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LITTLEWOOD–PALEY CHARACTERIZATION OF WEIGHTED HARDY SPACES ASSOCIATED WITH OPERATORS

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Abstract

Let (X, d, μ) be a metric measure space endowed with a distance d and a nonnegative, Borel, doubling measure μ. Let L be a nonnegative self-adjoint operator on $L^2(X)$. Assume that the (heat) kernel associated to the semigroup e^{-tL} satisfies a Gaussian upper bound. In this paper, we prove that for any $p \in (0, \infty)$ and $w \in A_\infty$, the weighted Hardy space $H^p_{L,S,w}(X)$ associated with L in terms of the Lusin (area) function and the weighted Hardy space $H^p_{L,G,w}(X)$ associated with L in terms of the Littlewood–Paley function coincide and their norms are equivalent. This improves a recent result of Duong *et al.* ['A Littlewood–Paley type decomposition and weighted Hardy spaces associated with operators', *J. Geom. Anal.* **26** (2016), 1617–1646], who proved that $H^p_{L,S,w}(X) = H^p_{L,G,w}(X)$ for $p \in (0, 1]$ and $w \in A_\infty$ by imposing an extra assumption of a Moser-type boundedness condition on L. Our result is new even in the unweighted setting, that is, when $w \equiv 1$.

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1. Introduction

In recent years the study of Hardy spaces associated with operators has attracted a lot of attention. This topic was initiated by Auscher *et al.* [1], who introduced the Hardy space $H^1_L(\mathbb{R}^n)$ associated with an operator L whose heat kernel satisfies a pointwise Poisson upper bound. Later, Duong and Yan [11, 12] introduced BMO-type spaces (spaces of functions of bounded mean oscillation) associated with operators and investigated the duality between $H^1_L(\mathbb{R}^n)$ and $\mathrm{BMO}_{L^*}(\mathbb{R}^n)$, where L^* denotes the adjoint of L in $L^2(\mathbb{R}^n)$. Recently, Auscher *et al.* [2] studied the Hardy space H^1 associated with the Hodge Laplacian on a Riemannian manifold. Meanwhile Hofmann and Mayboroda [21] investigated Hardy spaces associated with a second-order divergence form elliptic operator L on $\mathbb{R}^n$ with complex coefficients. The theory of the Hardy spaces $H^p_L(X)$, $1 \le p < \infty$, on a metric measure space (X, d, μ) associated with a nonnegative self-adjoint operator L satisfying Davies–Gaffney estimates was developed in [20]. For more developments of the theory of Hardy spaces associated with operators, we refer to [3, 8, 13, 22, 23, 30] and the references therein. Although the assumptions on the operators L may vary from one paper to another, mostly the considered operators are of three types: (a) operators satisfying pointwise Gaussian (or Poisson) heat kernel bounds, (b) operators satisfying Davies–Gaffney-type estimates and (c) operators satisfying bounded H_∞ functional calculus. In the case where $L = -\Delta + V$ is a Schrödinger operator with a locally integrable nonnegative potential V, the H^p and BMO spaces associated with L were investigated by Dziubański and Zienkiewicz [15, 16]. Note that such a Schrödinger operator L is a special example of nonnegative self-adjoint operators satisfying Gaussian heat kernel upper bounds. But it should be mentioned that the theory of Hardy spaces associated with such a special L is more satisfactory. For instance, $H^1_{-\Delta+V}(\mathbb{R}^n)$ can be characterized by the generalized Riesz transform $\nabla(-\Delta + V)^{-1/2}$; see [14].

It is also natural to consider weighted Hardy spaces $H^p_{L,w}$ associated with an operator L with an appropriate weight w. This was first done by Song and Yan [26], who introduced the weighted Hardy spaces $H^1_{L,w}(\mathbb{R}^n)$ associated with a Schrödinger operator $L = -\Delta + V$ and proved that the generalized Riesz transform $\nabla L^{-1/2}$ is bounded from $H^1_{L,w}(\mathbb{R}^n)$ to $H^1_w(\mathbb{R}^n)$, where $H^1_w(\mathbb{R}^n)$ is the classical weighted Hardy space introduced by Garcia-Cuerva [17]. Recently, Bui and Duong [4] studied weighted Hardy spaces $H^p_{L,w}(X)$, $0 < p \le 1$, on a metric measure space (X, d, μ) associated with a nonnegative self-adjoint operator L satisfying Davies–Gaffney estimates. The spaces $H^p_{L,w}(X)$ in [4] were defined by means of the Lusin (area) function associated with the heat semigroup generated by L. In a more recent paper [9], Duong *et al.* considered two kinds of weighted Hardy spaces on a metric measure space (X, d, μ) associated with an operator L whose heat kernel satisfies the Gaussian upper bound. One kind is defined by means of the Lusin (area) function associated with the heat semigroup generated by L, while the other kind is defined by means of the Littlewood–Paley function associated with the heat semigroup generated by L. A main contribution in [9] is to establish the equivalence between these two kinds of weighted spaces. However, to achieve this goal, Duong *et al.* in [9] needed to impose an extra assumption of a Moser-type boundedness condition on the operator L.

Our aim in the present paper is to prove that the two kinds of weighted Hardy spaces associated with operators introduced in [9] are equivalent, without assuming the Moser-type boundedness condition on the operator L. Before we state our main result, let us fix our setting. Let (X, d, μ) be a metric measure space, that is, d is a distance on a set X and μ is a Borel measure with respect to the topology induced by the distance d. We always assume that $\mu(X) = \infty$. Let $B(x, r)$ denote the open ball with center $x \in X$ and radius $r > 0$, and set $V(x, r) = \mu(B(x, r))$, the volume of $B(x, r)$. We often just use B instead of $B(x, r)$. We assume that the metric measure space (X, ρ, μ) satisfies the doubling condition, that is, there exists a constant $C > 0$ such that

$$V(x, 2r) \le CV(x, r) \tag{1.1}$$

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for all $x \in X$ and $r > 0$. Recall that a weight $w \geq 0$ on X is said to belong to the Muckenhoupt class A_p for a given p, $1 < p < \infty$, if

$$\left(\frac{1}{\mu(B)}\int_B w(x)\,d\mu(x)\right)\left(\frac{1}{\mu(B)}\int_B w^{-1/(p-1)}(x)\,d\mu(x)\right)^{p-1} \leq C$$

with the constant C independent of the ball B. The class A_1 is defined by letting $p \to 1$, that is, for every ball $B \subset X$,

$$\left(\frac{1}{\mu(B)}\int_B w(x)\,d\mu(x)\right)\|w^{-1}\|_{L^\infty(B)} \leq C$$

with the constant C independent of the ball B. Let $A_\infty := \bigcup_{1\leq p<\infty} A_p$ and, for any $w \in A_\infty$, define

$$q_w := \inf\{q \in [1,\infty) : w \in A_p\},$$

the critical index of w. For $1 < p < \infty$, the weighted Lebesgue space $L^p_w(X)$ is defined to be the space of all Lebesgue measurable functions f for which

$$\|f\|_{L^p_w(X)} := \int_X |f(x)|^p w(x)\,d\mu(x) < \infty.$$

In the present paper, we assume that L is a densely defined operator on $L^2(X)$ satisfying the following two properties:

(H1) L is a nonnegative self-adjoint operator on $L^2(X)$;

(H2) the kernel of e^{-tL}, denoted by $p_t(x,y)$, is a measurable function on $X \times X$ and satisfies a Gaussian upper bound, that is,

$$|p_t(x,y)| \leq \frac{C}{V(x,\sqrt{t})}\exp\left(-\frac{d^2(x,y)}{ct}\right) \tag{1.2}$$

for all $t > 0$ and $x, y \in X$, where C and c are positive constants.

