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TERNARY QUADRATIC FORMS AND BRANDT MATRICES

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Introduction

In a recent paper [9] the author showed (among other results) estimates on the asymptotic behaviour of the representation numbers of positive definite integral ternary quadratic forms, in particular, that for n in a fixed square class tZ^2 and lattices L, K in the same spinor genus one has $r(n, L) = r(n, K) + O(n^{1/4+\varepsilon})$. The main tool utilized for the proof was the theory of modular forms of weight 3/2, especially Shimura's lifting from the space of cusp forms of weight 3/2 to the space of modular forms of weight 2.

It is the purpose of this note to show that the aforementioned estimate can be obtained without explicitly using Shimura's lifting. Instead, we employ Eichler's Anzahlmatrices. We prove that they are essentially the same as the (reduced) Brandt matrices (a result that has been demonstrated by Ponomarev in special cases [6]) and that the difference of two rows of the (reduced) Brandt matrix series belonging to lattices in the same spinor genus consists of cusp forms. From this the estimates on the asymptotic behaviour of the representation numbers can easily be deduced. The methods used allow us to state and prove our results for totally definite forms over the integers of an arbitrary totally real number field.

Although Shimura's lifting does not appear explicitly in the proof given here, there are close connections. In fact, multiplying the vector $(r(t, L_1), \cdots, r(t, L_h))$ with the reduced Brandt matrix series defines a lifting for the $\theta(z, L_i)$ that is essentially the same as Shimura's lifting and coincides with it in special cases [6].

§ 1. Preliminaries

Let V be a 3-dimensional vector space over the totally real number field F of degree d over $\boldsymbol{Q}$, $\mathfrak{o}$ the ring of integers of F, Σ the set of prime

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spots $\mathfrak{p}$ of F, π a prime element in the completion $\mathfrak{o}_\mathfrak{p}$. Let q be a quadratic form on V with associated bilinear form $B(x, y) = q(x + y) - q(x) - q(y)$ which is positive definite at all infinite spots of F, L a lattice on V with $q(L) \subset \mathfrak{o}$, $\mathfrak{d}(L)$ the reduced determinant ([3], § 12) of L. $C^+(V)$ is the second Clifford algebra of V, by $C^+(L)$ we denote the order of $C^+(V)$ associated to L (see [1], Satz 14.1). Let $L_1 = L, L_2, \cdots, L_h$ be a set of representatives of the classes in the genus of L, $\mathfrak{O}_1, \cdots, \mathfrak{O}_h$ the associated orders in $C^+(V)$. $\mathfrak{O}_1, \cdots, \mathfrak{O}_h$ then is a set of representatives of the types of orders in $C^+(V)$ which are locally everywhere conjugate to $C^+(L)$. A left $\mathfrak{O}$-ideal $\mathfrak{J}$ for an order $\mathfrak{O}$ of $C^+(V)$ is a lattice $\mathfrak{J}$ on $C^+(V)$ with $\mathfrak{O} = \{A \in C^+(V) \,|\, A\mathfrak{J} \subsetneqq \mathfrak{J}\}$ ($\mathfrak{O}$ is the left order of $\mathfrak{J}$) and $\mathfrak{J}_\mathfrak{p} = \mathfrak{O}_\mathfrak{p} A_\mathfrak{p}$ with some $A_\mathfrak{p} \in C^+(V)_\mathfrak{p}$ for all $\mathfrak{p} \in \Sigma$, i.e., $\mathfrak{J} = C^+(V) \cap \mathfrak{O}_A A$ for some $A \in C^+(V)_A^x$ (the subscript A denoting adelization). The normalizer $\mathfrak{N}(C^+(L))$ is by Satz 14.2 of [1] the same as the set of all $A \in C^+(V)^x$ for which the map $x \mapsto A^{-1}xA$ is a unit of L.

§ 2. Brandt matrix and Anzahlmatrix

Let $\mathfrak{J}_{11}, \cdots, \mathfrak{J}_{1r_1}, \cdots, \mathfrak{J}_{h1}, \cdots, \mathfrak{J}_{hr_h}$ be a set of representatives of the classes of left $\mathfrak{O}_1$-ideals where $\mathfrak{J}_{ij}$ has right order of the type of $\mathfrak{O}_i$. As usual [4] for an integral $\mathfrak{o}$-ideal $\mathfrak{n}$ the Brandt matrix $B(\mathfrak{n})$ is the quadratic matrix of $r_1 + \cdots + r_h$ rows with entry $b_{ij,kl}(\mathfrak{n})$ equal to the number of integral left $\mathfrak{O}_i$-ideals of norm $\mathfrak{n}$ equivalent to $\mathfrak{J}_{ij}^{-1}\mathfrak{J}_{kl}$. An equivalent definition of the $b_{ij,kl}(\mathfrak{n})$ is the following: Let $e_k = (\mathfrak{O}_k^x : \mathfrak{o}^x)$, let $n_1, \cdots, n_s$ be a set of representatives of the totally positive numbers n with $nN(\mathfrak{J}_{ij}^{-1}\mathfrak{J}_{kl}) = \mathfrak{n}$ modulo squares of units in $\mathfrak{o}^x$ (if such an n exists). Then

