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THE EMPTY SPHERE

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In 1924 at the Toronto meeting of the International Congress of Mathematicians, B. N. Delone introduced his empty sphere method for lattices. We have titled our paper after this method as a tribute to his memory.

1. Introduction. We have studied the sets of integer solutions of equations of the form

$$(1)\quad f(x) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{i,j=1}^{n} a_{ij} x_i x_j = 0,$$

$$(a_{ij} = a_{ji};\ a_{ij},\ a_i,\ a_0,\ x_i \in \mathbf{R})$$

where f satisfies the following condition in which $\mathbf{Z}$ denotes the integers,

$$(2)\quad f(z) \geqq 0, \quad z \in \mathbf{Z}$$

and have resolved this problem using the theory of L-types of lattices [**3**, **4**, **11**]. We have been able to give a complete description of all such integer solutions when $n \leqq 4$.

This paper is a more lengthy discussion, with proofs of some of the results announced in [**9**].

Throughout, all of our functions will satisfy the above two conditions. If f is one such function then:

1.1. *Definition.* The *root figure* of f is the collection of integer solutions of $f(x) = 0$ and we denote it by R_f.

For a given n, a complete solution of our problem amounts to a list of all the possible root figures, R_f, that can occur.

Condition (2) on f forces the coefficient matrix $A_f = \{a_{ij}\}$ to be positive semi-definite and requires that the portion of $\mathbf{R}^n$ whose elements, x, satisfy the inequality $f(x) < 0$ to be free of integer points.

Suppose for the moment that R_f is non-empty and that A_f is positive definite. Under these circumstances the surface determined by the

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equation $f = 0$ is an ellipsoid, E_f. If the interior of E_f is not empty, since f assumes negative values there, it must be free of integer points. In both cases, empty or non-empty, the interior of E_f is free of integer points and we say that E_f is an *empty ellipsoid in* $\mathbf{R}^n$. The root figure, R_f, consists of the integer points lying on the empty ellipsoid, E_f, in $\mathbf{R}^n$. Moreover, any collection $R \subset \mathbf{Z}^n$, equal to the integer points on some empty ellipsoid in $\mathbf{R}^n$ is a possible root figure in $\mathbf{Z}^n$. All such root figures are finite.

If Γ is a general n-dimensional lattice and we blow up a sphere in one of its interstices in such a way that its interior is always free of lattice points, this *empty sphere* will eventually be held rigidly by the lattice when no further expansion is possible. At this time it must pass through at least $n + 1$ affinely independent lattice points. The convex hull of the lattice points lying on this sphere is a convex polytope which we will call an *L-polytope.*

1.2. THEOREM. *A root figure, R, corresponding to the collection of integer points lying on some empty ellipsoid in $\mathbf{R}^n$, is affinely equivalent to the vertex set, V, of some L-polytope*; dim V = dim $R \leqq n$.

We take as the dimension of a discrete set, D, that of its affine hull: dim D = dim aff(D). The affine hull of a set of points, X, is the smallest affine space containing X.

Proof. Suppose that R is the collection of integer points lying on the empty ellipsoid E'. Let

$$\Gamma = \mathbf{Z}^n \cap \operatorname{aff}(R), \quad E = E' \cap \operatorname{aff}(R).$$

Then R is also the collection of elements of Γ lying on the empty ellipsoid E.

Let T be some affinity (invertible affine transformation) mapping E onto a sphere. Then $T(E)$ is an empty sphere and the elements of $T(\Gamma)$ lying on this sphere are precisely $T(R)$. By construction dim $T(R)$ = dim $T(\Gamma)$ and therefore by the definition of L-polytope the convex hull of $T(R)$ is an L-polytope. Thus R is affinely equivalent to the vertex set of some L-polytope.

To illustrate the above ideas consider the equation

$$f_0(x, y) = 2 - 4x - 4y + 2x^2 + 2y^2 + 2xy = 0.$$

The curve defined by this equation is an ellipse passing through the three points (1, 0), (0, 1), (1, 1), whose interior is free of elements of $\mathbf{Z}^2$. Since it is only on the interior of this ellipse that the values of f_0 become negative, f_0 satisfies condition (2), and $f_0 = 0$ is an equation of the type we have studied. The root figure of f_0 consists of the three integer solutions of this equation which lie at the vertices of a triangle. The triangle appears as an L-polytope in two dimensional lattices (see Figure 1 of Section 4).

Regarding the overall structure of the paper we have the following comments. Due to its length, the proof of Theorem 2.1 of Section 2 was put into an appendix; a more complete discussion of some of this material will appear shortly (see reference [**2**]). Many of the results presented in the appendix appeared previously in preprint form (see [**1**]).

With the exception of Theorem 4.4 the material in Sections 3 and 4 is not new. It is a brief description of a portion of the results contained in [**4**, **6**] and included in order that our treatment be complete ([**4**] is difficult to obtain). The main result in Section 5, Theorem 5.1, is well known but we have supplied a new proof (see [**4**]).

We have broken the proof of our main result, Theorem 6.2, into two parts. The first of these parts is included in this article and the second, due to its length, will be published separately.

2. The root figures and z-equivalence. Infinite root figures are possible when kernel $(A_f) \neq 0$. In this case, it is easy to show that the surface $f = 0$ is a cylinder. The analysis of the infinite root figures is delicate and is analyzed in our appendix. From the major result obtained there (Theorem A.1) we have:

2.1. THEOREM. *The non-empty finite root figures $R \subset \mathbf{Z}^n$ consist of all possible collections of integer points in $\mathbf{R}^n$ lying on empty ellipsoids. The infinite root figures consist of all possible collections of integer points of the form $R + \Gamma$ where,*

(1) *R is a finite root figure,*

(2) $\Gamma \neq \{0\}$ *is a sublattice of* $\mathbf{Z}^n$ *which contains* 0,

(3) *Any element in the lattice* $\mathbf{Z}^n \cap \mathrm{aff}(R + \Gamma)$ *can be written uniquely as* $\rho + \gamma$ *where*

$$\rho \in \mathbf{Z}^n \cap \mathrm{aff}(R), \quad \gamma \in \Gamma.$$

Since all of the infinite root figures are obtained from the finite ones by a simple construction, enumeration of the possible root figures amounts to classifying the finite ones. Affine equivalence is not sufficiently discriminating for such a classification. In all dimensions n, there are root figures consisting of $n + 1$ elements, the vertices of a simplex. However, for $n = 5$ there are two geometrically distinct types of such figures. This follows from the construction of the proof of Theorem 1.2 which relates L-polytopes to root figures, and the fact that with $n = 5$ there are two distinct types of simplexes which appear as L-polytopes. The edge vectors of the first generate the ambient lattice whereas those of the second generate a sublattice of index 2. For larger values of n the number of distinct geometrical types of "simplicial" root figures increases.

The notion of z-equivalence is useful for classifying root figures. Besides the root figure it takes the ambient lattice into account.

2.2. *Definition*. Let $\Gamma_1 \subset \mathbf{R}^n$, $\Gamma_2 \subset \mathbf{R}^m$ be lattices and R_1, R_2 be subsets of Γ_1, Γ_2 respectively. We say that R_1 and R_2 are *z-equivalent* if there is an affinity (an invertible affine transformation)

$$T:\mathrm{aff}(R_1) \to \mathrm{aff}(R_2)$$

such that $T(R_1) = R_2$ and

$$T(\Gamma_1 \cap \mathrm{aff}(R_1)) = \Gamma_2 \cap \mathrm{aff}(R_2).$$

Typically R_1 and R_2 are root figures of say f_1 and f_2 with $\Gamma_1 = \mathbf{Z}^n$, $\Gamma_2 = \mathbf{Z}^m$. We will use the full generality of our definition when we consider L-polytopes whose vertices form a subset of some general lattice. We will say that two such polytopes are z-equivalent if their vertex sets are z-equivalent.

The statement of Theorem 1.2 can now be strengthened appropriately so that it becomes useful in the classification of the root figures (the proof remains as before).

2.3. THEOREM. *A finite root figure, $R \subset \mathbf{Z}^n$ is z-equivalent to the vertex set of some L-polytope.*

By Theorem 2.3 and the comments following Theorem 2.1 it follows that a complete solution of our problem for a given n is equivalent to classifying all of the L-polytopes of dimensions $1, 2, \dots, n$ up to z-equivalence.

3. L-decompositions and the classification of root figures. Suppose that L_1 and L_2 are two distinct L-polytopes of some lattice Γ. Then their interiors are disjoint and their intersection, if it is not empty, consists of an entire face, common to both of them, which is of some dimension less than n. The collection of all possible L-polytopes forms a decomposition of n-dimensional space called an *L-decomposition*:

Typically, all of the L-polytopes of some n-dimensional lattice, Γ_1, are simplexes and its L-decomposition is simplicial. Such a lattice is called *general*. Sufficiently small deformations of general lattices will not change the affine structure of its L-decomposition, i.e., the L-decomposition of the perturbed lattice will be affinely equivalent to that of Γ_1. However, under more persistent deformations suddenly new combinatorial types of L-polytopes are formed as some of the simplexes join to form more complicated polytopes. The affine structure of the L-decomposition undergoes an abrupt change as a special lattice, Γ_2, is formed. (A *special lattice* is one which has among its L-polytopes some which are not simplicial.)

General lattices can be classified by the affine structure of their respective simplicial L-decompositions. A pair of general lattices whose L-decompositions are affinely equivalent belong to the same *L-type* and the collection of lattices of a given type is called an *L-type domain*. An L-type domain is topologically connected in that any one of its members can be continuously deformed into any other in such a way that the affine structure of the intermediate L-decompositions remains constant.

