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On the Structure of the Full Lift for the Howe Correspondence of $(Sp(n), O(V))$ for Rank-One Reducibilities

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Abstract. In this paper we determine the structure of the full lift for the Howe correspondence of $(Sp(n), O(V))$ for rank-one reducibilities.

Introduction

In this paper we study the Howe correspondence for the dual pair $(Sp(n), O(V_r))$, where $n \geq 0$, V_r is a quadratic space in a fixed, but arbitrary even dimensional Witt tower (*cf.* §1) with a nonarchimedean local field F of characteristic different from 2. To explain our results, let us write $\omega_{n,r} = \omega^{\psi}_{n,r}$ for the oscillator representation associated to that pair and a fixed additive character ψ of F. Let χ be a quadratic character associated to V_r. Let $\sigma \in \mathrm{Irr}(Sp(n))$. Then the σ-isotypic component $\omega_{n,r}$ we denote by $\Theta(\sigma, r)$ (see §2). This is a smooth representation of $O(V_r)$. We call it the *full lift* of σ. The basic problems in the theory are: to determine when $\Theta(\sigma, r) \neq 0$, to prove the Howe duality conjecture (see [W] when residue characteristic is different than two), and to describe the structure of the unique irreducible quotient of $\Theta(\sigma, r) \neq 0$. In our paper [M], we succeeded solving all three problems for discrete series representations. Although the methods in [M] are rather powerful for discrete series, they are not sufficient for tempered (but not in discrete series) representations and non-tempered representations. In this paper we present a different approach based on a deep result of Bernstein [Be] about the existence of a right-adjoint functor to the functor of induction in the category of smooth representations [Be]. This method also gives the information about the structure of the full lift $\Theta(\sigma, r)$.

To describe our main results, let $\sigma \in \mathrm{Irr}(Sp(n))$ be a supercuspidal representation. Then it is known that at the first occurrence r, the lift $\Theta(\sigma, r) = \tau$ is a supercuspidal irreducible representation[MVW]. Assume that $\rho \in \mathrm{Irr}(GL(j, F))$ is supercuspidal. Let P_j, (resp., Q_j) be a maximal parabolic subgroup of $O(V_{r+j})$, (resp., $Sp(n + j)$) with a Levi factor isomorphic to $GL(j, F) \times O(V_r)$ (resp. $GL(j, F) \times Sp(n)$). The main results of this paper, Theorems 2.1 and 2.3, describe the structure of the full Howe lift to $Sp(n + j)$, (resp., $O(V_{r+j})$) of all irreducible subquotients of

$$\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\rho \otimes \tau), \quad (\text{resp., } \mathrm{Ind}_{Q_j}^{Sp(n+j)}(\rho \otimes \sigma)).$$

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As a result, the reduction and composition series of these induced representations are described in terms of the Howe correspondence. The results of the present paper (and the results about the lift of supercuspidal representations [MVW]) also suggest the following conjecture:

Conjecture *Assume that σ is in the discrete series; then $\Theta(\sigma, r)$ is irreducible or zero.*

Theorems 2.1 and 2.3 are proved in Section 2. The basic method is to consider Jacquet modules of $\omega_{n+j,r+j}$ (*cf.* [Ku]), but computing certain isotypic components in Jacquet modules of $\omega_{n+j,r+j}$ (Proposition 2.4) using [Be]. Section 3 is devoted to these computations.

In Section 4 we apply our results to get some explicit and new reducibility results for the inner form of the split $SO(2n)$ (Theorem 4.1), although more can be done using chains of supercuspidal representations as discussed in [Ku1].

This paper is an outgrowth of unpublished previous work [M1]. In a sequel to this paper [M2] we will pursue the approach to Howe correspondence adopted here. The present paper is the first in that direction, and it is fundamental since it treats the generalized rank-one case.

1 Preliminaries

Let F be a nonarchimedean field of characteristic different from 2. Let $\mathbb{Z}_+$, $\mathbb{R}$, and $\mathbb{C}$ be the set of non-negative rational integers, the field of real numbers, and the field of complex numbers, respectively.

Let G be an l-group (*cf.* [BZ]). Then we will write $\mathcal{A}(G)$ for the category of all smooth representations of G. Let $P = MN$ be a closed subgroup of G, given as the semi-direct product of closed subgroups M and N, M normalizes N. Assume that N is a union of its open compact subgroups. Then we have normalized induction and localization functors $\mathrm{Ind}_P^G \colon \mathcal{A}(M) \to \mathcal{A}(G)$ and $R_P \colon \mathcal{A}(G) \to \mathcal{A}(M)$. They are related by Frobenius reciprocity $\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\pi')) \cong \mathrm{Hom}_M(R_P(\pi), \pi')$; and Ind_P^G and R_P are exact functors.