$$b_{ij,kl}(\mathfrak{n}) = \frac{1}{e_k} \sum_{\nu=1}^{s} r(n_\nu, \mathfrak{J}_{kl}^{-1}\mathfrak{J}_{ij})$$

where $r(n_\nu, \mathfrak{J}_{kl}^{-1}\mathfrak{J}_{ij})$ denotes the number of representations of n by the lattice $\mathfrak{J}_{kl}^{-1}\mathfrak{J}_{ij}$ equipped with the reduced norm as quadratic form.

Let $\mathfrak{c}_1, \cdots, \mathfrak{c}_g$ be a set of integral representatives of the ideal classes of $\mathfrak{o}$, denote by $\mathfrak{c}_{ij,kl}$ that ideal in this set of representatives belonging to the class of $N(\mathfrak{J}_{ij}^{-1}\mathfrak{J}_{kl})$ and by $c_{ij,kl}^{(1)}, \cdots, c_{ij,kl}^{(s)}$ a set of representatives of the totally positive numbers c with $(c) = \mathfrak{c}_{ij,kl} N(\mathfrak{J}_{kl}^{-1}\mathfrak{J}_{ij})$ modulo squares of units in $\mathfrak{o}$. Finally, let $\sigma_1, \cdots, \sigma_d$ be the embeddings of F into $\boldsymbol{R}$. We define the Brandt matrix series

$$\theta_{ij,kl}(z_1, \cdots, z_d) = \frac{s}{e_k} + \sum_{\substack{n \gg 0 \\ n \in \mathfrak{c}_{ij,kl}^{-1}}} b_{ij,kl}(n \cdot \mathfrak{c}_{ij,kl}) \exp(2\pi i \operatorname{tr}(nz))$$

and have

$$\theta_{ij,kl}(z_1, \cdots, z_d) = \frac{1}{e_k} \cdot \sum_{\nu=1}^{s} \theta(\mathfrak{J}_{kl}^{-1}\mathfrak{J}_{ij}, \sigma_1(c_{ij,kl}^{(\nu)})^{-1}z_1, \cdots, \sigma_d(c_{ij,kl}^{(\nu)})^{-1}z_d) \tag{1}$$

where $\theta(\mathfrak{J}_{kl}^{-1}\mathfrak{J}_{ij}, z_1, \cdots, z_d)$ is the theta series of the lattice $\mathfrak{J}_{kl}^{-1}\mathfrak{J}_{ij}$ with the reduced norm and $n \gg 0$ means n totally positive. $\theta_{ij,kl}(z_1, \cdots, z_d)$ therefore is a (Hilbert) modular form of weight 2. As in [6] we define the reduced Brandt matrix $\bar{B}(\mathfrak{n})$ to be the $(h \times h)$-matrix with entry

$$\bar{b}_{ik}(\mathfrak{n}) = \sum_{l=1}^{r_k} b_{ij,kl}(\mathfrak{n})$$

where j $(1 \leq j \leq r_i)$ is arbitrary.

Equivalently, $\bar{b}_{ik}(\mathfrak{n})$ is the number of integral left $\mathfrak{O}_i$-ideals of norm $\mathfrak{n}$ with right order of type $\mathfrak{O}_k$.

Finally, the primitive Anzahlmatrix $P_0(\mathfrak{n}^2)$ is the $(h \times h)$-matrix with entry $\pi_{ik}^{(0)}(\mathfrak{n}^2)$ equal to the number of lattices isomorphic to L_k and contained in $\mathfrak{n}^{-1}L_i$ but not in $\mathfrak{m}^{-1}L_i$ for any proper divisor $\mathfrak{m}$ of $\mathfrak{n}$ and $P(\mathfrak{n}^2) = (\pi_{ik}(\mathfrak{n}^2))$ is the sum of the $P_0(\mathfrak{m}^2)$ with $\mathfrak{m} \mid \mathfrak{n}$ and $\mathfrak{m}^{-1}\mathfrak{n}$ the square of an ideal. As in [8] let $Z_\mathfrak{p}(L)$ for $\mathfrak{p} \nmid 2^{-1}\mathfrak{d}$ be the graph whose vertices are the lattices K on V with $K_\mathfrak{q} = L_\mathfrak{q}$ for $\mathfrak{q} \in \Sigma - \{\mathfrak{p}\}$ and $K_\mathfrak{p} \cong L_\mathfrak{p}$ and in which two vertices K, K' are joined by an edge if $K \subseteq \mathfrak{p}^{-1}K'$, $K' \subseteq \mathfrak{p}^{+1}K$, and $K \neq K'$. Then $\pi_{ij}(\mathfrak{p}^2)$ is the number of neighbours of L_i in $Z_\mathfrak{p}(L_i)$ that are isomorphic to L_j. We have

