), is proved in a more general set-up (see Proposition 1.1 and Proposition 2.2).

1. Probability Measures on Groups

Let X be a completely regular space. Let $P(X)$ be the space of all compact-regular Borel probability measures on X endowed with the weak* topology with respect to all bounded continuous real valued functions on X. We shall call X a *Prohorov space* if it satisfies the following: A subset $\mathcal{F}$ of $\mathcal{P}(X)$ is relatively compact if and only if for any $\varepsilon > 0$ there exists a compact set L of X such that $\mu(X \setminus L) \leqslant \varepsilon$ for all $\mu \in \mathcal{F}$. Complete separable metric spaces and locally compact spaces are Prohorov spaces (see [**10**] and [**8**, Theorem 1.1.11])

Let G be a topological group and λ be a probability measure on G. We define $\check{\lambda}$, the adjoint of λ by $\check{\lambda}(E) = \lambda(\{x \mid x^{-1} \in E\})$.

The following gives a sufficient condition for the existence of ρ.

Proposition 1.1. *Let G be a Prohorov topological group and S be a closed convex subsemigroup of $\mathcal{P}(G)$. Let λ and μ be in S. Suppose there exist a sequence (g_n) and compact sets L_1 and L_2 such that for some $\delta > 0$, $\mu^n(g_n^{-1}L_1) > \delta$ and $\check{\lambda}^n(g_n^{-1}L_2) > \delta$ for all n. Then there exists a probability measure $\rho \in S$ such that $\lambda * \rho * \mu = \rho$.*

Proof: It is clear that $\sup_{x\in G} \check{\lambda}^n(x^{-1}L_2) \not\to 0$ and $\sup_{x\in G} \mu^n(x^{-1}L_1) \not\to 0$. This implies that for any $\eta > \delta$ there exist compact sets C_η and L_η such that $\sup_{x\in G} \check{\lambda}^n(x^{-1}C_\eta) > \eta$ and $\sup_{s\in G} \mu^n(x^{-1}L_\eta) > \eta$ for all n. Thus, there exist sequences $(x_{n,\eta})$ and $(y_{n,\eta})$ such that $\check{\lambda}^n(x_{n,\eta}^{-1}C_\eta) > \eta$ and $\mu^n(y_{n,\eta}^{-1}L_\eta) > \eta$ for all n. This implies that $x_{n,\eta}^{-1}C_\eta \cap g_n^{-1}L_2 \neq \emptyset$ and $y_{n,\eta}^{-1}L_\eta \cap g_n^{-1}L_1 \neq \emptyset$ for all n and hence $x_{n,\eta}^{-1} \in g_n^{-1}L_2C_\eta^{-1}$ and $y_{n,\eta}^{-1} \in g_n^{-1}L_1L_\eta^{-1}$ for all n. Thus, $\check{\lambda}^n(g_n^{-1}L_2C_\eta^{-1}C_\eta) > \eta$ and $\mu^n(g_n^{-1}L_1L_\eta^{-1}L_\eta) > \eta$ for all n. Since G is Prohorov, $(g_n\mu^n)$ and $(g_n\check{\lambda}^n)$ are relatively compact and hence the sequence $(\lambda^n * \mu^n)$ is relatively compact. Then the sequence $(1/n)\Big(\sum_{k=1}^{n}(\lambda^k * \mu^k)\Big)$ is also relatively compact. Since S is convex, we have $(1/n)\Big(\sum_{k=1}^{n}(\lambda^k * \mu^k)\Big) \in S$. It is easy to see that

$$\left\|\frac{1}{n}\sum_{k=1}^{n}(\lambda^k * \mu^k) - \lambda * \frac{1}{n}\sum_{k=1}^{n}(\lambda^k * \mu^k) * \mu\right\| \to 0,$$

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where $\|\cdot\|$ is the total variation norm. Let ρ be a weak* limit point of $(1/n)\Big(\sum_{k=1}^{n} (\lambda^k * \mu^k)\Big)$. Then $\lambda * \rho * \mu = \rho$. Since S is closed, ρ is in S. This proves the proposition. □