The lattices lying on the boundary of an L-type domain are special. Thus the collection of all lattices is composed of possibly several connected regions, L-type domains, and the boundaries, which correspond to special lattices.

For $n = 1, 2, 3$ there is a single unique L-type, and the affine structures of the L-decompositions of any pair of general lattices coincide for these cases. For $n = 4$ there are 3 and for $n = 5$ there are 221 distinct L-type domains [**8**].

We can now describe the program proposed in this paper to enumerate the L-polytopes up to z-equivalence. It follows directly from the definition of z-equivalence that corresponding L-polytopes in affinely equivalent L-decompositions are z-equivalent. Thus for a given n we need only look at the collection of L-polytopes occurring in each affinely inequivalent L-decomposition and then group these into z-equivalence classes.

A list of z-inequivalent simplexes for a given n is established by looking at the L-decompositions of representative general lattices from each of the L-type domains that occur and classifying the simplexes appearing in these L-decompositions up to z-equivalence. We have found that for each of the cases $n \leqq 4$ all of the simplexes that occur are z-equivalent.

A list of the non-simplicial L-polytopes is obtained by examining the boundaries of the L-type domains. The L-polytopes found in these special lattices must be sorted into z-equivalence classes.

4. The first L-type domain. Let Γ be some n-dimensional lattice with a system of vectors, $\{a_1, a_2, \dots, a_n, a_{n+1}\}$, satisfying the three conditions:

(1) $\{a_1, a_2, \dots, a_n\}$ is a lattice basis for Γ

(2) $a_1 + a_2 + \dots + a_{n+1} = 0$

(3) $(a_i, a_j) < 0$, $i \neq j$; $((.\,,.)$ is the Euclidian scalar product).

The first L-type domain of Voronoi consists of all such lattices; it occurs for $n \geqq 2$.

Since $\{a_1, a_2, \dots, a_n)$ is a basis and a_{n+1} satisfies (2) it follows that any choice of n vectors from the system forms a basis for Γ.

Saying that lattices are orthogonally equivalent when they differ only by an orthogonal transformation, then, up to orthogonal equivalence the lattices of the first L-type domain may be parametrized by the quantities

$$|a_i|^2, \quad i = 1, \dots, n+1, \quad (a_i, a_j) < 0, \quad i \neq j \quad i, j = 1, \dots, n.$$

Special lattices on the boundary of this domain satisfy one or more equalities of the form:

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$(a_i, a_j) = 0.$

The collection of L-polytopes of a lattice, Γ, meeting at $\gamma \in \Gamma$ is the *star* of the L-decomposition at γ. Any other L-polytope must be orthogonally equivalent to one contained in this star.

Suppose now that Γ is a general n-dimensional lattice belonging to the first L-type domain with system $\{a_1, a_2, \dots, a_n, a_{n+1}\}$. Starting at $\gamma_0 \in \Gamma$, then moving to $\gamma_1 = \gamma_0 + a_1$, then to $\gamma_2 = \gamma_1 + a_2, \dots$, we finally arrive at $\gamma_n = \gamma_{n-1} + a_n$. Since

$$a_1 + a_2 + \dots + a_{n+1} = 0$$

it follows that $\gamma_0 = \gamma_n + a_{n+1}$. The convex hull of the $n + 1$ points $\gamma_0, \gamma_1, \dots, \gamma_n$ is a simplex. What is unanticipated is that the circumscribed sphere is an empty sphere, i.e., the simplex is an L-polytope of Γ. This simplex is denoted by $\langle a_1, a_2, \dots, a_{n+1}\rangle$.

By permuting the system vectors $\{a_1, a_2, \dots, a_{n+1}\}$ and carrying out similar constructions, $(n + 1)!$ simplicial L-polytopes can be constructed at γ_0. These are all denoted by symbols of the form

$$\langle a'_1, a'_2, \dots, a'_{n+1}\rangle$$

where the sequence in brackets is some permutation of the original. The entire collection of $(n + 1)!$ simplexes forms the star of the L-decomposition at γ_0.

Simplexes in the L-decomposition of Γ not belonging to the star at γ_0 are denoted by symbols of the form

$$\langle a'_1, a'_2, \dots, a'_n, a'_{n+1}\rangle_b.$$

This simplex belongs to the star at $\gamma_0 + b$ and can be constructed as above but starting at the center of this star. It is the convex hull of the lattice points

$$\gamma'_0 = \gamma_0 + b, \gamma'_1 = \gamma'_0 + a'_1, \gamma'_2 = \gamma'_1 + a'_2, \dots, \gamma'_n = \gamma'_{n-1} + a'_n.$$

We will need the following easily established result on the combinatorial structure of the L-decomposition of the first L-type domain:

4.1. PROPOSITION. *In the star at γ_0 two simplexes have an $(n - 1)$-dimensional face in common if and only if the order of the system vectors in their symbol differ by a single transposition of two adjacent vectors. Each simplex in this star has n such neighbours.*

If $\langle a'_1, a'_2, \dots, a'_n, a'_{n+1}\rangle$ is an arbitrary member of the star at γ_0 then $\langle a'_{n+1}, a'_2, a'_3, \dots, a'_n, a'_1\rangle_{b'}$ $(b' = a'_1 - a'_{n+1})$ is a neighbour with which it shares a $(n - 1)$-dimensional face. Each simplex has one such neighbour.

Thus $\langle a_1, a_2, a_3, \dots, a_n, a_{n+1}\rangle$ has as neighbours the n simplexes

$$\langle a_2, a_1, a_3, \dots, a_n, a_{n+1}\rangle, \langle a_1, a_3, a_2, \dots, a_n, a_{n+1}\rangle, \dots,$$

$$\langle a_1, a_2, a_3, \ldots, a_{n+1}, a_n \rangle$$

as well as the simplex

$$\langle a_{n+1}, a_2, a_3, \ldots, a_n, a_1 \rangle_b \quad (b = a_1 - a_{n+1}).$$

On the boundary of the first L-type domain where one or more of the pairs of system vectors are orthogonal some of these neighbours join to form more complicated polytopes. These polytopes can be constructed in a systematic way using the:

4.2. GLUEING THEOREM. *When* $a_i \perp a_j$

(a) *Any simplex in the star at* γ_0 *with system vectors* a_i, a_j *lying adjacent in its symbol joins to its neighbour whose symbol is obtained by transposition of* a_i *and* a_j.

(b) *Any simplex in the star at* γ_0 *with* a_i *lying first and* a_j *lying last joins to its neighbour in the star at* $\gamma_0 + a_i - a_j$ *whose symbol is obtained by transposing* a_i *and* a_j.

(c) *Any simplex in the star at* γ_0 *with* a_j *lying first and* a_i *lying last joins its neighbour in the star at* $\gamma_0 + a_j - a_i$ *whose symbol is obtained by transposition of* a_i *and* a_j.

The geometrical content of this theorem can be visualized in the following way. Imagine a general lattice Γ being deformed until a point on the boundary is achieved where $a_i \perp a_j$. Just before this boundary point is achieved the empty spheres circumscribing the pairs of neighbouring simplexes mentioned in the glueing theorem are distinct. At the exact moment when this boundary point is achieved these spheres coincide and the resulting L-polytopes have simplicial decompositions which contain these pairs of neighbouring simplexes. The simplicial decompositions of these more complicated L-polytopes could contain additional simplexes depending upon whether or not other pairs of system vectors are orthogonal.

As a simple consequence of the glueing theorem we have:

4.3. COROLLARY. *The affine structure of the L-decomposition of a lattice,* Γ, *belonging to the first L-type domain with system* $\{a_1, a_2, \ldots, a_{n+1}\}$ *is completely determined by the orthogonalities occuring among the system vectors.*

Regarding the z-equivalence of L-polytopes we have the following result:

4.4. THEOREM. *If two L-polytopes belonging to lattices of the first L-type domain or its boundary are affinely equivalent, they are z-equivalent.*

Proof. Suppose that L is simplicial in $\Gamma \subset \mathbf{R}^n$ belonging to the first L-type domain, i.e.,

$$L = \langle a_1', a_2', \ldots, a_{n+1}' \rangle_b.$$

Then it is clear that the difference set formed from the vertex set of L, $V(L) - V(L)$, contains a lattice basis for Γ and thus L determines Γ.

Now consider some more general L-polytope, L_1, belonging to Γ_1, on the boundary of the first L-type domain. Since L_1 is the union of simplexes of the form $\langle a_1', a_2', \ldots, a_{n+1}' \rangle_b$, the difference set $V(L_1) - V(L_1)$ also contains a lattice basis and thus L_1 determines Γ_1.

Thus arbitrary L-polytopes belonging to lattices of the first L-type domain or to its boundary determine their lattices and if T is some affinity mapping one L-polytope onto another it necessarily maps the lattice determined by the first onto the second and the two L-polytopes must be z-equivalent.