Given an operator L satisfying (H1)–(H2) and a function $f \in L^2(X)$, consider the following Lusin (area) function $S_L(f)$ and Littlewood–Paley function $G_L(f)$ associated with the heat semigroup generated by L:

$$S_L(f)(x) := \left(\int_0^\infty \int_{d(y,x)<t} |t^2Le^{-t^2L}f(y)|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\right)^{1/2}$$

and

$$G_L(f)(x) := \left(\int_0^\infty |t^2Le^{-t^2L}f(x)|^2\frac{dt}{t}\right)^{1/2}.$$

We shall be concerned with the weighted Hardy spaces $H^p_{L,S,w}(X)$ and $H^p_{L,S,w}(X)$ for $0 < p < \infty$ and $w \in A_\infty$, which we define by means of the Lusin (area) function and the Littlewood–Paley function as follows.

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Definition 1.1. Let L be an operator satisfying (H1)–(H2) and $w \in A_\infty$.

(i) The Hardy space $H^p_{L,S,w}(X)$, $0 < p < \infty$, is defined as the completion of $\{f \in L^2(X) : \|S_L f\|_{L^p_w(X)} < \infty\}$ with norm

$$\|f\|_{H^p_{L,S,w}(X)} := \|S_L f\|_{L^p_w(X)}.$$

(ii) The Hardy space $H^p_{L,G,w}(X)$, $0 < p < \infty$, is defined as the completion of $\{f \in L^2(X) : \|G_L f\|_{L^p_w(X)} < \infty\}$ with norm

$$\|f\|_{H^p_{L,G,w}(X)} := \|G_L f\|_{L^p_w(X)}.$$

Under the assumptions (H1)–(H2) of L, Duong *et al.* [9] proved that $H^p_{L,S,w}(X) \subset H^p_{L,G,w}(X)$ for $p \in (0, 1]$ and $w \in A_\infty$. However, in order to show the converse inclusion $H^p_{L,G,w}(X) \subset H^p_{L,S,w}(X)$, they needed to impose the following extra assumption on the operator L:

(H3) there exists some $0 < q < 1$ such that every solution $u(x, t)$ to the equation

$$\widetilde{L}u := -u_{tt} + Lu = 0$$

on $X \times (0, \infty)$ satisfies the following estimate for each ball $B(Y_0, r) \subset X\times$:

$$\sup_{Y\in B(Y_0,r)} |u(Y)| \le C\Big(\frac{1}{V(y_0, r)r} \int_{B(Y_0,2r)} |u(Y)|^q \, dY\Big)^{1/q}$$

for any $Y_0 = (y_0, t_0) \in X \times (0, \infty)$ and $0 < r < t_0/2$.

In [9], the assumption (H3) is called the Moser-type boundedness condition. Under the assumptions (H1)–(H3) of L, Duong *et al.* [9] showed that $H^p_{L,G,w}(X) \subset H^p_{L,S,w}(X)$ for $p \in (0, 1]$ and $w \in A_\infty$. However, as mentioned in [9], it was not clear whether or not $H^p_{L,G,w}(X) \subset H^p_{L,S,w}(X)$ when L merely satisfies (H1)–(H2). The aim of the present paper is to give an affirmative answer to this question. Our main result can be stated as follows.

Theorem 1.2. *Assume that L satisfies* (H1)–(H2). *Let $0 < p < \infty$ and $w \in A_\infty$. Then the spaces $H^p_{L,S,w}(X)$ and $H^p_{L,G,w}(X)$ coincide and their norms are equivalent.*

As noted in [9], under the assumptions (H1)–(H2), the Lusin function $S_L(f)$ and the Littlewood–Paley function $G_L(f)$ do not satisfy the standard regularity of the so-called Calderón–Zygmund operators; thus, standard techniques of Calderón–Zygmund theory [7, 29] are not applicable. The lack of smoothness of the kernel was indeed the main difficulty of our problem. To overcome this obstacle, we shall establish a generalized sub-mean-value inequality (see Lemma 3.4 below), which is inspired by the ideas of Bui *et al.* [5, 6].

The layout of the paper is as follows. In Section 2, we recall some basic facts and known results. In Section 3, we give the proof of Theorem 1.2.

Throughout, the letters 'c' and 'C' will denote positive constants which are independent of the essential variables involved, but whose values may vary from one occurrence to the next. By writing $a \approx b$, we mean that the variables a and b are equivalent, that is, there exist two positive constants C_1 and C_2 independent of a and b such that $C_1 a \le b \le C_2 a$.

2. Preliminaries

In this section we recall some basic facts and known results which will be needed in the next section.

First note that the doubling condition (1.1) implies that there exist constants C and n such that for all $x \in X$, $r > 0$ and $\lambda \geq 1$,

$$V(x, \lambda r) \leq C\lambda^n V(x, r). \tag{2.1}$$

The constant n plays the role of a dimension, though it need not even be an integer. In the sequel we want to consider n as small as possible. Note that in general one cannot take the infimum over such exponents n in (2.1). There also exist constants C and D, $0 \leq D \leq n$, so that

$$V(y, r) \leq C\Big(1 + \frac{d(x, y)}{r}\Big)^D V(x, r) \tag{2.2}$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (2.2) with $D = n$ is a direct consequence of (2.1) and the triangle inequality for the metric d. In the cases of the Euclidean spaces and Lie groups of polynomial growth, D can be chosen to be 0. Using the doubling condition (1.1), it is easy to show that for any $N > n$, there exists a constant C (depending on N) such that for all $x \in X$ and $t > 0$,

$$\int_X (1 + t^{-1}d(x, y))^{-N}\, d\mu(y) \leq CV(x, t). \tag{2.3}$$

The following lemma is standard and thus we skip the proof.

Lemma 2.1. *Suppose that $N > n + D$. Then there exists a constant $C > 0$ such that for all measurable functions f on X, $t > 0$ and $x \in X$,*

$$\int_X \frac{|f(y)|}{V(y, t)(1 + t^{-1}d(x, y))^N}\, d\mu(y) \leq C\mathcal{M}(f)(x),$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator on (X, d, μ) defined by

$$\mathcal{M}(f)(x) := \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y).$$

Given a weight w on X, $0 < p \leq \infty$ and $0 < q \leq \infty$, we denote by $L^p_w(\ell^q)$ the space of all sequences $\{f_j\}_j$ of measurable functions on X such that

$$\|\{f_j\}_j\|_{L^p_w(\ell^q)} := \|\|\{f_j(\cdot)\}_j\|_{\ell^q}\|_{L^p_w(X)} < \infty.$$

Lemma 2.2 [18, Section 6.6]. *Suppose that $1 < p < \infty$, $1 < q \leq \infty$ and $w \in A_p$. Then there exists a constant $C > 0$ such that*

$$\|\{\mathcal{M}(f_j)\}_j\|_{L^p_w(\ell^q)} \leq C\|\{f_j\}_j\|_{L^p_w(\ell^q)}.$$

Given $f \in L^2(X)$, $\alpha > 0$ and $x \in X$, we define

$$\mathcal{G}^*_{\alpha,L}(f)(x) := \Big(\int_0^\infty \int_X \Big(\frac{t}{t + d(x, y)}\Big)^{n\alpha} |t^2Le^{-t^2L}f(y)|^2 \frac{d\mu(y)}{V(x, t)} \frac{dt}{t}\Big)^{1/2}.$$

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Lemma 2.3. *Assume that L satisfies* (H1)–(H2). *Let $w \in A_\infty$, $0 < p < \infty$ and $\alpha > 2q_w/\min\{p, 2\}$. Then there exists a constant $C > 0$ such that for all $f \in L^2(X)$,*

$$\|\mathcal{G}^*_{\alpha,L}(f)\|_{L^p_w(X)} \le C\|S_L(f)\|_{L^p_w(X)}.$$

Proof. The proof is standard; we refer the reader to [28, Theorem 4 in Ch. 4] and [19, Lemma 3.2]. □

Let $E(\lambda)$ be the spectral resolution of L. For any bounded Borel measurable function $F : [0, \infty) \to \mathbb{C}$, by the spectral theory we can define the operator

$$F(L) = \int_0^\infty F(\lambda)\, dE(\lambda),$$

which is bounded on $L^2(X)$. The following result will be important to us.