LEMMA 1.1. *Let ν be an adapted probability measure on a noncompact locally compact group G. Suppose $\mathcal{H}(\nu)$ is compact. Then $\mathcal{H}(\nu)$ is the largest compact subgroup of G.*

PROOF: Suppose $\mathcal{H}(\nu)$ is compact. Then $G/\mathcal{H}(\nu)$ is discrete and isomorphic to $\mathbb{Z}$ (see [4, Proposition 1.6]). This implies that any compact subgroup K of G is contained in $\mathcal{H}(\nu)$ and hence $\mathcal{H}(\nu)$ is the largest compact subgroup of G. □

We now prove the following:

THEOREM 1.1. *Let G be a noncompact locally compact group. Let S be a closed convex commutative subsemigroup of $\mathcal{P}(G)$. Let μ and λ be in S. Suppose μ and λ are adapted probability measures on G. Then the following are equivalent:*

(1) *the subgroups $\mathcal{H}(\mu)$, $\mathcal{H}(\lambda)$ and $\mathcal{H}(\lambda, \mu)$ are all compact and the same and the measures μ and $\check{\lambda}$ are supported on $g\mathcal{H}(\mu)$ for all g in the support of μ, in particular, λ and μ are concentrated;*

(2) *there exist compact sets L_1 and L_2 and a sequence (g_n) in G such that for some $\delta > 0$, $\mu^n(g_n^{-1}L_1) > \delta$ and $\check{\lambda}^n(g_n^{-1}L_2) > \delta$ for all n;*

(3) *there exists $\rho \in S$ such that $\lambda * \rho * \mu = \rho$.*

PROOF: That (1) implies (2) is obvious because $\mu^n\big(g^n\mathcal{H}(\mu)\big) = 1$ and $\check{\lambda}^n(g^n\mathcal{H}(\mu)) = 1$. That (2) implies (3) follows from Proposition 1.1.

Now assume (3). Then since S is commutative, we have that $\mu * \lambda * \rho = \lambda * \mu * \rho = \rho$. Let $I(\rho) = \{g \in G \mid g\rho = \rho = \rho g\}$. Then $\lambda * \mu = \mu * \lambda$ is supported on $I(\rho)$ and $I(\rho)$ is a compact group (see [13]) and hence λ is supported on $g^{-1}I(\rho)$ and $I(\rho)g^{-1}$ for any g in the support of μ and μ is supported on $x^{-1}I(\rho)$ and $I(\rho)x^{-1}$ for any x in the support of λ. This implies that μ is supported on $gI(\rho)$ and $I(\rho)g$ for any g in the support of μ. Thus, for each n, $\mu^n * \check{\mu}^n$ and $\check{\mu}^n * \mu^n$ are supported on $I(\rho)$ and hence $\mathcal{H}(\mu) \subset I(\rho)$ (see [1]). Similarly we can prove that $\mathcal{H}(\lambda) \subset I(\rho)$. Thus, by Lemma 1.1, both $\mathcal{H}(\lambda)$ and $\mathcal{H}(\mu)$ are largest compact subgroups of G and hence $\mathcal{H}(\mu) = H(\lambda)$. Thus, $\mathcal{H}(\mu) = H(\lambda) = \mathcal{H}(\lambda, \mu)$ and $\check{\lambda}$ and μ are supported on the coset $g\mathcal{H}(\mu)$ for any g in the support of μ. □

REMARK 1.1. Let G be a connected real reductive Lie group and K be a maximal compact subgroup of G. Then the semigroup S of all K-biinvariant probability measures on G is a closed convex commutative semigroup and hence Theorem 1.1 holds. In this case condition (1) may be replaced by the following: there exists a $g \in G$ such that $\lambda = g\omega_K$ and $\mu = g^{-1}\omega_K$.