As an illustration of the above material we describe the L-polytopes in the plane (the 1-dimensional case is trivial). Any general lattice, Γ, in the plane belongs to the first L-type domain and has a system of lattice vectors satisfying the conditions set forth at the beginning of this section. The star of its L-decomposition is easily constructed:

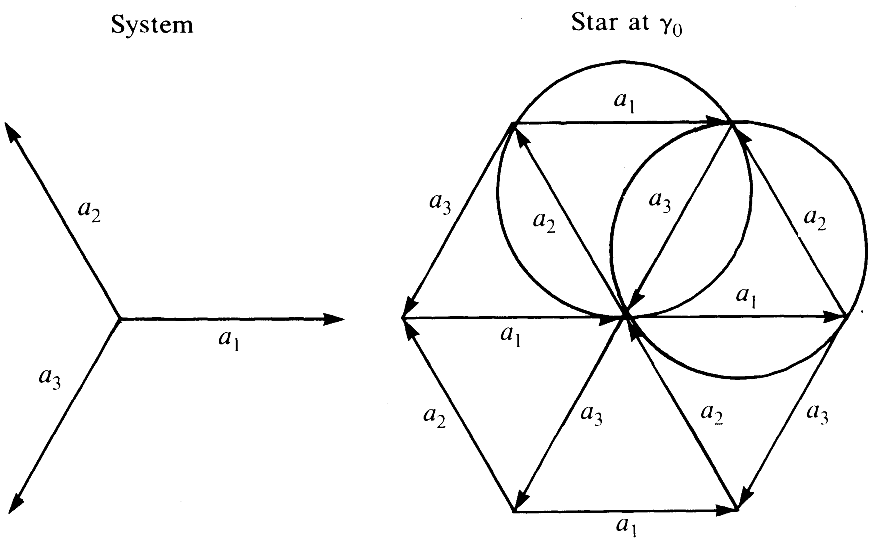


Figure 1

We have drawn empty spheres around the simplexes $\langle a_1, a_2, a_3 \rangle$ and $\langle a_2, a_1, a_3 \rangle$.

Now imagine that we deform the lattice in such a way that we move from the interior of the first L-type domain to a point on the boundary where $(a_1, a_2) = 0$ (see figure 2):

Deformation of Lattice

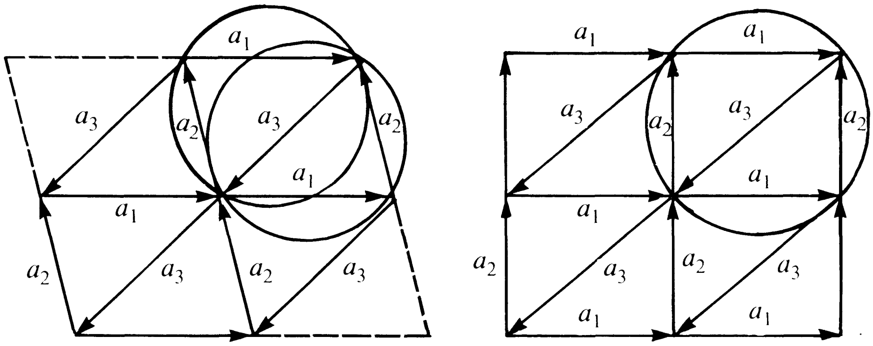


Figure 2

As soon as this boundary point is achieved the empty spheres circumscribing $\langle a_1, a_2, a_3 \rangle$ and $\langle a_2, a_1, a_3 \rangle$ coincide and a new type of L-polytope is formed; a rectangle. This figure is obtained by joining the two neighbours $\langle a_1, a_2, a_3 \rangle$ and $\langle a_2, a_1, a_3 \rangle$ as described in the glueing theorem. The three pairs of neighbours

$$\{ \langle a_3, a_1, a_2 \rangle, \langle a_3, a_2, a_1 \rangle \},$$

$$\{ \langle a_2, a_3, a_1 \rangle, \langle a_1, a_3, a_2 \rangle_b \quad (b = a_2 - a_1) \} \text{ and}$$

$$\{ \langle a_1, a_3, a_2 \rangle, \langle a_2, a_3, a_1 \rangle_{b'} \quad (b' = a_1 - a_2) \}$$

also join to form rectangles. This new star is composed of 4 affinely equivalent (and therefore by Theorem 4.4, z-equivalent) rectangles.

It follows by the symmetry of the construction that we also obtain four rectangles if, instead of $(a_1, a_2) = 0$ we require that either $(a_1, a_3) = 0$ or $(a_2, a_3) = 0$. Requiring that two distinct pairs of system vectors be orthogonal leads to a contradiction. For example, if $(a_1, a_2) = (a_1, a_3) = 0$ then by using the equation $a_1 + a_2 + a_3 = 0$ it follows that

$$|a_1|^2 = -(a_1, a_2) - (a_1, a_3) = 0,$$

a contradiction.

Thus by the above argument and by Theorem 4.4 we have:

4.5. THEOREM. *Up to z-equivalence there are two types of L-polytopes in the plane*: *the triangle and the rectangle.*

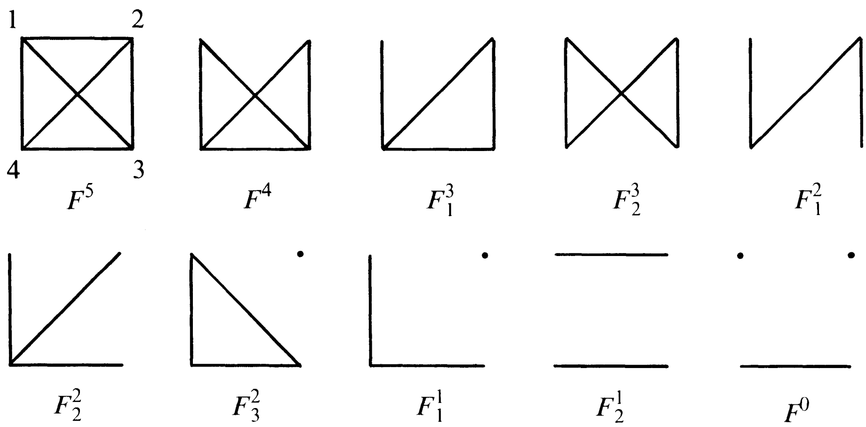
5. L-polytopes in 3-space. All of the general lattices in $\mathbf{R}^3$ belong to the first L-type domain and the special lattices to the boundary of this domain. Up to affine equivalence the L-decompositions of these lattices can be enumerated with the help of the Delone symbol. For any one of these lattices let $\{a_1, a_2, a_3, a_4\}$ be a system of lattice vectors as described in Section 4. We represent this system by a graph, the Delone symbol, which is constructed using the following rules:

(1) the graph has 4 vertices, one each for the four system vectors.

(2) We pencil in an edge joining vertex i and j if and only if $(a_i, a_j) < 0$.

Thus a general lattice is represented by a graph with 4 vertices which is complete and a special lattice by such a graph with edges missing. A pair of Delone symbols which differ by a permutation of vertices correspond to lattices with affinely equivalent L-decompositions. Up to a permutation of vertices, the following forms a complete list of the graphs on 4 vertices. (We assume that the vertices of all these graphs are numbered as F^5.)

TABLE I
Graphs on Four Vertices



There are no lattices corresponding to the graphs F_3^2 through F^0 of Table I. For example, F_3^2, would represent a lattice, Γ, with system vectors, $\{a_1, a_2, a_3, a_4\}$, satisfying the conditions

$$(a_1, a_2) = (a_2, a_3) = (a_2, a_4) = 0.$$

By virtue of the fact that $a_1 + a_2 + a_3 + a_4 = 0$ it follows that

$$|a_2|^2 = -(a_2, a_1 + a_3 + a_4) = 0$$

which is clearly impossible.

Also graphs F_1^2 and F_2^2 represent lattices whose L-decompositions are affinely equivalent. If $\{a_1, a_2, a_3, a_4\}$ is a system for Γ which is represented by F_2^2 then these vectors satisfy the conditions:

(1) $(a_1, a_2) = (a_1, a_3) = (a_2, a_3) = 0;$

(2) $(a_1, a_4), (a_2, a_4), (a_3, a_4) < 0.$

But $b_1 = a_1, b_2 = a_3 - a_2, b_3 = a_2$ also forms a basis for Γ and by adding $b_4 = -a_1 - a_3$ we have another system, $\{b_1, b_2, b_3, b_4\}$, for Γ since

$$b_1 + b_2 + b_3 + b_4 = 0 \quad \text{and}$$

(1) $(b_1, b_2) = (b_1, b_3) = (b_3, b_4) = 0;$

(2) $(b_2, b_3) = -|a_2|^2, (b_2, b_4) = -|a_3|^2,$

$$(b_1, b_4) = -|a_1|^2 < 0.$$

But this system has the graph F_1^2.

Thus up to affine equivalence, graphs F^5 through F_1^2 correspond to the five distinct types of L-decompositions which occur in $\mathbf{R}^3$.

We are now in a position to enumerate the L-polytopes occurring in these L-decompositions and start with a system $\{a_1, a_2, a_3, a_4\}$ for a general lattice Γ whose projection on some plane is:

System for $\mathbf{R}^3$

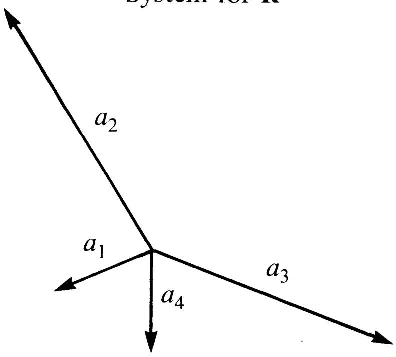


Figure 4

Now consider the simplex $\langle a_1, a_2, a_3, a_4\rangle$ belonging to the star at γ_0 of this general lattice Γ. Without modifying the two-dimensional representation of the system vectors imagine that various combinations of pairs of system vectors become orthogonal as indicated by the Delone symbols F^5 through F_1^2. This results in the joining of other simplexes to the original one as described in the glueing theorem. All in all five distinct L-polytopes are formed in this way, one each for the five Delone symbols F^5 through F_1^2 (see Table II).

By further investigation the entire stars of the five types of L-decompositions may be constructed and when this is done it is found that up to affine equivalence the only L-polytopes which occur are those which appear in our list of five (Table II). The numbers of these various types of L-polytopes appearing in these stars is recorded in Table III. Therein we denote the various L-polytopes by their numbers of vertices denoting the octahedron by 6_0 and the triangular prism by 6_1.