Lemma 2.4 [10, Lemma 4.3]. *Assume that L satisfies* (H1)–(H2). *Let $R > 0$ and $s > 0$. For any $\varepsilon > 0$, there exists a constant $C = C(s, \varepsilon)$ such that*

$$\int_X |K_{F(\sqrt{L})}(x, y)|^2 (1 + R\, d(x, y))^s\, d\mu(x) \le \frac{C}{V(y, R^{-1})} \|\delta_R F\|^2_{W^\infty_{(s/2)+\varepsilon}(\mathbb{R})}$$

for all even functions $F \in W^\infty_{(s/2)+\varepsilon}(\mathbb{R})$ such that $\operatorname{supp} F \subset [-R, R]$, *where $K_{F(\sqrt{L})}(x, y)$ is the kernel of the operator $F(t\sqrt{L})$, $\delta_R F(\lambda) := F(R\lambda)$ and $\|F\|_{W^\infty_s(\mathbb{R})} := \|(I - d^2/d\lambda^2)^{s/2} F\|_{L^\infty(\mathbb{R})}$.*

Note that Lemma 2.4 is slightly different from [10, Lemma 4.3], in which the function F is required to be supported in $[R/4, R]$. But a careful examination of the proof of [10, Lemma 4.3] shows that the assertion is still true if we assume that F is even and $\operatorname{supp} F \subset [-R, R]$. See also [24, Theorem 7.18].

We will also need the following two fundamental results.

Lemma 2.5 [25, Lemma 2]. *Let w be an arbitrary weight on X, $0 < p, q \le \infty$ and $\delta > 0$. Let $\{g_j\}_{j=-\infty}^\infty$ be a sequence of nonnegative measurable functions on X and put*

$$h_\ell(x) = \sum_{j=-\infty}^{\infty} 2^{-|j-\ell|\delta} g_j(x), \quad x \in X, \ell \in \mathbb{Z}.$$

Then there exists a constant $C = C(p, q, \delta)$ such that

$$\|\{h_\ell\}_{\ell=-\infty}^\infty\|_{L^p_w(\ell^q)} \le C\|\{g_j\}_{j=-\infty}^\infty\|_{L^p_w(\ell^q)}.$$

Lemma 2.6 [25, Lemma 3]. *Let $0 < r \le 1$, and let $\{b_j\}_{j=-\infty}^\infty$ and $\{d_\ell\}_{\ell=-\infty}^\infty$ be two sequences taking values in respectively $(0, \infty]$ and $(0, \infty)$. Assume that there exists $N_0 > 0$ such that $d_\ell = O(2^{\ell N_0})$ as $\ell \to \infty$, and that for every $N > 0$ there exists a finite constant C_N such that*

$$d_\ell \le C_N \sum_{j=\ell}^{\infty} 2^{-(j-\ell)N} b_j d_\ell^{1-r}, \quad \ell \in \mathbb{Z}.$$

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Then, for every $N > N_0$,

$$d_\ell^r \le C_N \sum_{j=\ell}^{\infty} 2^{-(j-\ell)Nr} b_j, \quad \ell \in \mathbb{Z},$$

with the same constants C_N.

3. Proof of Theorem 1.2

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz class on $\mathbb{R}$. For $M \in \mathbb{N}$ and $\Phi \in \mathcal{S}(\mathbb{R})$, we set

$$\|\Phi\|_{\mathcal{S}_M(\mathbb{R})} := \sup_{0\le\nu\le M} \sup_{\lambda\in\mathbb{R}} (1 + |\lambda|)^{M+2n+1} |\Phi^{(\nu)}(\lambda)|,$$

where $\Phi^{(\nu)}$ is the νth-order derivative of Φ. We have the following smooth functional calculus.

Lemma 3.1. *Assume that L satisfies* (H1)–(H2). *Then, for any $N > 0$, there exists a constant $C > 0$ such that for every even function $\Phi \in \mathcal{S}(\mathbb{R})$, the kernel $K_{\Phi(t\sqrt{L})}(x, y)$ of the operator $\Phi(t\sqrt{L})$ satisfies the estimate*

$$|K_{\Phi(t\sqrt{L})}(x, y)| \le C\|\Phi\|_{\mathcal{S}_M(\mathbb{R})} V(x, t)^{-1}(1 + t^{-1}d(x, y))^{-N},$$

where M is the smallest even integer that satisfies $M > N + n + 1$.

Proof. First we show that, for any $N > 0$ and $m > n/2$, there exists a constant $C = C(N, m)$ such that

$$|K_{(I+t^2L)^{-m}}(x, y)| \le CV(x, t)^{-1}(1 + t^{-1}d(x, y))^{-N}. \tag{3.1}$$

Indeed, by the formula

$$(I + t^2L)^{-m} = \frac{1}{\Gamma(m)} \int_0^\infty e^{-ut^2L} e^{-u} u^{m-1}\, du$$

and the Gaussian upper bound (1.2),

$$\begin{aligned}
|K_{(I+t^2L)^{-m}}(x, y)| &\le C \int_0^\infty |K_{e^{-ut^2L}}(x, y)| e^{-u} u^{m-1}\, du \\
&\le C \int_0^\infty \frac{1}{V(x, u^{1/2}t)} \exp\Big(-\frac{d^2(x, y)}{cut^2}\Big) e^{-u} u^{m-1}\, du \\
&\le C \int_0^\infty \frac{1}{V(x, u^{1/2}t)} \Big(1 + \frac{d(x, y)}{u^{1/2}t}\Big)^{-N} e^{-u} u^{m-1}\, du.
\end{aligned}$$

From this, the fundamental inequality

$$\Big(1 + \frac{d(x, y)}{u^{1/2}t}\Big)^{-N} \le \Big(1 + \frac{d(x, y)}{t}\Big)^{-N} (1 + u^{1/2})^N$$

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and the fact that

$$V(x,t) = V(x, u^{-1/2}u^{1/2}t) \le C(1+u^{-1/2})^n V(x, u^{1/2}t)$$

(which is a consequence of (2.1)),

$$\begin{aligned}
&|K_{(I+t^2L)^{-m}}(x,y)| \\
&\quad \le \frac{C}{V(x,t)}\Big(1+\frac{d(x,y)}{t}\Big)^{-N}\int_0^\infty (1+u^{-1/2})^n(1+u^{1/2})^N e^{-u}u^{m-1}\,du \\
&\quad \le \frac{C}{V(x,t)}\Big(1+\frac{d(x,y)}{t}\Big)^{-N},
\end{aligned}$$

which verifies (3.1).