We say that a locally compact group G is a group of Deriennic and Lin type or G is in $\mathcal{G}_{DL}$ if it satisfies the following: For an adapted probability measure ν in $\mathcal{P}(G)$, either the concentration function $\sup_{x\in G} \nu^n(xC) \to 0$ for all compact subsets C of G or ν is supported on a coset of a compact normal subgroup of G. This class was introduced by Bartoszek in [**1**]. This class contains all nilpotent and Tortrat groups: a locally compact group G is called a *Tortrat group* if a sequence of the form $(x_n\lambda x_n^{-1})$ where $\lambda \in \mathcal{P}(G)$ and (x_n) is a sequence in G has an idempotent limit point only if λ is an idempotent (see [**5**] and [**7**] for more details on Tortrat groups). We now prove the following:

THEOREM 1.2. *Let G be a noncompact locally compact group G. Let λ, μ be adapted probability measures in $\mathcal{P}(G)$. Suppose G is in $\mathcal{G}_{DL}$ or λ and μ are normal (that is, $\lambda * \check\lambda = \check\lambda * \lambda$ and $\mu * \check\mu = \check\mu * \mu$). Then the following are equivalent.*

(1) *the subgroups $\mathcal{H}(\mu)$, $\mathcal{H}(\lambda)$ and $\mathcal{H}(\lambda, \mu)$ are all compact and the same and the measures μ and $\check\lambda$ are supported on $g\mathcal{H}(\mu)$ for all g in the support of μ, in particular, λ and μ are concentrated;*

(2) *there exist compact sets L_1 and L_2 and a sequence (g_n) in G such that for some $\delta > 0$, $\mu^n(g_n^{-1}L_1) > \delta$ and $\check\lambda^n(g_n^{-1}L_2) > \delta$ for all n;*

(3) *there exists a probability measure $\rho \in \mathcal{P}(G)$ such that $\lambda * \rho * \mu = \rho$.*

PROOF: It is enough to prove (3) $\Rightarrow$ (1). Suppose $\lambda * \rho * \mu = \rho$. Then there exist sequences (x_n) and (y_n) in G such that $(\lambda^n x_n)$ as well as $(y_n\mu^n)$ is relatively compact. We now claim that $\mathcal{H}(\lambda)$ and $\mathcal{H}(\mu)$ are compact and the same. Suppose $G \in \mathcal{G}_{DL}$. Since $(\lambda^n x_n)$ is relatively compact, the concentration function $\sup_{x\in G} \check\lambda^n(xC) \not\to 0$ for some compact subset C of G and hence $\mathcal{H}(\lambda)$ is compact. Suppose λ is normal. Since $(\lambda_n x_n)$ is relatively compact $(\lambda^n * \check\lambda^n)$ is relatively compact and hence by [**6**, Theorem 2.2] there exists a compact subgroup H of G such that λ is supported on H and $xH = Hx$ for all x in the support of λ. Since λ is adapted, H is normal and hence $\mathcal{H}(\lambda)$ is compact. Similarly we can prove that $\mathcal{H}(\mu)$ is compact. By Lemma 1.1, $\mathcal{H}(\lambda) = \mathcal{H}(\mu) = K$, say, and hence $\check\lambda$ and μ are supported on gK for any g in the support of μ. ∎

2. Probability Measures on Semigroups

Let G be a Hausdorff topological semigroup. Let A and B be subsets of G. Then $A^{-1}B$ and BA^{-1} denote the set of all $x \in G$ such that $ax \in B$ and $xa \in B$ for some $a \in A$ respectively. A Hausdorff semigroup G is said to satisfy the *compactness condition* if CL^{-1} and $L^{-1}C$ are compact whenever L and C are compact.

REMARK 2.1. We observe that the semigroup $\mathcal{P}(G)$ of regular Borel probability measures on a Prohorov topological group G is a Prohorov semigroup satisfying the compactness condition, which may be seen as follows: By [**14**, Theorem 1], $\mathcal{P}(G)$ is a Prohorov space and $\mathcal{P}(G)$ satisfies the compactness conditions follows from the fact that if two nets $(\lambda_i)_{i\in I}$ and $(\mu_i)_{i\in I}$ are relatively compact and there exists a net $(\nu_i)_{i\in I}$ such that $\mu_i * \nu_i = \lambda_i$, then $(\nu_i)_{i\in I}$ is relatively compact (see [**11**]).