LEMMA 1. *If $\mathfrak{n}$ is prime to $\mathfrak{d}$ then $\pi_{ik}(\mathfrak{n}^2) = \bar{b}_{ik}(\mathfrak{n})$.*

Proof. By the definitions, $\pi_{ik}(\mathfrak{n}^2)$ is the number of classes of adeles $A \in (\mathfrak{O}_i)_A \cap \mathfrak{N}(\mathfrak{O}_i)_A A_{ik} C^+(V)^x$ with $N(A)\mathfrak{o} = \mathfrak{n}$ where $A_{ik}^{-1}L_i A_{ik} = L_k$ and A, B are in the same class if $A \in \mathfrak{N}(\mathfrak{O}_i)_A B$. On the other hand, $\bar{b}_{ik}(\mathfrak{n})$ is the number of classes of adeles $A \in (\mathfrak{O}_i)_A \cap \mathfrak{N}(\mathfrak{O}_i)_A A_{ik} C^+(V)^x$ with $N(A)\mathfrak{o} = \mathfrak{n}$ where now A and B are in the same class if $A \in (\mathfrak{O}_i)_A^x B$. Now let A and B be in the same class under the first equivalence relation. Then for $\mathfrak{p} \mid \mathfrak{d}$ evidently $A_\mathfrak{p}, B_\mathfrak{p} \in (\mathfrak{O}_i)_\mathfrak{p}^x$ while for $\mathfrak{p} \nmid \mathfrak{d}$ one has $\mathfrak{N}(\mathfrak{O}_i)_\mathfrak{p} = F_\mathfrak{p}^x(\mathfrak{O}_i)_\mathfrak{p}^x$ and thus elements of $\mathfrak{N}(\mathfrak{O}_i)_\mathfrak{p}$ the norm of which is a unit are in $(\mathfrak{O}_i)_\mathfrak{p}^x$. A and B therefore are in the same class under the second equivalence relation, which proves the lemma. The following proposition is crucial for the rest of this paper.

PROPOSITION. *Let $L_i, L_{i'}$ be in the same spinor genus. For $1 \leq \mu \leq g$ put*

$$\theta^{\mu}_{ik}(z_1, \cdots, z_d) = \frac{s \cdot t_k}{e_k} + \sum_{\substack{n>>0 \\ n \in \mathfrak{c}_\mu^{-1}}} \bar{b}_{ik}(n \cdot \mathfrak{c}_\mu) \exp(2\pi i \operatorname{tr}(nz)) = \sum_l \theta_{ij,kl}(z_1, \cdots, z_d)$$

where j is arbitrary, the second sum is extended over those l $(1 \le l \le r_k)$ with $N(\mathfrak{J}_{ij}^{-1}\mathfrak{J}_{kl})$ in the ideal class of $\mathfrak{c}_\mu$ and t_k is the number of such l. Then $\theta^{\mu}_{ik}(z_1, \cdots, z_d) - \theta^{\mu}_{i'k}(z_1, \cdots, z_d)$ is a cusp form.

Proof. By definition of the spinor genus we may assume without loss of generality that

$$A_{ii'} \in \operatorname{Spin}(V)_A = \{A \in C^+(V)^x_A \mid N(A_\mathfrak{p}) = 1 \text{ for all } \mathfrak{p} \in \Sigma\}.$$

For $l = 1, \cdots, r_k$ the ideals $A_{ii'}^{-1}\mathfrak{J}_{i1}^{-1}\mathfrak{J}_{kl}$ run through a set of representatives of the ideals with left order $\mathfrak{O}_{i'}$ and right order of the type of $\mathfrak{O}_k$, i.e., there is a permutation τ of $\{1, \cdots, r_k\}$ with

$$A_{ii'}^{-1}\mathfrak{J}_{i1}^{-1}\mathfrak{J}_{kl} = \mathfrak{J}_{i'1}^{-1}\mathfrak{J}_{k\tau(l)}A_l$$

for certain $A_l \in C^+(V)^x$.

Since $A_{ii'} \in \operatorname{Spin}(V)_A$, the map $x \to x(A_{ii'})_\mathfrak{p}$ is an isometry from $(\mathfrak{J}_{kl}^{-1}\mathfrak{J}_{i1})_\mathfrak{p}$ onto $(A_l^{-1}\mathfrak{J}_{k\tau(l)}^{-1}\mathfrak{J}_{i'1})_\mathfrak{p}$, i.e., the lattices $\mathfrak{J}_{kl}^{-1}\mathfrak{J}_{i1}$ and $A_l^{-1}\mathfrak{J}_{k\tau(l)}^{-1}\mathfrak{J}_{i'1}$ on $C^+(V)$ belong to the same genus. Since the difference of the theta series of lattices in the same genus is a cusp form and since the definition of $\theta_{ij,kl}$ depends only on the class of the ideal $\mathfrak{J}_{ij}^{-1}\mathfrak{J}_{kl}$, the assertion follows. As an application of the proposition we give the following corollary:

COROLLARY. *Lel $\mathfrak{p} \nmid 2^{-1}\mathfrak{d}$, $1 \le i, j \le h$ be such that there are neighbours of L_i in $Z_\mathfrak{p}(L)$ belonging to the spinor genus of L_j. Then all neighbours of L_i in $Z_\mathfrak{p}(L)$ belong to the spinor genus of L_j and with $o(L_j) = \sharp O(L_j)$ one has*

$$\pi_{ji}(\mathfrak{p}^2) = \frac{N^F_Q(\mathfrak{p}) + 1}{o(L_i)} \left(\sum_k \frac{1}{o(L_k)}\right)^{-1} + O((N^F_Q \mathfrak{p})^\alpha)$$

where the sum is taken over those k with L_k in the spinor genus of L_j and α is such that $|a_m| = O(N^F_Q(m)^\alpha)$ holds for the m-th Fourier coefficient of a Hilbert cusp form of weight 2 over F.

Remark. For $F = \boldsymbol{Q}$ one can choose $\alpha = 1/2 + \varepsilon$ by Eichler's proof of the generalized Ramanujan-Petersson-conjecture [2]. For general F it appears that Gundlach's [5] estimate $\alpha = 7/8 + \varepsilon$ is still best available.

Proof. It is easily seen that any two neighbours of L_i in $Z_\mathfrak{p}(L)$ belong to the same spinor genus. By the proposition this implies

$$\pi_{ji}(\mathfrak{p}^2) = \pi_{ki}(\mathfrak{p}^2) + O((N_Q^F \mathfrak{p})^\alpha) \qquad (1 \leq j, k \leq h).$$

Since one has

$$o(L_j)\pi_{ij}(\mathfrak{p}^2) = o(L_i)\pi_{ji}(\mathfrak{p}^2)$$

([9]), formula 1 of § 3) and

$$\sum_{j=1}^{n} \pi_{ij}(\mathfrak{p}^2) = N_Q^F(\mathfrak{p}) + 1$$

([8]) Satz 1 and Bemerkung 2), the assertion follows.

Remark 2. The corollary may be applied to replace the somewhat ad hoc proof of Satz 4 in [9] by a more natural one: The equation

$$\sum_{K \in \operatorname{gen} L} c_p(L, k)\pi_t(\vartheta(K, z) - \vartheta(L, z)) = 0 \qquad \text{for all } t$$

(p. 294 of [9]) transforms on using the assertion of the corollary (with p large enough and such that all lattices in $Z_p(L)$ belong to the same spinor genus) into

$$\pi_t(\vartheta(\operatorname{spn} L, z) - \vartheta(L, z)) = 0 \qquad \text{for all } t,$$

which is just another form of the assertion of Satz 4.

§ 3. Representation numbers of ternary quadratic forms

Before we can prove our main result we need a couple of auxiliary lemmas. We recall the definition of $Z_\mathfrak{p}(L)$ from Section 2 and denote by $\underline{r}(t, \operatorname{gen} L)$ the vector $(r(t, L_1), \cdots, r(t, L_h))$. Finally let $E_t = F(\sqrt{-2t \det V})$.

LEMMA 2. *Let $\mathfrak{p} \in \Sigma$ be odd, $\mathfrak{p} \nmid \mathfrak{d}$, $\mu \in N$.*
Then

$$(2) \quad \underline{r}(t, \operatorname{gen}(\mathfrak{p}^{-\mu}L))$$
$$= \bar{B}(\mathfrak{p}^\mu)\underline{r}(t, \operatorname{gen} L) - \begin{cases} \left(\dfrac{E_t/F}{\mathfrak{p}}\right)\bar{B}(\mathfrak{p}^{\mu-1})\underline{r}(t, \operatorname{gen} L) & \text{if } t\mathfrak{o}_\mathfrak{p} = q(L_\mathfrak{p})\mathfrak{o}_\mathfrak{p} \\ 0 & \text{if } t\mathfrak{o}_\mathfrak{p} = q(L_\mathfrak{p})\mathfrak{p} \\ N_Q^F(\mathfrak{p})\bar{B}(p^{\mu-1})\underline{r}(t, \operatorname{gen} L) & \text{if } t\mathfrak{o}_\mathfrak{p} \subseteq q(L_\mathfrak{p})\mathfrak{p}^2. \end{cases}$$

If $\mathfrak{p}$ is dyadic, $\mathfrak{p} \nmid 2^{-1}\mathfrak{d}$, the results of (2) *for the case $t\mathfrak{o}_\mathfrak{p} \subseteq q(L_\mathfrak{p})\mathfrak{p}$ still hold.*