Assume that G and G' are l-groups. Let $V \in \mathcal{A}(G \times G')$. If $\rho \in \mathrm{Irr}(G)$ is an admissible representation, then we write $\Theta(\rho, V) \in \mathcal{A}(G')$ for the ρ-isotypic part of V (*cf.* [MVW, Ch. II, Lemme III.4]). More precisely, set $V' = \bigcap_f \ker(f)$, $f \in \mathrm{Hom}_G(V, \rho)$; then

$$V/V' \cong \rho \otimes \Theta(\rho, V).$$

For convenience, let us state the next simple lemma.

Lemma 1.1 *The dual G'-module $\Theta(\rho, V)^*$ is isomorphic to the obvious (not necessarily smooth) G'-module $\mathrm{Hom}_G(V, \rho)$. Hence, we have an isomorphism of the corresponding smooth modules $\widetilde{\Theta(\rho, V)} = \Theta(\rho, V)^*_\infty \cong \mathrm{Hom}_G(V, \rho)_\infty$.*

Now, we will describe the groups which we will consider. We follow [MVW, Ch. I]. Fix an anisotropic even dimensional inner product F-space $(V_0, (\ ,\ )_0)$. Then its dimension can only be 0, 2, or 4. (For more details see [MVW, Ch. I].) Let $\chi = \chi_{V_0}$ be the quadratic character of $F^\times$ associated to the quadratic space V_0. (See [Ku, p. 240, (2.5)], or [Ku1, Proposition 4.3].) If V_0 is trivial or four dimensional space, then χ is the trivial character.

For each $r \in \mathbb{Z}_+$, let V_r be the orthogonal direct sum of V_0 with r hyperbolic planes. We will fix a Witt decomposition

$$V_r = V_r^{(1)} \oplus V_0 \oplus V_r^{(2)}, \tag{1.1}$$

where $V^{(i)} = Fv_1^{(i)} \oplus \cdots \oplus Fv_r^{(i)}$, $i = 1, 2$, satisfying $(v_k^{(i)}, v_l^{(i)}) = 0$ and $(v_k^{(1)}, v_l^{(2)}) = \delta_{kl}$. Let $O(V_r)$, (resp., $SO(V_r)$) be the corresponding orthogonal, (resp., special orthogonal) group. If $V_0 = 0$, then we let $O(V_0) = SO(V_0)$ be the trivial group. Let $\nu = \nu_r$ be the determinant character of $O(V_r)$.

The decomposition (1.1) gives us a set of standard parabolic subgroups in $O(V_r)$ and $SO(V_r)$. We will describe only maximal parabolic subgroups. For j, $1 \leq j \leq r$, let $V_j^{(i,r)} = Fv_{r-j+1}^{(i)} \oplus \cdots \oplus Fv_r^{(i)}$, $i = 1, 2$. Then we have a Witt decomposition

$$V_r = V_j^{(1,r)} \oplus V_{r-j} \oplus V_j^{(2,r)}. \tag{1.2}$$

Let P_j be the parabolic subgroup of $O(V_r)$ which stabilizes $V_j^{(1,r)}$. There is a Levi decomposition $P_j = M_j N_j$, where $M_j \cong GL(V^{(1,r)}) \times O(V_{r-j})$. (Beware of the difference between this choice of a Levi factor and that of [Ku, p. 233]. There, $GL(V_j^{(2,r)})$ is considered instead of $GL(V_j^{(1,r)})$.) Fix the isomorphism $GL(n, F) \cong GL(V_j^{(1,r)})$ using the above fixed basis of $V_j^{(1,r)}$.

Let ϵ be the element of $O(V_r)$ defined as follows. First, $\epsilon(v_j^{(i)}) = v_j^{(i)}$, for $1 \leq j \leq r-1$. Then, if $V_0 \neq 0$, we let $\epsilon(v_r^{(i)}) = v_r^{(i)}$, and let ϵ be any element α of $O(V_0)$ on V_0, $\alpha^2 = 1$, $\nu(\alpha) = -1$. If $V_0 = 0$, then we let $\epsilon(v_r^{(1)}) = v_r^{(2)}$ and $\epsilon(v_r^{(2)}) = v_r^{(1)}$. Clearly, $\nu(\epsilon) = -1$.

