2
Lorentz transformations

The equations of the Standard Model must be consistent with Einstein's principle of relativity, which states that the laws of Nature take the same form in every inertial frame of reference. An inertial frame is one in which a free body moves without acceleration. An earth-bound frame approximates to an inertial frame if the gravitational field of the earth is introduced as an external field. We shall assume that the reader is familiar with rotations, and with proper Lorentz transformations and the relativistic mechanics of particle collisions. This chapter is very largely about notation, which may make for dry reading; however an appropriate notation is crucial to the exposition of any theory, and particularly so to a relativistic theory, such as the Standard Model.

2.1 Rotations, boosts and proper Lorentz transformations

The time and space coordinates of an event measured in different inertial frames of reference are related by a Lorentz transformation. A rotation is a special case of a Lorentz transformation. Consider, for example, a frame K' that is rotated about the z-axis with respect to a frame K, by an angle θ. If $(t, \mathbf{r})$ are the time and space coordinates of an event observed in K, then in K' the event is observed at $(t', \mathbf{r'})$ and

$$\begin{aligned} t' &= t \\ x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z. \end{aligned} \tag{2.1}$$

Lorentz transformations also relate events observed in frames of reference that are moving with constant velocity, one with respect to the other. Consider, for example, an inertial frame K' moving in the z-direction in a frame K with velocity v, the spatial axes of K and K' being coincident at $t = 0$. If $(t, \mathbf{r})$ are the time and

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space coordinates of an event observed in K, and $(t', \mathbf{r}')$ are the coordinates of the same event observed in K', the transformation takes the form

$$\begin{aligned} ct' &= \gamma(ct - \beta z) \\ x' &= x \\ y' &= y \\ z' &= \gamma(z - \beta ct), \end{aligned} \tag{2.2}$$

where c is the velocity of light, $\beta = v/c, \;\; \gamma = (1 - \beta^2)^{-1/2}$.

Putting $x^0 = ct, \; x^1 = x, \; x^2 = y, \; x^3 = z$, the x^μ are dimensionally homogeneous, and an event in K is specified by the set x^μ, where $\mu = 0, 1, 2, 3$. Greek indices in the text will in general take these values. With this more convenient notation, we may write the Lorentz transformation (2.2) as

$$\begin{aligned} x'^0 &= x^0 \cosh\theta - x^3 \sinh\theta \\ x'^1 &= x^1 \\ x'^2 &= x^2 \\ x'^3 &= -x^0 \sinh\theta + x^3 \cosh\theta, \end{aligned} \tag{2.3}$$

where we have put $\beta = v/c = \tanh\theta$; then $\gamma = \cosh\theta$.

Transformations to a frame with parallel axes but moving in an arbitrary direction are called *boosts*. A general Lorentz transformation between inertial frames K and K' whose origins coincide at $x^0 = x'^0 = 0$ is a combination of a rotation and a boost. It is specified by six parameters: three parameters to give the orientation of the K' axes relative to the K axes, and three parameters to give the components of the velocity of K' relative to K. Such a general transformation is of the form

$$x'^\mu = L^\mu{}_\nu x^\nu, \tag{2.4}$$

where the elements $L^\mu{}_\nu$ of the transformation matrix are real and dimensionless. We use here, and subsequently, the *Einstein summation convention*: a repeated 'dummy' index is understood to be summed over, so that in (2.4) the notation $\sum_{\nu=0}^{3}$ has been omitted on the right-hand side. The matrices $L^\mu{}_\nu$ form a group, called the *proper Lorentz group* (Problem 2.6 and Appendix B). The significance of the placing of the superscript and the subscript will become evident shortly.