Proof. This is equivalent to the fact that the effect of the Hecke operator $T(\mathfrak{p}^2)$ on the vector $(\theta(L_1, z), \cdots, \theta(L_h, z))$ is given by multiplying this vector by $\bar{B}(\mathfrak{p})$ (also known as Eichler's commutation relation), and follows for $\mu = 1$ and odd $\mathfrak{p}$ from the results given in formula (11.19) of [1]. For proofs see [6], [7], [8]. For dyadic $\mathfrak{p}$ the results given in Satz 2 of [8] are proved only for unramified $F_\mathfrak{p}/\boldsymbol{Q}_2$. However, the proof goes through for arbitrary dyadic $\mathfrak{p}$ if one restricts attention to t with $t\mathfrak{o}_\mathfrak{p} \subseteq q(L_\mathfrak{p})\mathfrak{p}$. The statement of the lemma for arbitrary μ follows by induction on using

$$\bar{B}(\mathfrak{p}^{\mu+1}) = \bar{B}(\mathfrak{p})\bar{B}(\mathfrak{p}^{\mu}) - N^F_{\boldsymbol{Q}}(\mathfrak{p})\bar{B}(\mathfrak{p}^{\mu-1})$$

(see [4], Theorem 5. The additional factor $A(\mathfrak{p})$ occuring in the formula given there becomes identity on passing from the $B(\mathfrak{p}^{\mu})$ to the $\bar{B}(\mathfrak{p}^{\mu})$).

For the remaining finitely many primes we restrict our attention to t with $\operatorname{ord}_\mathfrak{p} t$ sufficiently large (as we have already done for the dyadic primes in Lemma 1).

LEMMA 3. *Let $\mathfrak{p} \in \Sigma - \infty$. Then there are $\lambda_\mathfrak{p}(L) \in \boldsymbol{N}$ and a lattice L' on V with $L'_\mathfrak{q} = L_\mathfrak{q}$ for $\mathfrak{q} \in \Sigma - \{\mathfrak{p}\}$ such that for $\operatorname{ord}_\mathfrak{p} t \geq \lambda_\mathfrak{p}(L)$ one has $r(t, L) = r(t, L')$ and either*

(i) *$V_\mathfrak{p}$ is anisotropic and $L'_\mathfrak{p}$ is a $\mathfrak{p}^{\lambda_\mathfrak{p}}$-maximal lattice on $V_\mathfrak{p}$ or*

(ii) *$V_\mathfrak{p}$ is isotropic and $L'_\mathfrak{p}$ is similar to $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \perp \langle \pi^s \rangle$ for some $s \in \boldsymbol{N}$.*

Furthermore, for any $\varphi \in O(V_\mathfrak{p})$ one has $(\varphi L_\mathfrak{p})' = \varphi L'_\mathfrak{p}$.

Proof. When $V_\mathfrak{p}$ is anisotropic, we have only to take any maximal lattice contained in $L_\mathfrak{p}$. When $V_\mathfrak{p}$ is isotropic, we take any sublattice of $L_\mathfrak{p}$ which is similar to $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \perp \langle \pi^s \rangle$ and maximal with respect to the inclusion.

LEMMA 4. *Let $\mathfrak{p} \in \Sigma - \infty$, $L_\mathfrak{p} \cong \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \perp \langle \pi^s \rangle$ $(s \in \boldsymbol{N})$.*

Then there are exactly $s + 1$ lattices $M_0, \cdots, M_s$ on V for which $(M_i)_\mathfrak{p}$ is $\mathfrak{p}^s$-maximal and contained in $L_\mathfrak{p}$ and $(M_i)_\mathfrak{q} = L_\mathfrak{q}$ for $\mathfrak{q} \in \Sigma - \{\mathfrak{p}\}$.

In the graph $Z_\mathfrak{p}(M_i)$ they form a chain of length ($=$ distance of the endpoints) equal to s.

If $\varphi \in O_A(V)$ then the chain corresponding to φL is $\varphi M_0, \cdots, \varphi M_s$. For $\operatorname{ord}_\mathfrak{p} t \geq s$ one has

$$r(t, L) = \sum_{i=0}^{s} r(t, M_i) - \sum_{i=0}^{s-1} r(t, M_i \cap M_{i+1}) \,. \tag{3}$$

Proof. Let z_1, z_2, z_3 be a basis of $L_\mathfrak{p}$ corresponding to which $L_\mathfrak{p}$ has the matrix given above. Then replacing $L_\mathfrak{p}$ by

$$(M_i)_{\mathfrak{p}} = \mathfrak{p}^i z_1 + \mathfrak{p}^{s-i} z_2 + \mathfrak{o}_{\mathfrak{p}} z_3$$