A set of standard maximal parabolic subgroups of $SO(V_r)$ may be described as follows. For each j, $1 \leq j \leq r$, set $P_j^0 = P_j \cap SO(V_r)$ and $M_j^0 = M_j \cap SO(V_r)$. P_j^0 is a standard parabolic subgroup of $SO(V_r)$ with a Levi decomposition $P_j^0 = M_j^0 N_j$. If $V_0 \neq 0$, then P_j^0, $1 \leq j \leq r$, exhaust the set of all standard maximal parabolic subgroups. If $V_0 = 0$, then we need to add one more parabolic subgroup P_r^{0-}, with a Levi decomposition $P_r^{0-} = M_r^{0-} N_r^{0-}$, where $M_r^{0-} = \epsilon M_r^0 \epsilon^{-1}$ and $N_r^{0-} = \epsilon N_r^0 \epsilon^{-1}$. This will be used to identify $M_r^{0-} \cong GL(r, F)$.

Now, we will discuss the representation theory of $O(V_r)$. Since the algebraic group $O(V_r)$ is not connected (in the algebraic sense), we need to discuss some general results of [Ca]. First, we will recall the relationship between irreducible representations of $O(V_r)$, and those of $SO(V_r)$ using the next simple lemma (*cf.* [MVW, Ch. III]).

Lemma 1.2

(i) *If* $\pi \in \operatorname{Irr}(O(V_r))$, *then* $\pi' = \pi|_{SO(V_r)}$ *is irreducible if and only if* π *is not equivalent to* $\nu\pi$. *If* π' *is irreducible, then* $\operatorname{Ind}_{SO(V_r)}^{O(V_r)}(\pi') \cong \pi \oplus \nu\pi$, *and* $\pi' \cong \pi'^{\epsilon}$. *(Here* $\pi'^{\epsilon}(g) = \pi'(\epsilon g \epsilon^{-1})$, *for all* $g \in SO(V_r)$.*)*

(ii) *If* $\pi' \in \operatorname{Irr}(SO(V_r))$, *then* $\pi = \operatorname{Ind}_{SO(V_r)}^{O(V_r)}(\pi')$ *is irreducible if and only if* $\pi' \not\cong \pi'^{\epsilon}$. *Furthermore,* $\nu\pi \cong \pi$.

Lemma 1.2 enables us to define supercuspidal, square integrable and tempered representations of $O(V_r)$ by considering their restrictions to $SO(V_r)$. The usual characterizations in terms of Jacquet modules (*cf.* [Ca, Si]) hold. (See also [MVW, pp. 61–62].) For our purpose the next theorem is sufficient.

Theorem 1.3 *Let* $\rho\otimes\tau \in \operatorname{Irr}(GL(j,F)\times O(V_{r-j}))$ *be a supercuspidal representation. Then we have*

(i) $R_{P_j}(\operatorname{Ind}_{P_j}^{O(V_r)}(\rho\otimes\tau)) = \rho\otimes\tau + \widetilde{\rho}\otimes\tau$, *in the corresponding Grothendieck group. Hence, the length of* $\operatorname{Ind}_{P_j}^{O(V_r)}(\rho\otimes\tau)$ *is at most two.*

(ii) *If* $\operatorname{Ind}_{P_j}^{O(V_r)}(\rho\otimes\tau)$ *(*ρ *unitary) reduces, then it is the direct sum of two irreducible non-equivalent tempered representations. If it is reducible for nonunitary* ρ*, then it has a unique irreducible subrepresentation (with* R_{P_j}*-localization* $\rho\otimes\tau$*) and a unique irreducible quotient (with* R_{P_j}*-localization* $\widetilde{\rho}\otimes\tau$*). Moreover, we can decompose* $\rho = |\det|^{e(\rho)}\rho^u$*, where* $e(\rho)\in\mathbb{R}$*, and* ρ^u *is unitary. If* $e(\rho)>0$*, the unique irreducible subrepresentation is square integrable, and the other constituent is not tempered.*

(iii) *Let* τ_0 *be any irreducible subrepresentation of* $\tau|_{SO(V_{r-j})}$*. Then* $\operatorname{Ind}_{P_j^0}^{SO(V_r)}(\rho\otimes\tau_0)$ *reduces if and only if* $\operatorname{Ind}_{P_j}^{O(V_r)}(\rho\otimes\tau)$ *reduces, unless one of the following holds:*

(1) $V_0 = 0$, $r>1$, $r=j$ *is odd, and* $\rho\cong\widetilde{\rho}$*;*
(2) $j<r$, j *is odd,* $\rho\cong\widetilde{\rho}$*, and* $\tau_0^\epsilon\not\cong\tau_0$*.*

In both cases, $\operatorname{Ind}_{P_r^0}^{SO(V_r)}(\rho\otimes\tau_0)$ *is irreducible and* $\operatorname{Ind}_{P_r}^{O(V_r)}(\rho\otimes\tau)$ *is the direct sum of two irreducible non-equivalent tempered representations.*

We will also need the results from Remark 1.4.