The *interval* $(\Delta s)^2$ between events x^μ and $x^\mu + \Delta x^\mu$ is defined to be

$$(\Delta s)^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2. \tag{2.5}$$

It is a fundamental property of a Lorentz transformation that it leaves the interval between two events invariant:

$$(\Delta s')^2 = (\Delta s)^2. \tag{2.6}$$

We can express $(\Delta s)^2$ more compactly by introducing the *metric tensor* $(g_{\mu\nu})$:

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.7}$$

Then

$$(\Delta s)^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu, \tag{2.8}$$

where the repeated upper and lower indices are summed over. Note that $g_{\mu\nu} = g_{\nu\mu}$; it is a symmetric tensor. It has the same elements in every frame of reference.

2.2 Scalars, contravariant and covariant four-vectors

Quantities, such as $(\Delta s)^2$, which are invariant under Lorentz transformations are called *scalars*. We define a *contravariant four-vector* to be a set a^μ which transforms like the set x^μ under a proper Lorentz transformation:

$$a'^\mu = L^\mu{}_\nu a^\nu. \tag{2.9}$$

A familiar example of a contravariant four-vector is the energy–momentum vector of a particle $(E/c, \mathbf{p})$.

We define the corresponding *covariant four-vector* a_μ, carrying a subscript, rather than a superscript, by

$$a_\mu = g_{\mu\nu}a^\nu. \tag{2.10}$$

Hence if $a^\mu = (a^0, \mathbf{a})$, then $a_\mu = (a^0, -\mathbf{a})$.

We can write the invariant Δs^2 as

$$\Delta s^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu = \Delta x_\nu \Delta x^\nu.$$

More generally, if a^μ, b^μ are contravariant four-vectors, the *scalar product*

$$g_{\mu\nu}a^\mu b^\nu = a_\mu b^\mu = a^\mu b_\mu = a^0 b^0 - \mathbf{a}\cdot\mathbf{b} \tag{2.11}$$

is invariant under a Lorentz transformation.

We can define the contravariant metric tensor $g^{\mu\nu}$ so that

$$\alpha^\mu = g^{\mu\nu}a_\nu. \tag{2.12}$$

The elements of $g^{\mu\nu}$ are evidently identical to those of $g_{\mu\nu}$.

The transformation law for covariant vectors, which we write

$$a'_\mu = L_\mu{}^\nu a_\nu, \tag{2.13}$$

follows from that for contravariant vectors (Problem 2.1). Note that, in general, ${L_\mu}^\nu$ is not equal to ${L_\nu}^\mu$ (Problem 2.1). Using the invariance of the scalar product (2.11), we have

$$a'_\mu b'^\mu = {L_\mu}^\nu {L^\mu}_\rho a_\nu b^p = a_\nu b^\nu$$

and

$$a'^\mu b'_\mu = {L^\mu}_\nu {L_\mu}^\rho a^\nu b_\rho = a^\nu b_\nu .$$

Since the a_μ and b_μ are arbitrary, it follows that

$${L^\mu}_\nu {L_\mu}^\rho = {L_\mu}^\nu {L^\mu}_\rho = \delta^\rho_\nu \tag{2.14}$$

where

$$\delta^\rho_\nu = \delta^\nu_\rho = \begin{cases} 1, & \rho = \nu \\ 0, & \rho \neq \nu. \end{cases}$$

2.3 Fields

The Standard Model is a theory of fields. We shall be concerned with fields that at each point x of space and time transform as scalars, or vectors, or tensors (defined later in this section). We use x to stand for the set (x^0, x^1, x^2, x^3). For example, we shall see that the electromagnetic potentials form a four-vector field, and the electromagnetic field is a tensor field. We shall also be concerned with scalar fields $\phi(x)$, which by definition transform simply as

$$\phi'(x') = \phi(x), \tag{2.15}$$

where x' and x refer to the same point in space-time.