TABLE II
L-Polytopes in $\mathbf{R}^3$

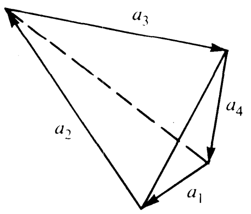
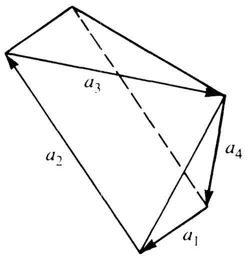
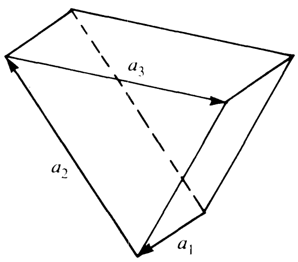
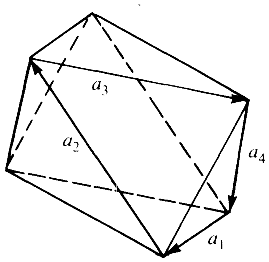
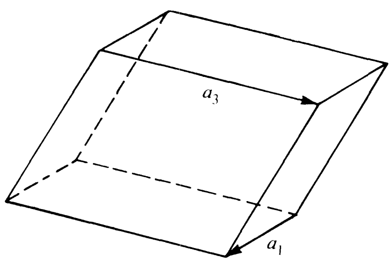
L-decomposition	Figure	Simplicial decomposition of figure	Description
F^5		$\langle a_1, a_2, a_3, a_4\rangle$	simplex
F^4		$\langle a_1, a_2, a_3, a_4\rangle$ $\langle a_2, a_1, a_3, a_4\rangle$	pyramid
F_1^3		$\langle a_1, a_2, a_3, a_4\rangle$ $\langle a_2, a_1, a_3, a_4\rangle$ $\langle a_2, a_3, a_1, a_4\rangle$	triangular prism
F_2^3		$\langle a_1, a_2, a_3, a_4\rangle$ $\langle a_2, a_1, a_3, a_4\rangle$ $\langle a_1, a_2, a_4, a_3\rangle$ $\langle a_2, a_1, a_4, a_3\rangle$	octahedron
F_1^2		$\langle a_1, a_2, a_3, a_4\rangle$ $\langle a_2, a_1, a_3, a_4\rangle$ $\langle a_1, a_2, a_4, a_3\rangle$ $\langle a_2, a_1, a_4, a_3\rangle$ $\langle a_2, a_3, a_1, a_4\rangle$ $\langle a_3, a_2, a_4, a_1\rangle$	box

TABLE III

L-decomposition	Stars of L-decompositions in $\mathbf{R}^3$				
	4	5	6_1	6_0	8
F^5	24	0	0	0	0
F^4	8	10	0	0	0
F_1^3	0	0	12	0	0
F_2^3	8	0	0	6	0
F_1^2	0	0	0	0	8

By virtue of Theorem 4.4 and the information appearing in Table II we have:

5.1. THEOREM. *Up to z-equivalence there are five distinct L-polytopes which occur in* $\mathbf{R}^3$. *These are pictured in Table II.*

6. The L-polytopes in $\mathbf{R}^4$. The general lattices in $\mathbf{R}^4$ belong to one of three different L-type domains **[9]**. We first enumerate all of the L-polytopes appearing in the first L-type domain and on its boundary as in the 3-dimensional case. Here the Delone symbol for such a lattice has five vertices and as before the absence of an edge corresponds to a pair of orthogonal system vectors. A complete list of the graphs on 5 vertices, up to permutations of vertices is given in Table IV. As before, only a portion of these graphs correspond to Delone symbols of lattices. In addition there are some instances where two or more of these graphs correspond to the same lattice. This of course can only happen when a lattice has two distinct systems of vectors with different orthogonality relations among them. In these cases a lattice has two or more Delone symbols.

The graphs F_6^5, F_6^4, F_2^3, F_3^3, F_5^3, F_1^2, F_2^2, F_3^2, F_4^2, F_1^1, F_2^1, F^0 do not correspond to lattices at all. Consider, for example, F_6^5. The system vectors $\{a_1, a_2, a_3, a_4, a_5\}$ of any such lattice would have to satisfy the equalities

$$(a_2, a_1) = (a_2, a_3) = (a_2, a_4) = (a_2, a_5) = 0,$$

but this is impossible.

Also some graphs represent lattices whose L-decompositions are affinely equivalent; F_1^5 and F_2^5 in such a case. If

$$\{a_1, a_2, a_3, a_4, -(a_1 + a_2 + a_3 + a_4)\}$$

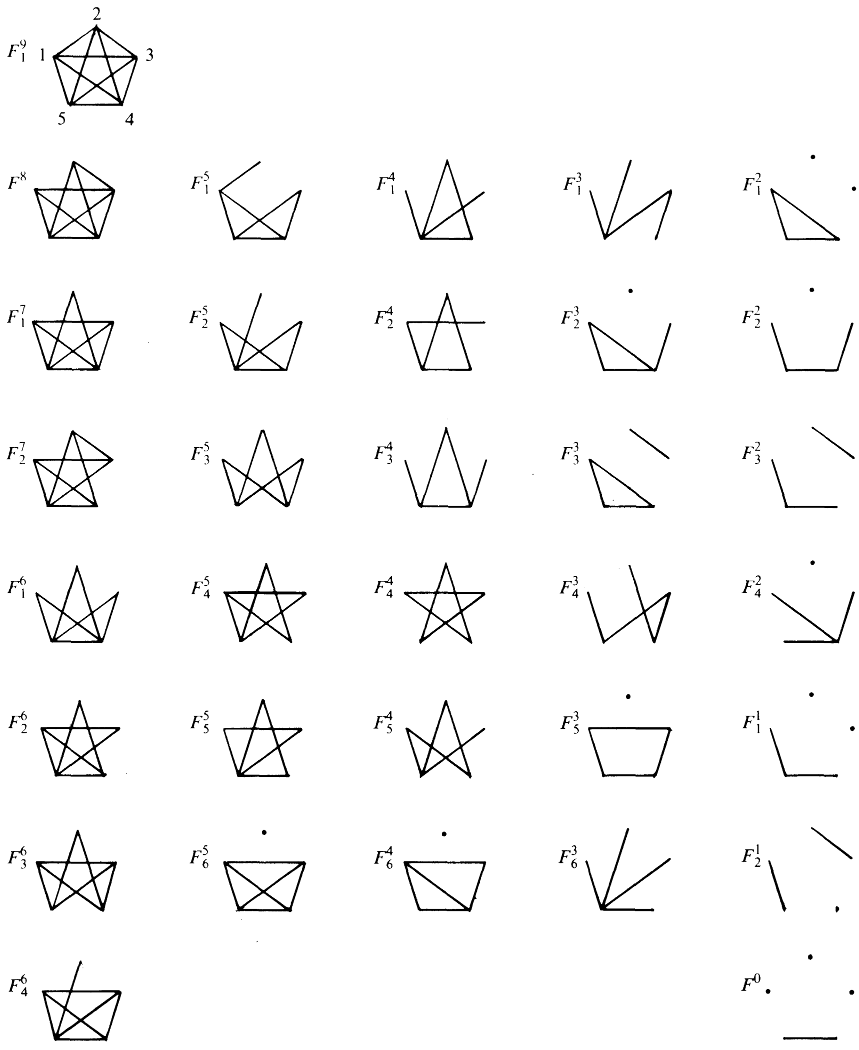
is a system for some lattice, Γ, represented by F_2^5 then

$$\{a_1 - a_2, a_2, a_3, a_4, -(a_1 + a_3 + a_4)\}$$

is also a system for Γ but this system has the graph F_1^5. In addition:

(a) F_1^4, F_2^4, F_3^4 represent lattices with affinely equivalent L-decompositions. For if

TABLE IV
Graphs on Five Vertices



$$\{a_1, a_2, a_3, a_4, -(a_1 + a_2 + a_3 + a_4)\}$$

is a system for Γ with symbol F_1^4 then

$$\{a_1 - a_3, a_2, a_3, a_4, -(a_1 + a_2 + a_4)\}$$

and

$$\{a_1, a_2, a_3, -a_3 + a_4, -(a_1 + a_2 + a_4)\}$$

are also systems for Γ. But these have graphs F_2^4 and F_3^4 respectively.

(b) F_1^3, F_4^3, F_6^3 represent lattices with affinely equivalent L-decompositions. If

$$\{a_1, a_2, a_3, a_4, -(a_1 + a_2 + a_3 + a_4)\}$$

is a system with symbol F_6^3 then

$$\{a_1, a_2, a_3 - a_4, a_4, -(a_1 + a_2 + a_3)\}$$

and

$$\{a_1, a_2, a_3 - a_4, a_4 - a_2, -(a_1 + a_3)\}$$

are also systems for the same lattice but with graphs F_1^3 and F_4^3 respectively.

Thus there are a total of 16 affinely inequivalent L-decompositions that can occur in $\mathbf{R}^4$ and these can be represented by the 16 Delone symbols F^9, F^8, F_1^7, F_2^7, F_1^6, F_2^6, F_3^6, F_4^6, F_2^5, F_3^5, F_4^5, F_5^5, F_1^4, F_4^4, F_5^4, F_6^3. Starting with the simplex $\langle a_1, a_2, a_3, a_4, a_5\rangle$ and proceeding through this list of possible L-decompositions, glueing simplexes as we go as dictated by the glueing theorem we obtain 16 distinct L-polytopes, one each for the 16 distinct types of L-decompositions that can occur. Two dimensional representations of all of these figures are included in Table V. We label the figures by the symbol for the L-decomposition in which they occur.