PROPOSITION 2.1. *Let G be a Hausdorff semigroup satisfying the compactness condition. Let ρ be a probability measure in $\mathcal{P}(G)$. Then $J(\rho) = \{g \in G \mid g\rho = \rho\}$ and $I(\rho) = \{g \in G \mid gx\rho = x\rho$ for all x in the support of $\rho\}$ are compact.*

PROOF: This proposition may be proved by arguing along the lines of [**8**, Theorem 1.2.4]. ∎

The following gives a sufficient condition for the existence of ρ.

PROPOSITION 2.2. *Let G be a Hausdorff topological semigroup satisfying the compactness condition such that G has the Prohorov property. Let S be a closed convex subsemigroup of $\mathcal{P}(G)$. Let λ and μ be in S. Suppose there exist compact sets C_1 and C_2 such that for some $\delta > 0$, $\mu^n(C_1) > \delta$ and $\lambda^n(C_2) > \delta$ for all n. Then there exists a measure $\rho \in S$ such that $\lambda * \rho * \mu = \rho$.*

PROOF: Suppose there exist compact sets C_1 and C_2 such that for some $\delta > 0$, $\lambda^n(C_2) > \delta$ and $\mu^n(C_1) > \delta$ for all n. Then $\sup_{x \in G} \mu^n(x^{-1}C_1) \not\to 0$ and $\sup_{x \in G} \lambda^n(x^{-1}C_2) \not\to 0$ and hence for $\eta > \delta$, there exists compact sets C_η and L_η and sequences $(x_{n,\eta})$ and $(y_{n,\eta})$ elements of G such that $\mu^n(x_{n,\eta}^{-1}C_\eta) > \eta$ and $\lambda^n(y_{n,\eta}^{-1}L_\eta) > \eta$ for all n. This implies that $x_{n,\eta}^{-1}C_\eta \cap C_1 \neq \emptyset$ and $y_{n,\eta}^{-1}L_\eta \cap C_2 \neq \emptyset$ for all n and hence $x_{n,\eta} \in C_\eta C_1^{-1}$ and $y_{n,\eta} \in L_\eta C_2^{-1}$ for all n. Thus, $\mu^n\big((C_\eta C_1^{-1})^{-1}C_\eta\big) > \eta$ and $\lambda^n\big((L_\eta C_2^{-1})^{-1}L_\eta\big) > \eta$ for all n. This implies that (μ^n) and (λ^n) are relatively compact. As in [**9**, Theorem 2.13], we can prove that $(1/n)\sum_{k=1}^{n} \mu^k \to \rho_1 \in S$ and $(1/n)\sum_{k=1}^{n} \lambda^k \to \rho_2 \in S$ and $\mu * \rho_1 = \rho_1 = \rho_1 * \mu$ and $\lambda * \rho_2 = \rho_2 = \rho_2 * \lambda$. This implies that $\mu * \rho_1 * \rho_2 * \lambda = \rho_1 * \rho_2$ and $\lambda * \rho_2 * \rho_1 * \mu = \rho_2 * \rho_1$. This proves the proposition. ∎

The following may be viewed as an analogue of Theorem 1.1 for measures on commutative semigroups.

THEOREM 2.1. *Let G be a locally compact Hausdorff second countable topological semigroup or an Abelian Hausdorff topological semigroup satisfying the compactness condition. Let S be a closed convex commutative semigroup of probability measures on G. Let μ and λ be in S. Consider the following:*

(1) *there exists a compact subsemigroup C of G such that μ is supported on Cx^{-1} for any x in the support of λ and λ is supported on $y^{-1}C$ for any y in the support of μ;*

(2) *there exist compact sets C_1 and C_2 in G such that for some $\delta > 0$, $\mu^n(C_1) > \delta$ and $\lambda^n(C_2) > \delta$ for all n;*

(3) *there exists a probability measure $\rho \in S$ such that $\mu * \rho * \lambda = \rho$.*

Then (2) $\Rightarrow$ (3) $\Rightarrow$ (1) holds.

PROOF: That (2) implies (3) follows from Proposition 2.2. When G is Abelian that (3) implies (1) follows from [**12**, Proposition 2.1] and when G is a locally compact second countable semigroup that (3) implies (1) follows from Proposition 2.1 and [**9**, Theorem 2.5]. ∎

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