$(0 \leq i \leq s)$ one obtains a chain of lattices $M_0, \cdots, M_s$ in $Z_{\mathfrak{p}}(M_i)$ with $(M_i)_{\mathfrak{p}}$ a $\mathfrak{p}^s$-maximal lattice contained in $L_{\mathfrak{p}}$ and $(M_i)_{\mathfrak{q}} = L_{\mathfrak{q}}$ for $\mathfrak{q} \in \Sigma - \{\mathfrak{p}\}$. Let $K_{\mathfrak{p}}$ be any $\mathfrak{p}^s$-maximal lattice contained in $L_{\mathfrak{p}}$. From $B(z_3, K_{\mathfrak{p}}) \subseteq B(z_3, L_{\mathfrak{p}}) \subseteq 2\mathfrak{p}^s$ we obtain $z_3 \in 2\mathfrak{p}^s K_{\mathfrak{p}}^{\#} \subseteq K_{\mathfrak{p}}$ and z_3 can be split off orthogonally in $K_{\mathfrak{p}}$. $K_{\mathfrak{p}} \cap F_{\mathfrak{p}} z_1 + F_{\mathfrak{p}} z_2$ thus is a $\mathfrak{p}^s$-maximal lattice contained in $L_{\mathfrak{p}}$ on $F_{\mathfrak{p}} z_1 + F_{\mathfrak{p}} z_2$ and therefore equal to one of the $\mathfrak{p}^i z_1 + \mathfrak{p}^{s-i} z_2$. This proves the first assertion of the lemma.

The second assertion follows from the uniqueness of the chain. Finally, any vector $x \in L_{\mathfrak{p}}$ with $\operatorname{ord}_{\mathfrak{p}} q(x) \geq s$ is contained in at least one of the M_i. Since the set of M_i containing x forms a subchain of the M_i ([8], Lemma 3]), a vector contained in exactly j of the M_i is counted $j-1$ times in the second sum on the right hand side of (3), which proves the assertion.

LEMMA 5. *Let L, $\mathfrak{p}$, M_i be as in Lemma* 4, $\operatorname{ord}_{\mathfrak{p}} t = s + \nu$ $(\nu \in N)$. *Then*

$$r(t, M_i \cap M_{i+1}) = \sum_{j=1}^{[\nu/2]} (-1)^{j-1} (r(t, \mathfrak{p}^j M_i) + r(t, \mathfrak{p}^j M_{i+1})) + (-1)^{[\nu/2]} r(t, \mathfrak{p}^{[\nu/2]}(M_i \cap M_{i+1})). \tag{4}$$

Proof. If x representing t is primitive in $M_i \cap M_{i+1}$ then from Lemma 4 of [8] it follows that x is primitive in exactly one of M_i, M_{i+1}. From this one gets

$$r(t, M_i \cap M_{i+1}) = r(t, \mathfrak{p} M_i) + r(t, \mathfrak{p} M_{i+1}) - r(t, \mathfrak{p}(M_i \cap M_{i+1}))\,,$$

and (4) follows by iterating this.

We are now ready to prove our main result:

THEOREM. *Let $L = L_1$ and L_i be in the same spinor genus, let $\alpha \geq 1/2$ be a number such that for the m-th Fourier coefficient of a Hilbert cusp form for F of weight* 2 *the estimate*

$$|a_m| = O(N_Q^F(m)^\alpha)$$

holds (*see Remark* 1 *of* § 2). *Then for $t \in q(L)\mathfrak{o}$ and $\varepsilon > 0$ there is a constant C depending on t, ε, and the genus of L such that for any integral $\mathfrak{o}$-ideal $\mathfrak{m}$ one has*:

$$|r(t, \mathfrak{m}^{-1} L_1) - r(t, \mathfrak{m}^{-1} L_i)| \leq C \cdot N_Q^F(\mathfrak{m})^{\alpha+\varepsilon}\,.$$

Proof. Let $\mathfrak{m} = \prod \mathfrak{p}^{\mu_\mathfrak{p}} = \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3$ where $\mathfrak{m}_1$ is prime to $2^{-1}\mathfrak{d}$ and a dyadic prime $\mathfrak{p} \nmid 2^{-1}\mathfrak{d}$ divides $\mathfrak{m}_1$ if and only if $t\mathfrak{o}_\mathfrak{p} \subseteq q(L_\mathfrak{p})\mathfrak{p}$, $\mathfrak{m}_2$ is relatively prime to $\mathfrak{m}_1$ and divisible only by primes $\mathfrak{p} | \mathfrak{d}$ with $V_\mathfrak{p}$ isotropic and $\mathfrak{m}_3$ is divisible only by primes $\mathfrak{p}$ for which $V_\mathfrak{p}$ is anisotropic.