By further investigation a complete list of the L-polytopes occurring in the star of any given L-decomposition can be obtained. In Table VII, along with the designation for the L-decomposition, we include a list of these additional L-polytopes. Since no new L-polytopes occur which have not been accounted for in the original list we have, by virtue of Theorem 4.4:

6.1. THEOREM. *Up to z-equivalence there are exactly* 16 *L-polytopes which occur in or on the boundary of the first L-type domain in* $\mathbf{R}^4$.

Three additional L-polytopes can be found on the boundaries of the other two L-type domains that occur in $\mathbf{R}^4$ but due to its length we will not include a description of how we obtained them. Thus the proof of our main Theorem 6.2 below is incomplete. All of this will be the subject of a second paper on the present topic.

We remark that the simplexes occurring in the L-decompositions of the second and third L-type domain are all z-equivalent to those occurring in the first. Thus any new figures that occur must appear on the boundaries of either the second or third L-type domain. But any portion of the boundary of either of these domains which is shared by the first L-type domain will yield no new L-polytopes since we have already enumerated all of these. Figure A, included in Table VI below occurs only on the boundary of the third L-type domain, Figures B and C lie on portions of the boundary of the third L-type domain shared by the second L-type domain.

6.2. THEOREM. *Up to z-equivalence there are a total of* 19 *distinct L-polytopes in* $\mathbf{R}^4$, *those appearing in Tables* V *and* VII.

Appendix. Kernel $(A_f) \neq 0$. We will prove the following:

A. 1. THEOREM. *Suppose that* $R_f \subset \mathbf{Z}^n$ *is not empty and* $0 \neq V = \ker(A_f)$. *Then* R_f *is of the form*

$$R_f = R + \Gamma,$$

where R, Γ *satisfy the three conditions*;

(1) R *is the collection of integer points lying on some empty ellipsoid in* $\mathbf{R}^n$.

(2) Γ *is a sublattice of* $\mathbf{Z}^n$ *containing* 0.

(3) *The two lattices* $\Gamma_R = \mathbf{Z}^n \cap \mathrm{aff}(R)$ *and* Γ *form a decomposition of* $\mathbf{Z}^n \cap \mathrm{aff}(R + \Gamma)$:

$$\mathbf{Z}^n \cap \mathrm{aff}(R + \Gamma) = \Gamma_R + \Gamma, \quad (\Gamma_R - \Gamma_R) \cap \Gamma = 0.$$

Such a root figure is finite or infinite depending upon whether $\dim(\Gamma) > 0$.

Moreover any subset of $\mathbf{Z}^n$ *of the form* $R + \Gamma$ *where* R, Γ *satisfy these three conditions is a root figure in* $\mathbf{R}^n$.

In the course of our proof we will make use of two subspaces of $\mathbf{R}^n$ which are equal to the affine hulls of their integer elements and are related to V. We define (a), V_0 to be the largest linear subspace of $\mathbf{R}^n$ which is the affine hull of its integer elements and is contained in V and (b), V_1 to be the smallest linear subspace which is the affine hull of its integer elements and contains V. Both are uniquely determined:

$$V_0 = \mathrm{aff}(\mathbf{Z}^n \cap V), \quad V_1 = (V^{\perp} \cap \mathbf{Z}^n)^{\perp},$$

where $\perp$ denotes the orthogonal complement with respect to the usual scalar product. The two containments $V_0 \subset V \subset V_1$ become equalities if and only if V is the affine hull of its integer elements.

The following three propositions, proved in [**2**], describe some useful properties of the subspaces V, V_1.

A.2. PROPOSITION. *If* $0 \neq V = \ker(A_f)$, *then* f *is constant on the translates of* V *by elements* $x \in \mathbf{R}^n : f(x + V) = f(x)$.

A.3. PROPOSITION. *Let* $0 \neq V = \ker(A_f)$. *Then the set* M, *where*

$$M = \{m \in \mathbf{R}^n | f(m) \leqq f(m + V_1)\},$$

is an affine subspace of $\mathbf{R}^n$ *with*

(1) $(M - M) \cap V_1 = V$,

(2) $\mathbf{R}^n = M + V_1$.

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A.3. PROPOSITION. *If* $0 \neq V = \ker(A_f)$, *then* f *is non-negative on the translates of* V_1 *by elements* $z \in \mathbf{Z}^n : f(z + V_1) \geqq 0$.

A.1. *Proof of theorem.* Consider f with properties as in the statement of Theorem A.1. Let

$$\Gamma_f = \mathbf{Z}^n \cap \operatorname{aff}(R_f).$$

By Proposition A.2 $R_f + \mathbf{Z}^n \cap V = R_f$, so

$$\Gamma_f + \mathbf{Z}^n \cap V = \Gamma_f.$$

Choose a sublattice $\Gamma_* \subset \Gamma_f$ so that $\Gamma = \mathbf{Z}^n \cap V$, Γ_* form a decomposition of Γ_f:

$$\Gamma_f = \Gamma_* + \Gamma, \quad (\Gamma_* - \Gamma_*) \cap \Gamma = 0.$$

Using this decomposition we write, for $m \in \Gamma_f$, $m = r + v$, where $r \in \Gamma_*$, $v \in \Gamma$. By Proposition A.2, $f(m) = f(r)$ and $m \in R_f$ if and only if $r \in R_f$. Defining R to be equal to $R_f \cap \Gamma_*$ it follows that

$$R_f = R + \Gamma.$$

Regarding the properties of Γ, R we have:

(1) If $m \in R$ then $f(m) = 0$ and by Proposition A.4

$$f(m) \leqq f(m + V_1);$$

m is therefore an element of M, the affine space defined in Proposition A.3. Since $\operatorname{aff}(R) \subset M$ property (1) of Proposition A.3 implies that

$$(\operatorname{aff}(R) - \operatorname{aff}(R)) \cap V_1 \subset V.$$

Since the two subspaces $(\operatorname{aff}(R) - \operatorname{aff}(R))$, V_1 both have integer bases the same is true of ther intersection and

$$(\operatorname{aff}(R) - \operatorname{aff}(R)) \cap V_1 \subset V_0,$$

which implies that

$$(\operatorname{aff}(R) - \operatorname{aff}(R)) \cap V \subset V_0.$$

But by the construction of Γ_* it follows that

$$(\operatorname{aff}(\Gamma_*) - \operatorname{aff}(\Gamma_*)) \cap V_0 = 0 \quad (V_0 = \operatorname{aff} \Gamma).$$

Since $R \subset \Gamma_*$ we conclude that

$$(\operatorname{aff}(R) - \operatorname{aff}(R)) \cap V = 0$$

and the restriction of the surface defined by the equation $f = 0$ to $\operatorname{aff}(R)$ must be an empty ellipsoid in $\operatorname{aff}(R)$; R is the collection of integer points lying on an empty ellipsoid in $\operatorname{aff}(R)$. This being the case R can also be realized as the collection of integer points lying on some empty ellipsoid in

$\mathbf{R}^n$ and therefore satisfies condition (1).

(2) Since $0 \in \Gamma$, Γ satisfies condition (2).

(3) Since $R \subset \Gamma_*$ and since

$$\dim(R) = \dim(R_f) - \dim(\Gamma) = \dim(\Gamma_f) - \dim(\Gamma) = \dim(\Gamma_*)$$

we must have $\mathbf{Z}^n \cap \operatorname{aff}(R) = \Gamma_*$. With this identity, condition (3) of the theorem is a restatement of the fact that Γ_*, Γ form a decomposition of Γ_f.

The last statement of the first part of the theorem regarding the finiteness of the root figure is obvious.

To begin the proof of the second part of the theorem assume that R, Γ satisfy conditions (1) through (3) of A.1. By condition (1), R can be realized as the integer points lying on some empty ellipsoid in $\mathbf{R}^n$. Let g be a function satisfying the two conditions stated in the introduction and such that the surface determined by the equation $g = 0$ coincides with this empty ellipsoid. Then $R = R_g$. Let

$$\Gamma_1 = \mathbf{Z}^n \cap \operatorname{aff}(R)$$

or, if need be, let Γ_1 be equal to some extension of $\mathbf{Z}^n \cap \operatorname{aff}(R)$ so that Γ_1, Γ form a decomposition of $\mathbf{Z}^n$:

$$\mathbf{Z}^n = \Gamma_1 + \Gamma, (\Gamma_1 - \Gamma_1) \cap \Gamma = 0.$$

That such a Γ_1 can be constructed is guaranteed by conditions (2), (3) on Γ, R. If $Y_1 = \operatorname{aff}(\Gamma_1)$, $Y = \operatorname{aff}(\Gamma)$ then an element $x \in \mathbf{R}^n$ can be written uniquely as $y_1 + y$ with $y_1 \in Y_1$, $y \in Y$. Now define the function h by the formula

$$h(x) = h(y_1 + y) = g(y_1).$$

Since y_1 is related to x by an affine transformation, $h(x)$ can be written as in equation (1) of the introduction. For $z \in \mathbf{Z}^n$,

$$h(z) = h(\gamma_1 + \gamma) = g(\gamma_1) \geqq 0$$

since $\gamma_1 \in \mathbf{Z}^n$ by the construction of Γ_1. Thus h satisfies condition (2) of the introduction and

$$R_h = \{z \in \mathbf{Z}^n | h(z) = 0\}$$

is a root figure. Since it is clear that $R + \Gamma = R_h$, $R + \Gamma$ is a root figure and our proof is complete.