Since $\Phi(t\sqrt{L}) = (I+t^2L)^m\Phi(t\sqrt{L})(I+t^2L)^{-m}$,

$$K_{\Phi(t\sqrt{L})}(x,y) = \int_X K_{(I+t^2L)^m\Phi(t\sqrt{L})}(x,z)K_{(I+t^2L)^{-m}}(z,y)\,d\mu(z).$$

Hence, for any $N>0$, by (3.1) and the fundamental inequality

$$(1+t^{-1}d(x,y))^N(1+t^{-1}d(z,y))^{-N} \le (1+t^{-1}d(x,z))^N,$$

$$\begin{aligned}
&V(x,t)(1+t^{-1}d(x,y))^N|K_{\Phi(t\sqrt{L})}(x,y)| \\
&\qquad \le C\int_X |K_{(I+t^2L)^m\Phi(t\sqrt{L})}(x,z)|(1+t^{-1}\,d(x,z))^N\,d\mu(z). \qquad (3.2)
\end{aligned}$$

Let $\varphi_0 \in C_0^\infty(\mathbb{R})$ be an even function such that $\varphi_0(\lambda)=1$ on $\{|\lambda|\le 1/2\}$ and $\varphi_0(\lambda)=0$ on $\{|\lambda|\ge 1\}$. Define $\varphi(\lambda)=\varphi_0(\lambda)-\varphi_0(2\lambda)$, $\lambda\in\mathbb{R}$. Then $\operatorname{supp}\varphi \subset \{1/4\le|\lambda|\le 1\}$, and $\varphi_0(\lambda)+\sum_{\ell=1}^\infty \varphi(2^{-\ell}\lambda)=1$ for $\lambda\in\mathbb{R}$. Let $m\in(n/2,3n/4)$, and set $F^0(\lambda)=\varphi_0(\lambda)(1+\lambda^2)^m\Phi(\lambda)$ and $F^\ell(\lambda)=\varphi(2^{-\ell}\lambda)(1+\lambda^2)^m\Phi(\lambda)$, $\ell=1,2,\ldots$. Then $\operatorname{supp}F^0(t\cdot)\subset[-t^{-1},t^{-1}]$, $\operatorname{supp}F^\ell(t\cdot)\subset[-2^\ell t^{-1},2^\ell t^{-1}]$, $F^0(t\sqrt{L})=\varphi_0(t\sqrt{L})(1+t^2L)^m\Phi(t\sqrt{L})$,

$$F^\ell(t\sqrt{L}) = \varphi(2^{-\ell}t\sqrt{L})(1+t^2L)^m\Phi(t\sqrt{L}), \quad \ell=1,2,\ldots$$

and

$$K_{(I+t^2L)^m\Phi(t\sqrt{L})}(x,z) = \sum_{\ell=0}^\infty K_{F^\ell(t\sqrt{L})}(x,z). \qquad (3.3)$$

By the Cauchy–Schwartz inequality, (2.3), Lemma 2.4 and (2.1), we have, for $\ell=1,2,\ldots,$

$$\begin{aligned}
&\int_X |K_{F^\ell(t\sqrt{L})}(x,z)|(1+t^{-1}d(x,z))^N d\mu(z) \\
&\quad \le C\Big(\int_X |K_{F^\ell(t\sqrt{L})}(x,z)|^2(1+2^\ell t^{-1}d(x,z))^{2(N+n+1)}d\mu(z)\Big)^{1/2}V(x,t)^{1/2} \\
&\quad \le CV(x,2^{-\ell}t)^{-1/2}V(x,t)^{1/2}\|\delta_{2^\ell t^{-1}}[F^\ell(t\cdot)]\|^2_{W^\infty_{N+n+1+\varepsilon}(\mathbb{R})} \\
&\quad \le C2^{\ell n/2}\|\delta_{2^\ell t^{-1}}[F^\ell(t\cdot)]\|^2_{W^\infty_{N-n-1+\varepsilon}(\mathbb{R})} \\
&\quad = C2^{\ell n/2}\|\lambda\mapsto\varphi(\lambda)(1+2^{2\ell}\lambda^2)^m\Phi(2^\ell\lambda)\|^2_{W^\infty_{N+n+1+\varepsilon}(\mathbb{R})}. \qquad (3.4)
\end{aligned}$$

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Recall that if s is an even integer, then $\|F\|_{W_s^\infty(\mathbb{R}^n)} \approx \sum_{k=0}^{s} \|d^k F/d\lambda^k\|_{L^\infty(\mathbb{R})}$; see, for example, [27, Ch. V, Section 6.6]. Since M is the smallest even integer that satisfies $M > N + n + 1$, by taking ε sufficiently small we have $M > N + n + 1 + \varepsilon$ and thus

$$\begin{aligned}
&\|\lambda \mapsto \varphi(\lambda)(1 + 2^{2\ell}\lambda^2)^m \Phi(2^\ell \lambda)\|_{W^\infty_{N+n+1+\varepsilon}(\mathbb{R})} \\
&\quad \leq \|\lambda \mapsto \varphi(\lambda)(1 + 2^{2\ell}\lambda^2)^m \Phi(2^\ell \lambda)\|_{W^\infty_M(\mathbb{R})} \\
&\quad \leq C\|\Phi\|_{\mathcal{S}_M(\mathbb{R})} 2^{\ell M} \sup_{1/4 \leq |\lambda| \leq 1} (1 + 2^\ell |\lambda|)^{2m-(M+2n+1)} \\
&\quad \leq C\|\Phi\|_{\mathcal{S}_M(\mathbb{R})} 2^{\ell(2m-2n-1)}.
\end{aligned}$$

Substituting this into (3.4) yields

$$\begin{aligned}
&\int_X |K_{F^\ell(t\sqrt{L})}(x,z)|(1 + t^{-1}d(x,z))^N \, d\mu(z) \\
&\quad \leq C\|\Phi\|_{\mathcal{S}_M(\mathbb{R})} 2^{-\ell[(3n/2)+1-2m]}.
\end{aligned} \tag{3.5}$$

Analogously,

$$\int_X |K_{F^0(t\sqrt{L})}(x,z)|(1 + t^{-1}d(x,z))^N \, d\mu(z) \leq C\|\Phi\|_{\mathcal{S}_M(\mathbb{R})}. \tag{3.6}$$

Since $m \in (n/2, 3n/4)$, we have $(3n/2) + 1 - 2m > 0$. Hence, combining (3.2), (3.3), (3.5) and (3.6),

$$\begin{aligned}
&V(x,t)(1 + t^{-1}d(x,y))^N |K_{\Phi(t\sqrt{L})}(x,y)| \\
&\quad \leq C \sum_{\ell=0}^{\infty} \int_X |K_{F^\ell(t\sqrt{L})}(x,z)|(1 + t^{-1}d(x,z))^N \, d\mu(z) \leq C\|\Phi\|_{\mathcal{S}_M(\mathbb{R})},
\end{aligned}$$

which readily implies the desired estimate. □

Lemma 3.2. *Assume that L satisfies* (H1)–(H2). *Suppose that $\Phi, \Psi \in \mathcal{S}(\mathbb{R})$ are even functions, and that*

$$\Psi^{(\nu)}(0) = 0, \quad \nu = 0, 1, \ldots, 2\kappa \tag{3.7}$$

for some positive integer κ. Then, for any $N > 0$, there exists a constant C such that for all $j, \ell \in \mathbb{Z}$ with $j \geq \ell$,

$$\begin{aligned}
|K_{\Phi(2^{-\ell}\sqrt{L})\Psi(2^{-j}\sqrt{L})}(x,y)| &\leq C\|\lambda \mapsto \lambda^{2\kappa}\Phi(\lambda)\|_{\mathcal{S}_{M_1}(\mathbb{R})} \|\lambda \mapsto \lambda^{-2\kappa}\Psi(\lambda)\|_{\mathcal{S}_{M_2}(\mathbb{R})} \\
&\qquad \times 2^{-2(j-\ell)\kappa} V(x, 2^{-\ell})^{-1}(1 + 2^\ell d(x,y))^{-N},
\end{aligned}$$

where M_1 (respectively M_2) is the smallest even integer that satisfies $M_1 > N + n + 1$ (respectively $M_2 > N + 2n + D + 2$).