Let $\mathfrak{p} | \mathfrak{m}_3$ be such that $\operatorname{ord}_\mathfrak{p} t \geq \lambda_\mathfrak{p}(\mathfrak{m}^{-1}L) + 2$ ($\lambda_\mathfrak{p}(L)$ defined as in Lemma 3). Then

$$\underline{r}(t, \operatorname{gen}(\mathfrak{p}\mathfrak{m}^{-1}L)) = \underline{r}(t, \operatorname{gen}\mathfrak{m}^{-1}L)$$

since a $\mathfrak{p}^j$-maximal lattice on an anisotropic $F_\mathfrak{p}$-space does not represent primitively any numbers contained in $\mathfrak{p}^{j+2}$. We can therefore assume that for $\mathfrak{p} | \mathfrak{m}_3$ we have $\mu_\mathfrak{p} \leq \gamma_\mathfrak{p}$ where $\gamma_\mathfrak{p}$ is some constant depending only on $\operatorname{gen} L$ and t.

Let $\mathfrak{p} | \mathfrak{m}_2$ be such that $\operatorname{ord}_\mathfrak{p} t \geq \lambda_\mathfrak{p}(\mathfrak{m}^{-1}L)$, the lattice $(\mathfrak{m}^{-1}L_\mathfrak{p})'$ is isomorphic to $\begin{pmatrix} 0 & \pi^{s_1} \\ \pi^{s_1} & 0 \end{pmatrix} \perp \langle \pi^{s_2} \rangle$ $(s_2 \geq s_1)$ and $\operatorname{ord}_\mathfrak{p} t \geq s_2 + 2$. Then we first replace $r(t, \mathfrak{m}^{-1}L)$ by $r(t, (\mathfrak{m}^{-1}L)')$ and express $r(t, (\mathfrak{m}^{-1}L)')$ on using Lemma 4 and Lemma 5 as a sum of representation numbers $r(t, \mathfrak{p}^j M_k)$ of t by integral multiples of the lattices of the chain associated to $(\mathfrak{m}^{-1}L)'$ plus a sum of representation numbers $r(t, \mathfrak{p}^j(M_k \cap M_{k+1}))$ where $t\mathfrak{o}_p \not\subseteq q(M_k \cap M_{k+1})\mathfrak{p}^{2j+2}$. Since corresponding members of the chains associated to $(\mathfrak{m}^{-1}L)'$ and $(\mathfrak{m}^{-1}L_i)'$ respectively belong to the same spinor genus and since the number of terms in either sum is bounded by a (bounded) power of $\log N_Q^F \mathfrak{m}_2$, it suffices to prove the assertion for each summand.

We are thus reduced to the case that $\mu_\mathfrak{p} \leq \gamma_\mathfrak{p}$ for all $\mathfrak{p} | \mathfrak{m}_2\mathfrak{m}_3$ with some constant $\gamma_\mathfrak{p}$ depending only on t and $\operatorname{gen} L$.

Denote by $\underline{\bar{b}}_i(\mathfrak{m})$ the i-th row of $B(\mathfrak{m})$, by $\mathfrak{m}_1'$ the product of the $\mathfrak{p} | \mathfrak{m}_1$ with $t\mathfrak{o}_\mathfrak{p} = q(L_\mathfrak{p})\mathfrak{o}_\mathfrak{p}$, and by $\mathfrak{m}_1''$ the product of the $\mathfrak{p} | \mathfrak{m}_1$ with $t\mathfrak{o}_\mathfrak{p} \subseteq q(L_\mathfrak{p})\mathfrak{p}^2$. Using Lemma 2 we obtain:

$$\begin{aligned} |r(t, \mathfrak{m}^{-1}L_1) - r(t, \mathfrak{m}^{-1}L_i)| = |&(\underline{\bar{b}}_1(m_1) - \underline{\bar{b}}(\mathfrak{m}_1))\underline{r}(t, \operatorname{gen} \mathfrak{m}_2^{-1}\mathfrak{m}_3^{-1}L) \\ &- \left(\frac{E_t/F}{\mathfrak{m}_i'}\right)\underline{\bar{b}}_1(\mathfrak{m}_1\mathfrak{m}_1'^{-1}) - \underline{\bar{b}}_i(\mathfrak{m}_1\mathfrak{m}_1'^{-1}))\underline{r}(t, \operatorname{gen} \mathfrak{m}_2^{-1}\mathfrak{m}_3^{-1}L) \\ &- N_Q^F(\mathfrak{m}_1'')(\underline{\bar{b}}_1(\mathfrak{m}_1\mathfrak{m}_1''^{-1}) - \underline{\bar{b}}_i(\mathfrak{m}_1\mathfrak{m}_1''^{-1}))\underline{r}(t, \operatorname{gen} \mathfrak{m}_1''\mathfrak{m}_2^{-1}\mathfrak{m}_3^{-1}L)| \,. \end{aligned}$$

$N_Q^F(\mathfrak{m}_1'')|\underline{r}(t, \operatorname{gen} \mathfrak{m}_1''\mathfrak{m}_2^{-1}\mathfrak{m}_3^{-1}L)|$ can be bounded by a constant times $|r(t, \operatorname{gen}(\mathfrak{m}_2^{-1}\mathfrak{m}_3^{-1}L))|$ by a computation of local densities, and $|\underline{r}(t, \operatorname{gen}(\mathfrak{m}_2^{-1}\mathfrak{m}_3^{-1}L))|$ can be bounded by a constant depending only on t and $\operatorname{gen} L$ since we are now assuming $\mu_\mathfrak{p}$ to be bounded for all $\mathfrak{p}$ dividing $\mathfrak{m}_2\mathfrak{m}_3$.