TABLE V
L-polytopes; first L-type domain

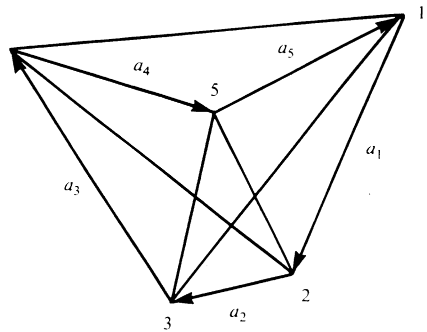
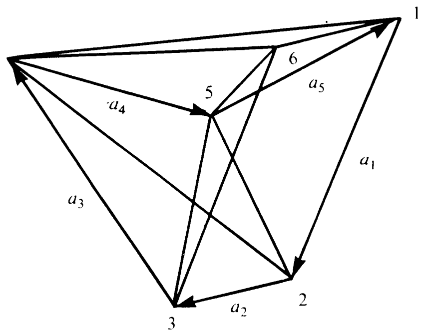
No.	L-Decomp. & No. Vertices	Figure	Simplicial Decomposition	Facets		Description
1.	F^9 5		$\langle a_1, a_2, a_3, a_4, a_5\rangle$	4:	(1, 2, 3, 4) (1, 2, 3, 5) (1, 2, 4, 5) (1, 3, 4, 5) (2, 3, 4, 5)	Simplex. Notice how the system vectors are drawn. This is perspective A.
2.	F^8 6		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$	4: 5:	(1, 2, 4, 5) (1, 4, 5, 6) (2, 3, 4, 5) (3, 4, 5, 6) (1, 2, 3, 4, 6) (1, 2, 3, 5, 6)	Pyramid with 3-dimensional pyramid as base. Add vertex 6 to fig. 1. Perspective A.

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TABLE V (continued)
L-polytopes; first L-type domain

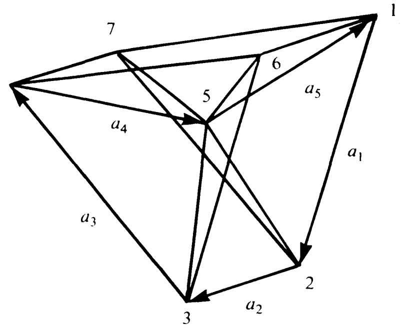
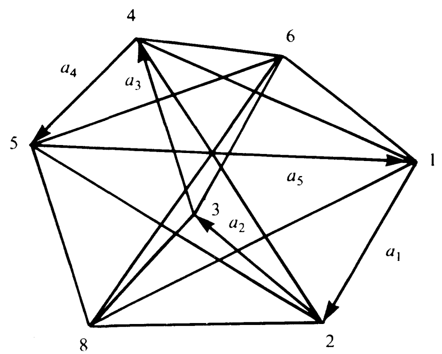
No.	L-Decomp. & No. Vertices	Figure	Simplicial Decomposition	Facets	Description
3.	F_1^7 7		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$ $\langle a_1, a_3, a_2, a_4, a_5\rangle$	4: (1, 2, 5, 7) (3, 4, 5, 6) 5: (1, 2, 3, 5, 6) (1, 4, 5, 6, 7) (2, 3, 4, 5, 7) 6_1: (1, 2, 3, 4, 6, 7)	Pyramid with 3-dimensional triangular prism as base. Add vertex 7 to fig. 2 Perspective A.
4.	F_2^7 7		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$ $\langle a_1, a_2, a_4, a_3, a_5\rangle$ $\langle a_2, a_1, a_4, a_3, a_5\rangle$	4: (1, 2, 4, 5) (1, 2, 5, 8) (1, 4, 5, 6) (1, 4, 6, 8) 5: (1, 2, 3, 4, 6) (1, 2, 3, 6, 8) (2, 3, 4, 5, 8) (3, 4, 5, 6, 8)	A 3-dimensional pyramid (1, 2, 3, 4, 6) with a copy of one of its edges, 5-8 lying above it. Add vertex 8 to fig. 2. This is perspective B.

TABLE V (continued)

L-polytopes; first L-type domain

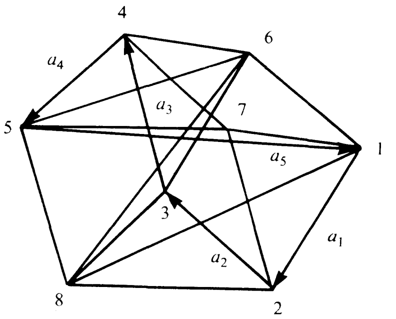
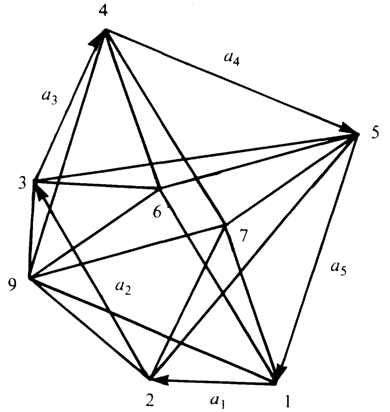
No.	L-Decomp. & No. Vertices	Figure	Simplicial Decomposition	Facets	Description
5.	F_2^6 8		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$ $\langle a_1, a_3, a_2, a_4, a_5\rangle$ $\langle a_1, a_2, a_4, a_3, a_5\rangle$ $\langle a_2, a_1, a_4, a_3, a_5\rangle$	4: (1, 5, 6, 8) 5: (1, 2, 3, 6, 8) (1, 2, 5, 7, 8) (1, 4, 5, 6, 7) (3, 4, 5, 6, 8) 6_1: (1, 2, 3, 4, 6, 7) (2, 3, 4, 5, 7, 8)	A 3-dimensional triangular prism, (1, 2, 3, 4, 6, 7), with a copy of one of its edges 5-8, lying above it. Add vertex 8 to fig. 3 or vertex 7 to fig. 4. Perspective B.
6.	F_3^6 8		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$ $\langle a_1, a_3, a_2, a_4, a_5\rangle$ $\langle a_1, a_2, a_3, a_5, a_4\rangle$ $\langle a_2, a_1, a_3, a_5, a_4\rangle$ $\langle a_1, a_3, a_2, a_5, a_4\rangle$	4: (1, 2, 5, 7) (1, 2, 7, 9) (3, 4, 5, 6) (3, 4, 6, 9) 5: (1, 2, 3, 5, 6) (1, 2, 3, 6, 9) (2, 3, 4, 5, 7) (2, 3, 4, 7, 9) 6_0: (1, 4, 5, 6, 7, 9)	Bipyramid with 3-dimensional triangular prism as base. Add vertex 9 to fig. 3. Perspective C.

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TABLE V (continued)
L-polytopes; first L-type domain

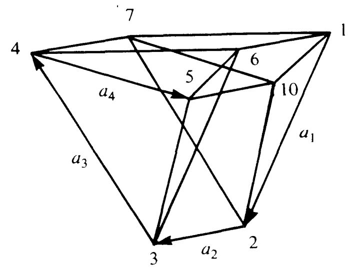
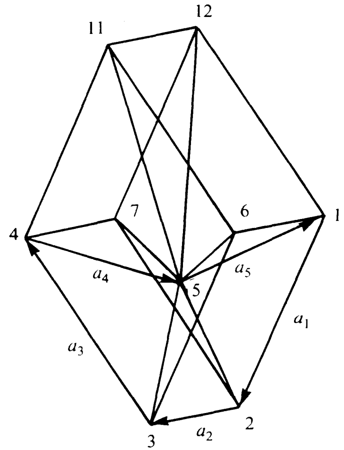
No.	L-Decomp. & No. Vertices	Figure	Simplicial Decomposition	Facets	Description
7.	F_4^6 8		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$ $\langle a_1, a_3, a_2, a_4, a_5\rangle$ $\langle a_1, a_3, a_4, a_2, a_5\rangle$	4: (1, 2, 7, 10) (3, 4, 5, 6) 6_1: (1, 2, 3, 4, 6, 7) (1, 2, 3, 5, 6, 10) (1, 4, 5, 6, 7, 10) (2, 3, 4, 5, 7, 10)	Prism with 3-dimensional simplex as base. Add vertex 10 to fig. 3. Perspective A.
8.	F_1^6 9		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$ $\langle a_1, a_3, a_2, a_4, a_5\rangle$ $\langle a_2, a_3, a_1, a_4, a_5\rangle$ $\langle a_3, a_1, a_2, a_4, a_5\rangle$ $\langle a_3, a_2, a_1, a_4, a_5\rangle$	5: (1, 2, 3, 5, 6) (1, 2, 5, 7, 12) (1, 5, 6, 11, 12) (2, 3, 4, 5, 7) (3, 4, 5, 6, 11) (4, 5, 7, 11, 12) 8: (1, 2, 3, 4, 6, 7, 11, 12)	Pyramid with 3-dimensional box as base and apex 5. Add vertices 11, 12 to fig. 3. Perspective A.