Proof. Since

$$\Phi(2^{-\ell}\sqrt{L})\Psi(2^{-j}\sqrt{L}) = 2^{-2(j-\ell)\kappa}[(2^{-2\ell}L)^\kappa \Phi(2^{-\ell}\sqrt{L})][(2^{-2j}L)^{-\kappa}\Psi(2^{-j}\sqrt{L})],$$

$$
\begin{aligned}
&K_{\Phi(2^{-\ell}\sqrt{L})\Psi(2^{-j}\sqrt{L})}(x,y)\\
&\quad = 2^{-2(j-\ell)\kappa}\int_X K_{(2^{-2\ell}L)^{\kappa}\Phi(2^{-\ell}\sqrt{L})}(x,z)K_{(2^{-2j}L)^{-\kappa}\Psi(2^{-j}\sqrt{L})}(z,y)\,d\mu(z).
\end{aligned}
$$

The condition (3.7) implies that the function $\lambda\mapsto\lambda^{-2\kappa}\Psi(\lambda)$ is smooth at 0 and belongs to $\mathcal{S}(\mathbb{R})$. Hence, by Lemma 3.1,

$$
\begin{aligned}
&|K_{(2^{-2\ell}L)^{\kappa}\Phi(2^{-\ell}\sqrt{L})}(x,z)|\\
&\quad\le C\|\lambda\mapsto\lambda^{2\kappa}\Phi(\lambda)\|_{\mathcal{S}_{M_1}(\mathbb{R})}V(x,2^{-\ell})^{-1}(1+2^{\ell}d(x,z))^{-N}
\end{aligned}
$$

and

$$
\begin{aligned}
&|K_{(2^{-2j}L)^{-\kappa}\Psi(2^{-j}\sqrt{L})}(z,y)|\\
&\quad\le C\|\lambda\mapsto\lambda^{-2\kappa}\Psi(\lambda)\|_{\mathcal{S}_{M_2}(\mathbb{R})}V(z,2^{-j})^{-1}(1+2^{j}d(z,y))^{-(N+n+D+1)},
\end{aligned}
$$

where M_1 (respectively M_2) is the smallest even integer that satisfies $M_1>N+n+1$ (respectively $M_2>N+2n+D+2$). From all above and the inequality $V(z,2^{-j})^{-1}\le(1+2^{j}d(z,y))^{D}V(y,2^{-j})^{-1}$ (which is a consequence of (2.2)),

$$
\begin{aligned}
&|K_{\Phi(2^{-\ell}\sqrt{L})\Psi(2^{-j}\sqrt{L})}(x,y)|\\
&\quad\le C2^{-2(j-\ell)\kappa}\|\lambda\mapsto\lambda^{2\kappa}\Phi(\lambda)\|_{\mathcal{S}_{M_1}(\mathbb{R})}\|\lambda\mapsto\lambda^{-2\kappa}\Psi(\lambda)\|_{\mathcal{S}_{M_2}(\mathbb{R})}V(x,2^{-\ell})^{-1}\\
&\quad\quad\times V(y,2^{-j})^{-1}\int_X(1+2^{\ell}d(x,z))^{-N}(1+2^{j}d(z,y))^{-(N+n+1)}d\mu(z). \qquad (3.8)
\end{aligned}
$$

By the inequality

$$
(1+2^{\ell}d(x,z))^{-N}(1+2^{j}d(z,y))^{-N}\le(1+2^{\ell}d(x,y))^{-N}\quad(j\ge\ell)
$$

and (2.3),

$$
\int_X(1+2^{\ell}\,d(x,z))^{-N}(1+2^{j}\,d(z,y))^{-(N+n+1)}\,d\mu(z)\le CV(y,2^{-j})(1+2^{\ell}\,d(x,y))^{-N}.
$$

Substituting this into (3.8) yields the desired estimate. □

For our purpose we introduce a Fefferman–Stein-type maximal function. Given $f\in L^2(X)$, $\alpha>0$ and $(x,t)\in X\times(0,\infty)$, define

$$
M^*_{\alpha,L}(f)(x,t):=\operatorname*{ess\,sup}_{y\in X}\frac{|t^2Le^{-t^2L}f(y)|}{(1+t^{-1}\,d(x,y))^{\alpha}}.
$$

LEMMA 3.3. *Let w be an arbitrary weight on X, $0<p<\infty$ and $\alpha>0$. Then there exists a constant $C>0$ such that for all $f\in L^2(X)$,*

$$
\|S_L(f)\|_{L^p_w(X)}\le C\bigg\|\bigg(\int_0^{\infty}[M^*_{\alpha,L}(f)(\cdot,t)]^2\frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^p_w(X)}.
$$

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Proof. Observe that for all $\alpha > 0$, $t > 0$ and $x \in X$,

$$\frac{1}{V(x,t)} \int_{B(x,t)} |t^2 L e^{-t^2 L} f(y)|^2 \, d\mu(y) \\ \leq \operatorname*{ess\,sup}_{y \in B(x,t)} |t^2 L e^{-t^2 L} f(y)|^2 \leq 2^{n\alpha} [M^*_{\sigma,L}(f)(x,t)]^2.$$

Applying the norm $\int_0^\infty |\cdot| dt/t$ on both sides gives the pointwise estimate

$$[S_L(f)(x)]^2 \leq 2^{n\alpha} \int_0^\infty [M^*_{\alpha,L}(f)(x,t)]^2 \frac{dt}{t},$$

which readily yields the desired estimate. □

Lemma 3.4. *Assume that L satisfies* (H1)–(H2). *Then, for any $\mu > 0$, $r > 0$ and $\alpha > D/2$, there exists a positive constant C such that for all $f \in L^2(X)$, $\ell \in \mathbb{Z}$, $x \in X$ and $t \in [1, 2]$,*

$$[M^*_{\alpha,L}(f)(x, 2^{-\ell} t)]^r \\ \leq C \sum_{j=\ell}^{\infty} 2^{-(j-\ell)\mu r} \int_X \frac{|(2^{-j}t)^2 L e^{-(2^{-j}t)^2 L} f(z)|^r}{V(z, 2^{-\ell})(1 + 2^\ell d(x,z))^{\alpha r}} \, d\mu(z). \tag{3.9}$$

Proof. We follow the ideas developed by Bui *et al.* [5, 6], which were simplified by Rychkov [25]. Let $\Phi_0(\lambda) := e^{-\lambda^2}$, $\Phi(\lambda) := \lambda^2 e^{-\lambda^2}$ for $\lambda \in \mathbb{R}$. Then

$$(2^{-j}t)^2 L e^{-(2^{-j}t)^2 L} f = \Phi(2^{-j} t \sqrt{L}) f.$$

Let $\eta_0, \eta \in \mathcal{S}(\mathbb{R})$ be nonnegative even functions such that $|\eta_0(\lambda)| \neq 0 \iff |\lambda| < 2$ and $|\eta(\lambda)| \neq 0 \iff 1/2 < |\lambda| < 2$. Then set $\omega(\lambda) := \eta_0(\lambda)\Phi_0(\lambda) + \sum_{\ell=1}^\infty \eta(2^{-\ell}\lambda)\Phi(2^{-\ell}\lambda)$, $\lambda \in \mathbb{R}$. Finally, let $\Psi_0(\lambda) := \eta_0(\lambda)/\omega(\lambda)$, $\Psi(\lambda) := \eta(\lambda)/\omega(\lambda)$, $\lambda \in \mathbb{R}$. Then Ψ_0, Ψ are even Schwartz functions on $\mathbb{R}$, $\operatorname{supp} \Psi_0 \subset [-2, 2]$, $\operatorname{supp} \Psi \subset [-2, -1/2] \cup [1/2, 2]$ and

$$\Phi_0(\lambda)\Psi_0(\lambda) + \sum_{j=1}^\infty \Phi(2^{-j}\lambda)\Psi(2^{-j}\lambda) = 1, \quad \forall \lambda \in \mathbb{R}. \tag{3.10}$$