The assertion of the theorem then follows from the proposition of Section 2.

Remark 2. If $r(t, \operatorname{spn} L)$ denotes Siegel's weighted average of the $r(t, L_i)$ over the L_i in the spinor genus of L then $r(t, \operatorname{spn} L) - r(t, L)$ is a linear combination of the $r(t, L_i) - r(t, L_j)$ (L_i, L_j in the spinor genus of L). The bound of the theorem therefore applies to this difference too.

$r(t, \operatorname{spn} L)$ has been computed in [10], in particular, conditions on t have been given under which $r(t, \operatorname{spn} L)$ can be bounded from below by some constant times $N_Q^F(t)^{1/2-\varepsilon}$ (Korollar 2 and 3 of [10] deal only with $F = \boldsymbol{Q}$ but the proof goes through for any F).

With the theorem and the estimates from [2], [5] quoted above $r(t, \mathfrak{m}^{-1}L) = r(t, \operatorname{spn}(\mathfrak{m}^{-1}L)) + r(t, \mathfrak{m}^{-1}L) - r(t, \operatorname{spn} \mathfrak{m}^{-1}L)$ thus gives an asymptotic formula for $r(t, \mathfrak{m}^{-1}L)$.

Remark 3. As in [6], putting

$$\underline{A}_\mu^{(t)}(L) = \theta^{(\mu)}(z_1, \cdots, z_d)\underline{r}(t, \operatorname{gen} L)$$

($1 \leq \mu \leq g$, θ^μ the matrix with entries θ_{ik}^μ (see the proposition)) one obtains a lifting (which we will call the Brandt lifting) from the space spanned by the theta series of the lattices in the genus of L to a subspace of the space of modular forms of weight two. By the proposition of Section 2, this lifting carries the difference of theta series belonging to lattices in the same spinor genus to a cusp form.

For $F = \boldsymbol{Q}$ it is clear that this lifting is essentially the same as Shimura's lifting [11], since for $p \nmid d$ the action of $T(p^2)$ on the vector $(\theta(L_1, z), \cdots, \theta(L_h, z))$ is given by multiplication with $\bar{B}(p)$ (Eichler's commutation relation, see Lemma 2) as is the action of $T(p)$ on the reduced Brandt matrix series (see [3]). The lifting therefore has the characteristic property of Shimura's lifting, viz. to commute with the Hecke operators in the sense that the lifting of $T(p^2)f$ is the same as $T(p)$ applied to the lifting of f.

A consequence of this is that the Brandt lifting carries a cusp form to a cusp form if and only if Shimura's lifting does so. Another consequence is that the difference between the two liftings of a given cusp form lies in the space that is generated by old forms. An explicit version of this fact in a special case was proved by Ponomarev in Theorem 2 of [6].

References

[1] Eichler, M., Quadratische Formen und orthogonale Gruppen, Berlin-Göttingen-Heidelberg: Springer 1952.

[2] ——, Quaternäre quadratische Formen und die Riemannsche Vermutung für Kongruenzzetafunktionen, Arch. Math., **5** (1954), 355–366.

[3] ——, The basis problem for modular forms and the traces of the Hecke operators, Modular functions of one variable I, Lecture notes in Mathematics, Vol. 320, pp. 75–151, Berlin-Heidelberg-New York: Springer 1973.

[4] ——, On theta functions in real algebraic number fields, Acta Arith., **33** (1977), 269–292.

[5] Gundlach, K.-B., Über die Darstellung der ganzen Spitzenformen zu den Idealstufen der Hilbertschen Modulgruppe und die Abschätzung ihrer Fourierkoeffizienten, Acta math., **92** (1954), 309–345.

[6] Ponomarev, P., Ternary quadratic forms and Shimura's correspondence, Nagoya Math. J., **81** (1981), 123–151.

[7] Rallis. S., The Eichler commutation relation and the continuous spectrum of the Weil representation, Non commutative harmonic analysis, Proc. Marseille-Luminy 1978, Lecture Notes in Mathematics, Vol. 728, pp. 211–244. Berlin-Heidelberg-New York: Springer 1979.

[8] Schulze-Pillot, R., Darstellung durch definite ternäre quadratische Formen, J. of Number Theory, **14** (1982), 237–250.

[9] ——, Thetareihen positiv definiter quadratischer Formen, Invent. Math., **75** (1984), (1984), 283–299.

[10] ——, Darstellungsmaße von Spinorgeschlechtern ternärer quadratischer Formen, J. reine und angew. Math., **252** (1984), 114–132.

[11] Shimura, G., On modular forms of half integral weight, Ann. of Math., **97** (1973), 440–481.

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