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TABLE V (continued)
L-polytopes; first L-type domain

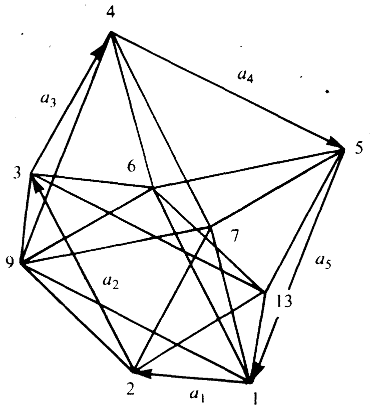
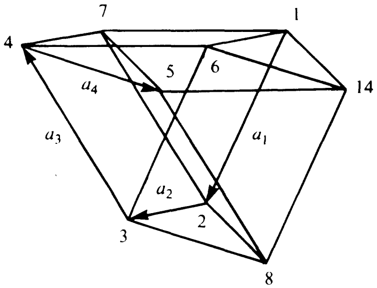
No.	L-Decomp. & No. Vertices	Figure	Simplicial Decomposition	Facets	Description
9.	F_4^5		$\langle a_1, a_2, a_3, a_4, a_5\rangle$	4: (1, 2, 7, 9)	A 3-dimensional octahedron
			$\langle a_2, a_1, a_3, a_4, a_5\rangle$	(1, 5, 6, 13)	(1, 2, 3, 6, 9, 13) with a copy,
	9		$\langle a_1, a_3, a_2, a_4, a_5\rangle$	(3, 4, 6, 9)	(4, 5, 7) of one of its faces
			$\langle a_1, a_2, a_3, a_5, a_4\rangle$	5: (1, 2, 5, 7, 13)	lying above it.
			$\langle a_2, a_1, a_3, a_5, a_4\rangle$	(2, 3, 4, 7, 9)	
			$\langle a_1, a_3, a_2, a_5, a_4\rangle$	(3, 4, 5, 6, 13)	Add vertex 13 to fig. 6.
			$\langle a_1, a_2, a_4, a_3, a_5\rangle$	6_0: (1, 2, 3, 6, 9, 13)	
			$\langle a_2, a_1, a_4, a_3, a_5\rangle$	(1, 4, 5, 6, 7, 9)	Perspective C.
				6_1: (2, 3, 4, 5, 7, 13)	
10.	F_5^5		$\langle a_1, a_2, a_3, a_4, a_5\rangle$	6_1: (1, 2, 3, 4, 6, 7)	The cartesian product of two
			$\langle a_2, a_1, a_3, a_4, a_5\rangle$	(1, 2, 3, 6, 8, 14)	triangles.
	9		$\langle a_1, a_3, a_2, a_4, a_5\rangle$	(1, 2, 5, 7, 8, 14)	Add vertex 14 to fig. 5.
			$\langle a_1, a_2, a_4, a_3, a_5\rangle$	(1, 4, 5, 6, 7, 14)	
			$\langle a_2, a_1, a_4, a_3, a_5\rangle$	(2, 3, 4, 5, 7, 8)	Perspective A.
			$\langle a_2, a_4, a_1, a_3, a_5\rangle$	(3, 4, 5, 6, 8, 14)	

TABLE V (continued)
L-polytopes; first L-type domain

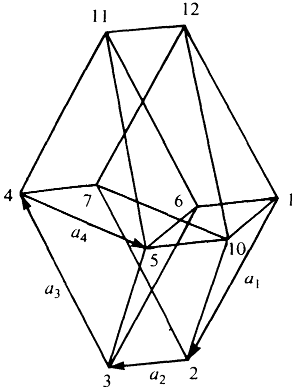
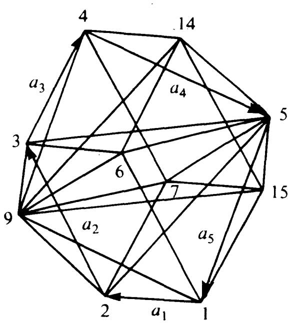
No.	L-Decomp. & No. Vertices	Figure	Simplicial Decomposition	Facets	Description
11.	F_2^5 10		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$ $\langle a_1, a_3, a_2, a_4, a_5\rangle$ $\langle a_2, a_3, a_1, a_4, a_5\rangle$ $\langle a_3, a_1, a_2, a_4, a_5\rangle$ $\langle a_3, a_2, a_1, a_4, a_5\rangle$ $\langle a_1, a_3, a_4, a_2, a_5\rangle$ $\langle a_3, a_1, a_4, a_2, a_5\rangle$	5: (1, 2, 7, 10, 12) (3, 4, 5, 6, 11) 6_1: (1, 2, 3, 5, 6, 10) (1, 5, 6, 10, 11, 12) (2, 3, 4, 5, 7, 10) (4, 5, 7, 10, 11, 12) 8: (1, 2, 3, 4, 6, 7, 11, 12)	Prism with 3-dimensional pyramid as base. Add vertex 10 to fig. 8 or vertices 11, 12 to fig. 7. Perspective A.
12.	F_3^5 10		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$ $\langle a_1, a_3, a_2, a_4, a_5\rangle$ $\langle a_1, a_2, a_3, a_5, a_4\rangle$ $\langle a_2, a_1, a_3, a_5, a_4\rangle$ $\langle a_1, a_3, a_2, a_5, a_4\rangle$ $\langle a_2, a_3, a_1, a_4, a_5\rangle$ $\langle a_3, a_1, a_2, a_4, a_5\rangle$ $\langle a_2, a_3, a_1, a_5, a_4\rangle$ $\langle a_3, a_1, a_2, a_5, a_4\rangle$ $\langle a_3, a_2, a_1, a_4, a_5\rangle$ $\langle a_3, a_2, a_1, a_5, a_4\rangle$	5: (1, 2, 3, 5, 6) (1, 2, 3, 6, 9) (1, 2, 5, 7, 15) (1, 2, 7, 9, 15) (1, 5, 6, 14, 15) (1, 6, 9, 14, 15) (2, 3, 4, 5, 7) (2, 3, 4, 7, 9) (3, 4, 5, 6, 14) (3, 4, 6, 9, 14) (4, 5, 7, 14, 15) (4, 7, 9, 14, 15)	Bipyramid with 3-dimensional box as base. Add vertices 14 & 15 to fig. 6. Perspective C.

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TABLE V (continued)
L-polytopes; first L-type domain

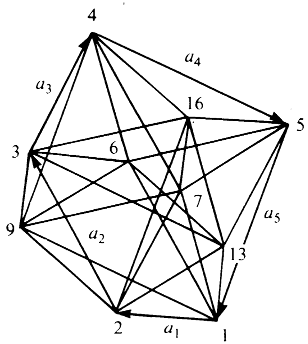
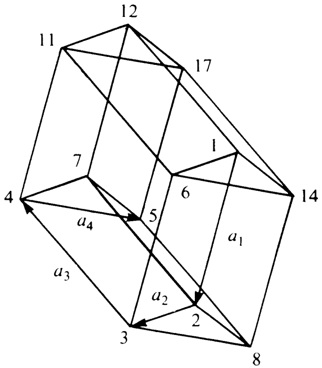
No.	L-Decomp. & No. Vertices	Figure	Simplicial Decomposition	Facets	Description
13.	F_4^4 10		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$ $\langle a_1, a_3, a_2, a_4, a_5\rangle$ $\langle a_1, a_2, a_3, a_5, a_4\rangle$ $\langle a_2, a_1, a_3, a_5, a_4\rangle$ $\langle a_1, a_3, a_2, a_5, a_4\rangle$ $\langle a_1, a_2, a_4, a_3, a_5\rangle$ $\langle a_2, a_1, a_4, a_3, a_5\rangle$ $\langle a_2, a_1, a_4, a_3, a_5\rangle$ $\langle a_5, a_2, a_3, a_4, a_1\rangle_{a_1-a_5}$ $\langle a_5, a_3, a_2, a_4, a_1\rangle_{a_1-a_5}$ $\langle a_5, a_2, a_4, a_3, a_1\rangle_{a_1-a_5}$	4: (1, 2, 7, 9) (1, 5, 6, 13) (3, 4, 6, 9) (2, 3, 13, 16) (4, 5, 7, 16) 6_0: (1, 2, 3, 6, 9, 13) (1, 2, 5, 7, 13, 16) (1, 4, 5, 6, 7, 9) (2, 3, 4, 7, 9, 16) (3, 4, 5, 6, 13, 16)	Octahedron (1, 2, 3, 6, 9, 13), with simplex, (4, 5, 7, 16), lying above it. Combinatorially equivalent to the convex hull of the midpoints of the 10 edges of the 4-simplex. Add vertex 16 to fig. 9. Perspective C.
14.	F_1^4 12		$\langle a_1, a_2, a_3, a_4, a_5\rangle$ $\langle a_2, a_1, a_3, a_4, a_5\rangle$ $\langle a_1, a_3, a_2, a_4, a_5\rangle$ $\langle a_1, a_2, a_4, a_3, a_5\rangle$ $\langle a_2, a_1, a_4, a_3, a_5\rangle$ $\langle a_2, a_4, a_1, a_3, a_5\rangle$ $\langle a_2, a_3, a_1, a_4, a_5\rangle$ $\langle a_3, a_1, a_2, a_4, a_5\rangle$ $\langle a_2, a_4, a_3, a_1, a_5\rangle$ $\langle a_3, a_2, a_1, a_4, a_5\rangle$ $\langle a_2, a_3, a_4, a_1, a_5\rangle$ $\langle a_3, a_2, a_4, a_1, a_5\rangle$	6_1: (1, 2, 3, 6, 8, 14) (1, 6, 11, 12, 14, 17) (2, 3, 4, 5, 7, 8) (4, 5, 7, 11, 12, 17) 8: (1, 2, 3, 4, 6, 7, 11, 12) (1, 2, 5, 7, 8, 12, 14, 17) (3, 4, 5, 6, 8, 11, 14, 17)	Cartesian product of triangle and rectangle. Add vertices 11, 12, 17 to fig. 10. Perspective A.