Replacing λ with $2^{-\ell} t\lambda$ in (3.10), we see that for all $\ell \in \mathbb{Z}$ and $t \in [1, 2]$,

$$\Phi_0(2^{-\ell} t\lambda)\Psi_0(2^{-\ell} t\lambda) + \sum_{j=1}^\infty \Phi(2^{-(j+\ell)} t\lambda)\Psi(2^{-(j+\ell)} t\lambda) = 1, \quad \forall \lambda \in \mathbb{R}.$$

It then follows from the spectral theory that for all $f \in L^2(X)$, $\ell \in \mathbb{Z}$ and $t \in [1, 2]$,

$$f = \Phi_0(2^{-\ell} t \sqrt{L})\Psi_0(2^{-\ell} t \sqrt{L}) f + \sum_{j=1}^\infty \Phi(2^{-(j+\ell)} t \sqrt{L})\Psi(2^{-(j+\ell)} t \sqrt{L}) f$$

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with convergence in the sense of $L^2(X)$ norm. Hence, for almost every $y \in X$,

$$\begin{aligned}
&\Phi(2^{-\ell} t\sqrt{L})f(y) \\
&\quad= \Phi_0(2^{-\ell} t\sqrt{L})\Psi_0(2^{-\ell} t\sqrt{L})\Phi(2^{-\ell} t\sqrt{L})f(y) \\
&\qquad+ \sum_{j=1}^{\infty} \Phi(2^{-\ell} t\sqrt{L})\Psi(2^{-(j+\ell)} t\sqrt{L})\Phi(2^{-(j+\ell)} t\sqrt{L})f(y) \\
&\quad= \int_X K_{\Phi_0(2^{-\ell} t\sqrt{L})\Psi_0(2^{-\ell} t\sqrt{L})}(y,z)\Phi(2^{-\ell} t\sqrt{L})f(z)\,d\mu(z) \\
&\qquad+ \sum_{j=1}^{\infty} \int_X K_{\Phi(2^{-\ell} t\sqrt{L})\Psi(2^{-(j+\ell)} t\sqrt{L})}(y,z)\Phi(2^{-(j+\ell)} t\sqrt{L})f(z)\,d\mu(z). \qquad (3.11)
\end{aligned}$$

Let $N \geq \alpha$ and let κ be an integer such that $2\kappa - \alpha - n/r \geq \mu$. Since Ψ vanishes near the origin, (3.7) is valid for all $\kappa \in \mathbb{N}$. Hence, by Lemma 3.2, there exists a constant $C = C(\kappa, N)$ such that for all $\ell \in \mathbb{Z}$, $j \in \{1, 2, \ldots\}$ and $t \in [1, 2]$,

$$\begin{aligned}
&|K_{\Phi(2^{-\ell} t\sqrt{L})\Psi(2^{-(j+\ell)} t\sqrt{L})}(y,z)| \\
&\quad\leq C\|\lambda \mapsto \lambda^{2\kappa}\Phi(t\lambda)\|_{\mathcal{S}_{M_1}(\mathbb{R})}\|\lambda \mapsto \lambda^{-2\kappa}\Psi(t\lambda)\|_{\mathcal{S}_{M_2}(\mathbb{R})} \\
&\qquad\times 2^{-2j\kappa} V(z, 2^{-\ell})(1 + 2^{\ell} d(y,z))^{-(N+D)},
\end{aligned}$$

where M_1 (respectively M_2) is the smallest even integer that satisfies $M_1 > N + n + D + 1$ (respectively $M_2 > N + 2n + 2D + 2$). Obviously, for fixed M_1 and M_2, there exists a constant $C = C(\Phi, \Psi, M_1, M_2)$ such that

$$\sup_{t\in[1,2]} \|\lambda \mapsto \lambda^{2\kappa}\Phi(t\lambda)\|_{\mathcal{S}_{M_1}(\mathbb{R})}\|\lambda \mapsto \lambda^{-2\kappa}\Psi(t\lambda)\|_{\mathcal{S}_{M_2}(\mathbb{R})} \leq C.$$

Hence,

$$\begin{aligned}
&|K_{\Phi(2^{-\ell} t\sqrt{L})\Psi(2^{-(j+\ell)} t\sqrt{L})}(y,z)| \\
&\quad\leq C2^{-2j\kappa} V(y, 2^{-\ell})^{-1}(1 + 2^{\ell} d(y,z))^{-(N+D)} \\
&\quad\leq C2^{-2j\kappa} V(z, 2^{-\ell})^{-1}(1 + 2^{\ell} d(y,z))^{-N}, \qquad (3.12)
\end{aligned}$$

where the constant C depends on Φ, Ψ and N, but does not depend on $\ell \in \mathbb{Z}$, $j \in \{1, 2, \ldots\}$ and $t \in [1, 2]$. Analogously,

$$|K_{\Phi_0(2^{-\ell} t\sqrt{L})\Psi_0(2^{-\ell} t\sqrt{L})}(y,z)| \leq CV(z, 2^{-\ell})^{-1}(1 + 2^{\ell} d(y,z))^{-N}. \qquad (3.13)$$

Putting (3.12) and (3.13) into (3.11),

$$\begin{aligned}
&|\Phi(2^{-\ell} t\sqrt{L})f(y)| \\
&\quad\leq C\sum_{j=0}^{\infty} 2^{-2j\kappa} \int_X \frac{|\Phi(2^{-(j+\ell)} t\sqrt{L})f(z)|}{V(z, 2^{-\ell})(1 + 2^{\ell}\, d(y,z))^N}\,d\mu(z) \\
&\quad= C\sum_{j=\ell}^{\infty} 2^{-2(j-\ell)\kappa} \int_X \frac{|\Phi(2^{-j} t\sqrt{L})f(z)|}{V(z, 2^{-\ell})(1 + 2^{\ell}\, d(y,z))^N}\,d\mu(z). \qquad (3.14)
\end{aligned}$$

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To prove (3.9), we first consider the case $0 < r \le 1$. Dividing both sides of (3.14) by $(1 + 2^\ell t^{-1} d(x, y))^\alpha$, in the left-hand side taking the supremum over $y \in X$, in the right-hand side making use of the inequalities $V(z, 2^{-\ell}) \ge V(z, 2^{-j})$ (for all $j \ge \ell$) and $(1 + 2^\ell t^{-1} d(x, y))(1 + 2^\ell d(y, z)) \ge C(1 + 2^\ell d(x, z))$ (for all $t \in [1, 2]$), we obtain, for all $t \in [1, 2]$ and $x \in X$,

$$M^*_{\alpha,L}(f)(x, 2^{-\ell}t) \le C \sum_{j=\ell}^{\infty} 2^{-2(j-\ell)\kappa} \int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|}{V(z, 2^{-j})(1 + 2^\ell\, d(x, z))^\alpha}\, d\mu(z). \tag{3.15}$$

To proceed, we further observe that

$$|\Phi(2^{-j}t\sqrt{L})f(z)| \le |\Phi(2^{-j}t\sqrt{L})f(z)|^r [M^*_{\alpha,L}(f)(x, 2^{-\ell}t)]^{1-r}(1 + 2^j t^{-1} d(x, z))^{\alpha(1-r)}. \tag{3.16}$$

From (3.15), (3.16) and the inequality

$$\frac{(1 + 2^j t^{-1} d(x, z))^{\alpha(1-r)}}{(1 + 2^\ell d(x, z))^\alpha} \le C \frac{2^{(j-\ell)\alpha}}{(1 + 2^j d(x, z))^{\alpha r}} \quad (j \ge \ell,\ t \in [1, 2]),$$

$$\begin{aligned} M^*_{\alpha,L}(f)(x, 2^{-\ell}t) &\le C \sum_{j=\ell}^{\infty} 2^{-(j-\ell)(2\kappa-\alpha)} \int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|^r}{V(z, 2^{-j})(1 + 2^j\, d(x, z))^{\alpha r}}\, d\mu(z) \\ &\quad \times [M^*_{\alpha,L}(f)(x, 2^{-\ell}t)]^{1-r}. \end{aligned} \tag{3.17}$$