We can construct a vector field from a scalar field. Consider the change of field $\mathrm{d}\phi$ in moving from x to a neighbouring point $x + \mathrm{d}x$, with $\mathrm{d}x$ infinitesimal. Then

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x^\mu} \mathrm{d}x^\mu$$

is invariant under a Lorentz transformation. Since the set $\mathrm{d}x^\mu$ make up an arbitrary contravariant infinitesimal vector, the set $\partial\phi/\partial x^\mu$ must make up a covariant vector (Problem 2.3). Following the subscript convention we write

$$\frac{\partial \phi}{\partial x^\mu} = \left(\frac{1}{c} \frac{\partial \phi}{\partial t}, \nabla \phi \right) = \partial_\mu \phi. \tag{2.16}$$

We can then also define the contravariant vector

$$\partial^\mu \phi = g^{\mu\nu} \partial_\nu \phi = \frac{\partial \phi}{\partial x_\mu} = \left(\frac{1}{c} \frac{\partial \phi}{\partial t}, -\nabla \phi \right). \tag{2.17}$$

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It follows that

$$\partial_\mu\phi\partial^\mu\phi = \left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2 - (\nabla\phi)^2 \tag{2.18}$$

and

$$\partial_\mu\partial^\mu\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi \tag{2.19}$$

are invariant under Lorentz transformations.

We can define, and we shall need, tensor quantities. Tensors $T^{\mu\nu}$, $T_{\mu\nu}$, $T^\mu{}_\nu$, $T^{\mu\nu}{}_\lambda$, etc., are defined as quantities which transform under a Lorentz transformation in the same way as $a^\mu a^\nu$, $a_\mu a_\nu$, $a^\mu a_\nu$, $a^\mu a^\nu a_\lambda$, etc. For example,

$$T'^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\lambda T^{\rho\lambda}.$$

The 'contraction' by summation of a repeated upper and lower index leaves the transformation properties determined by what remains. For example, $T^\mu{}_\mu$ is a scalar, $T^{\mu\nu}{}_\mu$ is a contravariant four-vector. The metric tensors $g_{\mu\nu}$, $g^{\mu\nu}$ conform with the definition, and this leads to the conditions on the matrix elements $L^\mu{}_\nu$:

$$g_{\mu\nu} = g_{p\lambda} L^\rho{}_\mu L^\lambda{}_\nu. \tag{2.20}$$

The conditions (2.20) and (2.14) are equivalent.

As well as scalars, vectors and tensors there are also very important objects called *spinors*, and *spinors fields*, which have well-defined rules of transformation under a Lorentz transformation of the coordinates. Their properties are discussed in Appendix B and Chapter 5.

2.4 The Levi–Civita tensor

The Levi–Civita tensor $\varepsilon_{\mu\nu\lambda\rho}$ is defined by

$$\varepsilon_{\mu\nu\lambda\rho} = \begin{cases} +1 & \text{if } \mu, \nu, \lambda, \rho \text{ is an even permutation of } 0, 1, 2, 3; \\ -1 & \text{if } \mu, \nu, \lambda, \rho \text{ is an odd permutation of } 0, 1, 2, 3; \\ 0 & \text{otherwise.} \end{cases} \tag{2.21}$$

For example, $\varepsilon_{1023} = -1$, $\varepsilon_{1203} = +1$, $\varepsilon_{0023} = 0$.

It is straightforward to verify that $\varepsilon_{\mu\nu\lambda\rho}$ satisfies

$$\begin{aligned} \varepsilon'_{\mu\nu\lambda\rho} &= L_\mu{}^\alpha L_\nu{}^\beta L_\lambda{}^\gamma L_\rho{}^\delta \varepsilon_{\alpha\beta\gamma\delta} \\ &= \varepsilon_{\mu\nu\lambda\mu} \det(L) = \varepsilon_{\mu\nu\lambda\mu}, \end{aligned}$$

using the definition of a determinant (Appendix A), and the result that the determinant of the transformation matrix is 1 (Problems 2.4 and 2.5).