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TABLE V (continued)
L-polytopes; first L-type domain

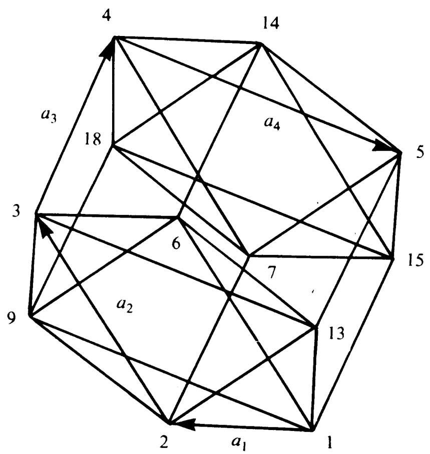
No.	L-Decomp. & No. Vertices	Figure	Simplicial Decomposition	Facets		Description
15.	F_5^4		$\langle a_1, a_2, a_3, a_4, a_5 \rangle$	6_0:	(1, 2, 3, 6, 9, 13)	A prism with 3-dimensional
			$\langle a_2, a_1, a_3, a_4, a_5 \rangle$		(4, 5, 7, 14, 15, 18)	octahedron as base.
	12		$\langle a_1, a_3, a_2, a_4, a_5 \rangle$	6_1:	(1, 2, 5, 7, 13, 15)	
			$\langle a_1, a_2, a_3, a_5, a_4 \rangle$		(1, 2, 7, 9, 15, 18)	Add vertices 13 and 18 to fig.
			$\langle a_2, a_1, a_3, a_5, a_4 \rangle$		(1, 5, 6, 13, 14, 15)	12 or add vertices 14, 15, 18
			$\langle a_1, a_3, a_2, a_5, a_4 \rangle$		(1, 6, 9, 14, 15, 18)	to fig. 9.
			$\langle a_1, a_2, a_4, a_3, a_5 \rangle$		(2, 3, 4, 5, 7, 13)	
			$\langle a_2, a_1, a_4, a_3, a_5 \rangle$		(2, 3, 4, 7, 9, 18)	Perspective C.
			$\langle a_2, a_3, a_1, a_4, a_5 \rangle$		(3, 4, 5, 6, 13, 14)	
			$\langle a_3, a_1, a_2, a_4, a_5 \rangle$		(3, 4, 6, 9, 14, 18)	
			$\langle a_2, a_3, a_1, a_5, a_4 \rangle$			
			$\langle a_3, a_1, a_2, a_5, a_4 \rangle$			
			$\langle a_3, a_2, a_1, a_4, a_5 \rangle$			
			$\langle a_3, a_2, a_1, a_5, a_4 \rangle$			
			$\langle a_4, a_1, a_2, a_5, a_3 \rangle_{a_3 - a_4}$			
			$\langle a_4, a_2, a_1, a_5, a_3 \rangle_{a_3 - a_4}$			

TABLE V (concluded)
L-polytopes; first L-type domain

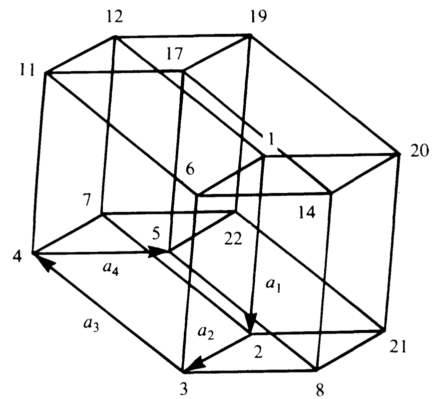
No.	L-Decomp. & No. Vertices	Figure	Simplicial Decomposition	Facets	Description
16.	F_6^3		$\langle a_1, a_2, a_3, a_4, a_5\rangle$	8: (1, 2, 3, 4, 6, 7, 11, 12)	4-dimensional analoque of
	16		$\langle a_1, a_2, a_4, a_3, a_5\rangle$	(1, 2, 3, 6, 8, 14, 20, 21)	box.
			$\langle a_1, a_3, a_2, a_4, a_5\rangle$	(1, 2, 7, 12, 19, 20, 21, 22)	
			$\langle a_1, a_3, a_4, a_2, a_5\rangle$	(1, 6, 11, 12, 14, 17, 19, 20)	Add vertices 19, 20, 21, 22, to
			$\langle a_1, a_4, a_2, a_3, a_5\rangle$	(2, 3, 4, 5, 7, 8, 21, 22)	fig. 14.
			$\langle a_1, a_4, a_3, a_2, a_5\rangle$	(3, 4, 5, 6, 8, 11, 14, 17)	
				(4, 5, 7, 11, 12, 17, 19, 22)	Perspective A.
				(5, 8, 14, 17, 19, 20, 21, 22)	
			$\langle a_2, a_1, a_3, a_4, a_5\rangle$		
			$\langle a_2, a_1, a_4, a_3, a_5\rangle$ $\langle a_3, a_4, a_1, a_2, a_5\rangle$		
			$\langle a_2, a_3, a_1, a_4, a_5\rangle$ $\langle a_3, a_4, a_2, a_1, a_5\rangle$		
			$\langle a_2, a_3, a_4, a_1, a_5\rangle$ $\langle a_4, a_1, a_2, a_3, a_5\rangle$		
			$\langle a_2, a_4, a_1, a_3, a_5\rangle$ $\langle a_4, a_1, a_3, a_2, a_5\rangle$		
			$\langle a_2, a_4, a_3, a_1, a_5\rangle$ $\langle a_4, a_2, a_1, a_3, a_5\rangle$		
			$\langle a_3, a_1, a_2, a_4, a_5\rangle$ $\langle a_4, a_2, a_3, a_1, a_5\rangle$		
			$\langle a_3, a_1, a_4, a_2, a_5\rangle$ $\langle a_4, a_3, a_1, a_2, a_5\rangle$		
			$\langle a_3, a_2, a_1, a_4, a_5\rangle$ $\langle a_4, a_3, a_2, a_1, a_5\rangle$		
			$\langle a_3, a_2, a_4, a_1, a_5\rangle$		

TABLE VI

L-polytopes; second and third L-type domain

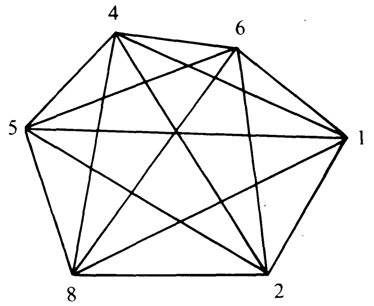
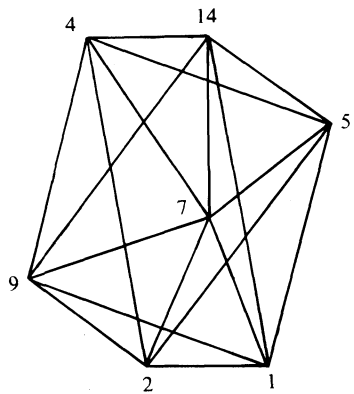
No.	No. Vertices	Figure	Facets	Description
A.	6		4: (1, 2, 4, 5) (1, 2, 4, 6) (1, 2, 5, 8) (1, 2, 6, 8) (1, 4, 5, 6) (1, 5, 6, 8) (2, 4, 5, 8) (2, 4, 6, 8) (4, 5, 6, 8)	The 4-dimensional cyclic polytope P_6 of David Gale [**5**]. It is 2-neighborly. Delete vertex 3 from fig. 4, table V. Perspective B.
B.	7		4: (1, 2, 5, 7) (1, 2, 7, 9) (1, 5, 7, 14) (1, 7, 9, 14) (2, 4, 5, 7) (2, 4, 7, 9) (4, 5, 7, 14) (4, 7, 9, 14) 6_0: (1, 2, 4, 5, 9, 14)	A pyramid with 3-dimensional octahedron as base. Delete vertices 3, 6, 15 from fig. 12, table V. Perspective C.

TABLE VI (concluded)
L-polytopes; second and third *L*-type domain

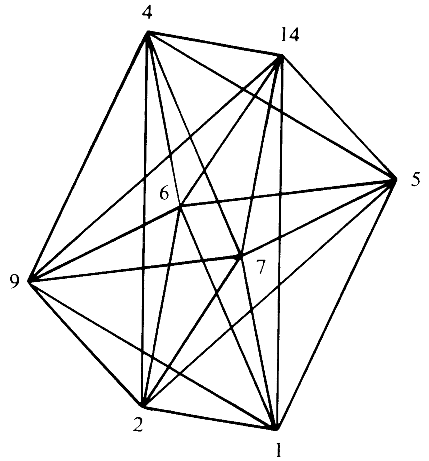
No.	No. Vertices	Figure	Facets	Description
C.	8		4: (1, 2, 5, 6) (4, 5, 6, 14) (1, 2, 5, 7) (4, 5, 7, 14) (1, 2, 6, 9) (4, 6, 9, 14) (1, 2, 7, 9) (4, 7, 9, 14) (1, 5, 6, 14) (1, 5, 7, 14) (1, 6, 9, 14) (1, 7, 9, 14) (2, 4, 5, 6) (2, 4, 5, 7) (2, 4, 6, 9) (2, 4, 7, 9)	A bipyramid with 3-dimensional octahedron as base. The 4-dimensional cross polytope. Add vertex 6 to fig. B. Perspective C.

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TABLE VII
Stars of L-decompositions first L-type domain

L-decomposition	Figure number															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
F^9	120															
F_1^8	60	36														
F_1^7	20	24	28													
F_2^7	40	24		14												
F_2^6	10		28		16											
F_3^6	20	24				16										
F_4^6							48									
F_1^6			72					18								
F_4^5	10		14						18							
F_5^5										36						
F_2^5							16				20					
F_3^5		36										10				
F_4^4	10												20			
F_1^4														24		
F_5^4							16								12	
F_6^3																16

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