We claim that for any fixed $\alpha > D/2$, $f \in L^2(X)$, $x \in X$ and $t > 0$, there exists $N_0 > 0$ such that

$$M^*_{\alpha,L}(f)(x, 2^{-\ell}t) < \infty, \quad \forall \ell \in \mathbb{Z} \tag{3.18}$$

and

$$M^*_{\alpha,L}(f)(x, 2^{-\ell}t) = O(2^{\ell N_0}) \quad (\ell \to \infty). \tag{3.19}$$

To see this, we first note that the Gaussian upper bound (1.2) for $p_t(x, y)$ is further inherited by the time derivatives of $p_t(x, y)$; in particular,

$$\left|\frac{\partial}{\partial t} p_t(x, y)\right| \le \frac{C}{tV(x, \sqrt{t})} \exp\left(-\frac{d^2(x, y)}{ct}\right), \quad \forall t > 0$$

for almost every $x, y \in X$; see, for example, [24, Theorem 6.18]. This implies that

$$|K_{u^2Le^{-u^2L}}(y, z)| = \left|\left[-t\frac{\partial}{\partial t} p_t(y, z)\right]\Big|_{t=u^2}\right| \le CV(y, u)^{-1}(1 + u^{-1} d(y, z))^{-(n+1)/2}.$$

Hence, by the Cauchy–Schwartz inequality and (2.3), for almost every $y \in X$,

$$\begin{aligned} |u^2Le^{-u^2L} f(y)| &\le \int_X |f(z)||K_{u^2Le^{-u^2L}}(y, z)|\, d\mu(z) \\ &\le \|f\|_{L^2(X)} V(y, u)^{-1}\left(\int_X (1 + u^{-1}\, d(y, z))^{-(n+1)}\, d\mu(z)\right)^{1/2} \\ &\le C\|f\|_{L^2(X)} V(y, u)^{-1/2}. \end{aligned}$$

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This along with (2.2) yields that for $\alpha > D/2$,

$$M^*_{\alpha,L}(f)(x,u) \le C \operatorname*{ess\,sup}_{y\in X}[(1+u^{-1}\,d(x,y))^{-\alpha}\|f\|_{L^2(X)}V(y,u)^{-1/2}]$$
$$\le C\|f\|_{L^2(X)}V(x,u)^{-1/2}.$$

Hence, (3.18) is true. Moreover, if $\ell \ge 0$, by (2.1),

$$M^*_{\alpha,L}(f)(x,2^{-\ell}t) \le C\|f\|_{L^2(X)}V(x,2^{-\ell}t)^{-1/2} \le C2^{\ell n/2}V(x,t)^{-1/2},$$

which shows (3.19) with $N_0 = n/2$. From (3.17), (3.18), (3.19) and Lemma 2.6,

$$\begin{aligned}
&[M^*_{\alpha,L}(f)(x,2^{-\ell}t)]^r\\
&\quad \le C\sum_{j=\ell}^{\infty} 2^{-(j-\ell)(2\kappa-\alpha)r}\int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|^r}{V(z,2^{-j})(1+2^j\,d(x,z))^{\alpha r}}\,d\mu(z)\\
&\quad \le C\sum_{j=\ell}^{\infty} 2^{-(j-\ell)(2\kappa-\alpha-n/r)r}\int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\,d(x,z))^{\alpha r}}\,d\mu(z),
\end{aligned}$$

where we used $V(z,2^{-j})^{-1} \le 2^{(j-\ell)n}V(z,2^{-\ell})^{-1}$ (for all $j \ge \ell$). This implies the desired estimate (3.9), since $2\kappa - \alpha - n/r \ge \mu$ and $\Phi(\lambda) = \lambda^2 e^{-\lambda^2}$.

The proof of (3.9) for $r > 1$ is much easier. Indeed, from (3.14) with $\mu + \varepsilon$ instead of 2κ (since we can take κ arbitrarily large), and with $\alpha + (D+n+1)/r'$ instead of N, where $1/r + 1/r' = 1$,

$$\begin{aligned}
&|\Phi(2^{-\ell}t\sqrt{L})f(y)|\\
&\quad \le C\sum_{j=\ell}^{\infty} 2^{-(j-\ell)(\mu+\varepsilon)}\int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|}{V(z,2^{-\ell})(1+2^\ell\,d(y,z))^{\alpha+(D+n+1)/r'}}\,d\mu(z)\\
&\quad \le C\sum_{j=\ell}^{\infty} 2^{-(j-\ell)(\mu+\varepsilon)}\Big(\int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\,d(y,z))^{\alpha r}}\,d\mu(z)\Big)^{1/r}\\
&\quad \le C\Big(\sum_{j=\ell}^{\infty} 2^{-(j-\ell)\mu r}\int \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\,d(y,z))^{\alpha r}}\,d\mu(z)\Big)^{1/r},
\end{aligned}$$

where we applied Hölder's inequality for the integrals and the sums, and used (2.2) and (2.3). Raising both sides to the power r, dividing both sides by $(1+2^\ell t^{-1}d(x,y))^{\alpha r}$, in the left-hand side taking the supremum over $y \in X$ and in the right-hand side using the inequality $(1+2^\ell t^{-1}d(x,y))^{\alpha r}(1+2^\ell td(y,z))^{\alpha r} \ge C(1+2^\ell d(x,z))^{\alpha r}$ (for all $t \in [1,2]$), we obtain the desired estimate (3.9) for $r > 1$. □

Lemma 3.5. *Assume that L satisfies* (H1)–(H2). *Let $w \in A_\infty$, $0 < p < \infty$ and $\alpha > (n+D)q_w/\min\{p,2\}$. Then there exists a constant $C > 0$ such that for all $f \in L^2(X)$,*

$$\Big\|\Big(\int_0^\infty [M^*_{\alpha,L}(f)(\cdot,t)]^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p_w(X)} \le C\|G_L(f)\|_{L^p_w(X)}.$$

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Proof. Let $\Phi(\lambda) := \lambda^2 e^{-\lambda^2}$, $\lambda \in \mathbb{R}$. Since $\alpha > (n+D)q_w/\min\{p,2\}$, there exists a number r such that $0 < r < \min\{p,2\}/q_w$ and $\alpha r > n + D$. From Lemma 3.4, we see that for any $\mu > 0$ there exists a constant C such that for all $f \in L^2(X)$, $\ell \in \mathbb{Z}$, $x \in X$ and $t \in [1,2]$,

$$[M^*_{\alpha,L}(f)(x, 2^{-\ell}t)]^r \le C \sum_{j=\ell}^{\infty} 2^{-(j-\ell)\mu r} \int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\, d(x,z))^{\alpha r}}\, d\mu(z).$$