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The corresponding Levi–Civita symbol in three dimensions, ε_{ijk}, is defined similarly. It is useful in the construction of volumes, since

$$\varepsilon_{ijk}A^i B^j C^k = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

is the volume of the parallelepiped defined by the vectors **A, B, C**. The four-dimensional Levi–Civita tensor enables one to construct four-dimensional volumes $\varepsilon_{\mu\nu\lambda\rho}a^\mu b^\nu c^\lambda d^\rho$. The contraction of indices leaves this a Lorentz scalar. In particular, taking *a,b,c,d* to be infinitesimal elements parallel to the axes $0x^\mu$ so that $a = (\mathrm{d}x^0, 0, 0, 0)$, $b = (0, \mathrm{d}x^1, 0, 0)$, $c = (0, 0, \mathrm{d}x^2, 0)$, $d = (0, 0, 0, \mathrm{d}x^3)$, it follows that the 'volume' element of space-time

$$\mathrm{d}^4x = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = cd^3x\ \mathrm{d}t$$

is a Lorentz invariant scalar (see also Problem 2.9).

2.5 Time reversal and space inversion

The operations of time reversal:

$$\begin{aligned} x'^0 &= -x^0, \\ x'^i &= x^i, \quad i = 1, 2, 3, \end{aligned}$$

and space inversion:

$$\begin{aligned} x'^0 &= x^0 \\ x'^i &= -x^i, \quad i = 1, 2, 3, \end{aligned}$$

also leave $(\Delta s)^2$ invariant, but these transformations are excluded from the proper Lorentz group. They are however of interest, and will arise in later chapters.

Problems

2.1 Show that $L_\mu{}^\nu = g_{\mu\rho}L^\rho{}_\lambda g^{\lambda\nu}$. Verify $L_0{}^1 = -L^1{}_0$.

2.2 Using (2.14), show that the inverse transformations to (2.9) and (2.13) are

$$a^\mu = a'^\nu L_\nu{}^\mu,\ \ a_\mu = a'_\nu L^\nu{}_\mu.$$

Hence show

$$L_\nu{}^\mu L^\rho{}_\mu = \delta^\rho_\nu.$$

2.3 Prove that if $\phi(x)$ is a scalar field, the set $(\partial\phi/\partial x^\mu)$ makes up a covariant vector field.

2.4 Using Problem 2.1, show that $\det(L^{\mu}{}_{\nu}) = \det(L_{\mu}{}^{\nu})$ and hence show, using equation (2.14), that

$$\det(L^{\mu}{}_{\nu}) = \pm 1.$$

2.5 Show that $\det(L^{\mu}{}_{\nu})$ for both the rotation (2.1) and the boost (2.3) is equal to $+1$. This is a general property of proper Lorentz transformations that distinguishes them from space reflections and time reversal (Section 2.5), for which the determinant of the transformation equals -1.

2.6 Show that the matrices $L_{\mu}{}^{\nu}$ corresponding to proper Lorentz transformations form a group.

2.7 Show that δ^{μ}_{ν} is a tensor.

2.8 The frequency ω and wave vector $\mathbf{k}$ of an electromagnetic wave in free space make up a contravariant four-vector

$$k = (\omega/c, \mathbf{k}).$$

The invariant $k_{\mu}k^{\mu} = 0$; this corresponds to the dispersion relation $\omega^2 = c^2\mathbf{k}^2$. Show that a wave propagating with frequency ω in the z-direction, if viewed from a frame moving along the z-axis with velocity v, is seen to be Doppler shifted in frequency, with

$$\omega' = \mathrm{e}^{-\theta}\omega = \sqrt{\frac{1 - v/c}{1 + v/c}}\,\omega.$$

2.9 By considering the Jacobian of the Lorentz transformation, show that the four-dimensional volume element $\mathrm{d}^4x = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$ is a Lorentz invariant.

2.10 Show that $\varepsilon_{\mu\nu\lambda\rho}$ is a *pseudo-tensor*, i.e. it changes sign under the operation of space inversion.