Taking the norm $(\int_1^2 |\cdot|^{2/r}(dt/t))^{r/2}$ on both sides and applying the Minkowski inequality and Lemma 2.1,

$$\begin{aligned}
&\bigg(\int_1^2 [M^*_{\alpha,L}(f)(x,2^{-\ell}t)]^2 \frac{dt}{t}\bigg)^{r/2} \\
&\quad \le C \sum_{j=\ell}^{\infty} 2^{-(j-\ell)\mu r} \int_X \frac{\big(\int_1^2 |\Phi(2^{-j}t\sqrt{L})f(z)|^2 \frac{dt}{t}\big)^{r/2}}{V(z,2^{-\ell})(1+2^\ell\, d(x,z))^{\alpha r}}\, d\mu(z) \\
&\quad \le C \sum_{j=\ell}^{\infty} 2^{-(j-\ell)\mu r} \mathcal{M}\bigg[\bigg(\int_1^2 |\Phi(2^{-j}t\sqrt{L})f(\cdot)|^2 \frac{dt}{t}\bigg)^{r/2}\bigg](x) \\
&\quad \le C \sum_{j=-\infty}^{\infty} 2^{-|j-\ell|\mu r} \mathcal{M}\bigg[\bigg(\int_1^2 |\Phi(2^{-j}t\sqrt{L})f(\cdot)|^2 \frac{dt}{t}\bigg)^{r/2}\bigg](x).
\end{aligned}$$

Applying Lemma 2.5 in the space $L^{p/r}_w(\ell^{2/r})$ and Lemma 2.2,

$$\begin{aligned}
&\bigg\|\bigg(\int_0^\infty [M^*_{\alpha,L}(f)(\cdot,t)]^2 \frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^p_w(X)} \\
&\quad = \bigg\|\bigg\{\bigg(\int_1^2 [M^*_{\alpha,L}(f)(\cdot,2^{-\ell}t)]^2 \frac{dt}{t}\bigg)^{r/2}\bigg\}_{\ell=-\infty}^{\infty}\bigg\|_{L^{p/r}_w(\ell^{2/r})}^{1/r} \\
&\quad \le C\bigg\|\bigg\{\mathcal{M}\bigg[\bigg(\int_1^2 |\Phi(2^{-j}t\sqrt{L})f(\cdot)|^2 \frac{dt}{t}\bigg)^{r/2}\bigg]\bigg\}_{j=-\infty}^{\infty}\bigg\|_{L^{p/r}_w(\ell^{2/r})}^{1/r} \\
&\quad \le C\bigg\|\bigg\{\bigg(\int_1^2 |\Phi(2^{-j}t\sqrt{L})f(\cdot)|^2 \frac{dt}{t}\bigg)^{r/2}\bigg\}_{j=-\infty}^{\infty}\bigg\|_{L^{p/r}_w(\ell^{2/r})}^{1/r} \\
&\quad = C\|G_L(f)\|_{L^p_w(X)},
\end{aligned}$$

where we used the fact that $p/r > q_w$ and $2/r > 1$. □

Lemma 3.6. *Assume that L satisfies* (H1)–(H2). *Let $w \in A_\infty$, $0 < p < \infty$ and $\alpha > 0$. Then there exists a constant $C > 0$ such that for all $f \in L^2(X)$ and $t \in [1,2]$,*

$$\bigg\|\bigg(\int_0^\infty [M^*_{\alpha+D/2,L}(f)(\cdot,t)]^2 \frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^p_w(X)} \le C\|\mathcal{G}^*_{(2/n)\alpha,L}(f)\|_{L^p_w(X)}.$$

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PROOF. Let $\Phi(\lambda) := \lambda^2 e^{-\lambda^2}$, $\lambda \in \mathbb{R}$. Let $\mu > \alpha + D/2$. By Lemma 3.4 with $r = 2$, we see that there exists a constant $C > 0$ such that for all $f \in L^2(X)$, $\ell \in \mathbb{Z}$ and $t \in [1, 2]$,

$$\begin{aligned}
&[M^*_{\alpha+D/2,L}(f)(x, 2^{-\ell}t)]^2 \\
&\quad\leq C \sum_{j=\ell}^{\infty} 2^{-2(j-\ell)\mu} \int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|^2}{V(z, 2^{-j}t)(1 + 2^{\ell}t^{-1}\, d(x,z))^{2\alpha+D}}\, d\mu(z) \\
&\quad\leq C \sum_{j=-\infty}^{\infty} 2^{-|j-\ell|(2\mu-2\alpha-D)} \int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|^2}{(1 + 2^{j}t^{-1}\, d(x,z))^{2\alpha}} \frac{d\mu(z)}{V(x, 2^{-j}t)},
\end{aligned} \tag{3.20}$$

where for the last line we used (2.2) and

$$1 + 2^{\ell}t^{-1}d(x,z) \geq 2^{-(j-\ell)}(1 + 2^{j}t^{-1}d(x,z)), \quad \forall j \geq \ell.$$

Taking the norm $\int_1^2 |\cdot|(dt/t)$ on both sides of (3.20) gives

$$\begin{aligned}
&\int_1^2 [M^*_{\alpha+D/2,L}(f)(x, 2^{-\ell}t)]^2 \frac{dt}{t} \\
&\quad\leq C \sum_{j=-\infty}^{\infty} 2^{-|j-\ell|(2\mu-2\alpha-D)} \int_1^2 \int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|^2}{(1 + 2^{j}t^{-1}\, d(x,z))^{2\alpha}} \frac{d\mu(z)}{V(x, 2^{-j}t)} \frac{dt}{t}.
\end{aligned}$$

Since $2\mu - 2\alpha - D > 0$, applying Lemma 2.5 in $L^{p/2}_w(\ell^1)$,

$$\begin{aligned}
&\bigg\|\bigg(\int_0^{\infty} [M^*_{\alpha+D/2,L}(f)(\cdot, t)]^2 \frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^p_w(X)} \\
&\quad= \bigg\|\bigg\{\int_1^2 [M^*_{\alpha+D/2,L}(f)(\cdot, 2^{-\ell}t)]^2 \frac{dt}{t}\bigg\}_{\ell=-\infty}^{\infty}\bigg\|^{1/2}_{L^{p/2}_w(\ell^1)} \\
&\quad\leq C\bigg\|\bigg\{\int_1^2 \int_X \frac{|\Phi(2^{-j}t\sqrt{L})f(z)|^2}{(1 + 2^{j}t^{-1}\, d(\cdot,z))^{2\alpha}} \frac{d\mu(z)}{V(x, 2^{-j}t)} \frac{dt}{t}\bigg\}_{j=-\infty}^{\infty}\bigg\|^{1/2}_{L^{p/2}_w(\ell^1)} \\
&\quad= C\|\mathcal{G}^*_{(2/n)\alpha,L}(f)\|_{L^p_w(X)}.
\end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 1.2. From the definition of $G_L(f)$ and $M^*_{\alpha,L}(f)(x,t)$, we see that for any $\alpha > 0$ and $f \in L^2(X)$,

$$G_L(f)(x) \leq \bigg(\int_0^{\infty} [M^*_{\alpha,L}(f)(x,t)]^2 \frac{dt}{t}\bigg)^{1/2}, \quad \text{a.e. } x \in X. \tag{3.21}$$

Fix two numbers α, α' such that $\alpha > nq_w/\min\{p, 2\}$ and $\alpha' > (n + D)q_w/\min\{p, 2\}$. Then, for all $f \in L^2(X)$, by (3.21) and Lemmas 3.6, 2.3, 3.3 and 3.5,

$$\begin{aligned}
\|G_L(f)\|_{L^p_w(X)} &\leq C\bigg\|\bigg(\int_0^{\infty} [M^*_{\alpha+D/2,L}(f)(\cdot, t)]^2 \frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^p_w(X)} \\
&\leq C\|\mathcal{G}^*_{(2/n)\alpha,L}(f)\|_{L^p_w(X)} \leq C\|S_L(f)\|_{L^p_w(X)} \\
&\leq C\bigg\|\bigg(\int_0^{\infty} [M^*_{\alpha',L}(f)(\cdot, t)]^2 \frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^p_w(X)} \leq C\|G_L(f)\|_{L^p_w(X)}.
\end{aligned}$$

Hence, $\|S_L(f)\|_{L^p_w(X)} \approx \|G_L(f)\|_{L^p_w(X)}$. The proof of Theorem 1.2 is complete. □

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