Remark 1.4

(i) Note that $N_{SO(V_r)}(M_j^0)$ has two elements, unless $V_0 = 0$, $r>1$ is odd, and $r=j$ (*cf.* [Go]). If w_0 is the nontrivial element of that group, then

$$w_0(\rho\otimes\tau_0) = \begin{cases}\widetilde{\rho}\otimes\tau_0^\epsilon & \text{if } j \text{ is odd},\\ \widetilde{\rho}\otimes\tau_0 & \text{if } j \text{ is even}.\end{cases}$$

If $V_0 = 0$, $r>1$ is odd, and $j=r$, then Goldberg [Go] has shown P_r^0 and P_r^{0-} are associated in $SO(V_r)$.

(ii) For each $\pi\in\operatorname{Irr}(O(V_r))$, $\pi\cong\widetilde{\pi}$. (*cf.* [MVW, Ch. III, Théorème II.1]).

(iii) The only group among $O(V_r)$, $r\geq 0$ which does not have supercuspidal representations is $O(V_1)$, but only when $V_0 = 0$.

Proof of Theorem 1.3 The theorem, except the part of (iii) related to the case $V_0 = 0$, $r>1$, $r=j$ is odd, is an easy consequence of Lemma 1.2 and the representation theory of $SO(V_r)$, and is left to reader. Let us prove that part of (iii). First, $\pi_1 = \operatorname{Ind}_{P_r^0}^{SO(V_r)}(\rho\otimes 1)$ is irreducible because a necessary condition for reducibility, namely that $N_{SO(V_r)}(M_r^0) \neq \{1\}$, [Ca, Theorem 7.1.4], does not hold. Also, π_1 is equivalent to $\pi_2 = \mathrm{Ind}_{P_r^{0-}}^{SO(V_r)}(\widetilde{\rho} \otimes 1)$. We have $\pi_1^\epsilon = \mathrm{Ind}_{P_r^{0-}}^{SO(V_r)}(\rho \otimes 1)$, and

$$\mathrm{Ind}_{P_r}^{O(V_r)}(\rho \otimes 1)|_{SO(V_r)} \cong \pi_1 \oplus \pi_1^\epsilon.$$

Again, by [Ca, Theorem 7.1.4], $\pi_1^\epsilon \cong \pi_2$ ($\cong \pi_1$) if and only if $\rho \cong \widetilde{\rho}$. Now, Lemma 1.2 implies the last part of (iii). ■

We will finish this section by briefly discussing symplectic groups $Sp(n) = Sp(n, F)$. (For more details see [Ku, p. 235] [MVW, T].) By the analogous geometric description, the groups $Sp(n)$ (n is the semisimple rank) have proper maximal parabolic subgroups parametrized by numbers i, $1 \leq i \leq n$. Write $Q_i = M_i' N_i'$ for the corresponding parabolic subgroup, where $M_i' = GL(i, F) \times Sp(n - i)$. Theorem 1.3(i) and (ii) have the similar form for $Sp(n)$ (*cf.* [T]). Finally, set $Sp(0) = \{1\}$.

2 Correspondence

In this section we prove our main results. To explain the results, we need to introduce more notation. Put

$$m_r = \dim_F (V_r)/2.$$

The pair $(Sp(n), O(V_r))$ is a dual pair in $Sp(n \cdot (2m_r), F)$ (*cf.* [MVW, Ku1]). We write

$$\omega_{n,r} = \omega_{n,r}^{\psi},$$

for the oscillator representation associated to that pair and a fixed additive character ψ of F. (Here $\omega_{0,r}$ is the trivial representation of $O(V_r)$, and if $V_0 = 0$, then $\omega_{n,0}$ is the trivial representation of $Sp(n)$.)

For each $\sigma \in \mathrm{Irr}(Sp(n))$ and $r \geq 0$, write $\Theta(\sigma, r)$, for a smooth representation of $O(V_r)$, defined as the σ-isotypic part of $\omega_{n,r}$ (*cf.* [MVW, Ch. II, Lemme III.4]). If $\tau \in \mathrm{Irr}(O(V_r))$, we will write $\Theta(\tau, n)$, for the analogously defined smooth representation of $Sp(n)$, $n \geq 0$.

It is known (*cf.* [MVW]) that if j is large enough, then $\Theta(\sigma, j) \neq 0$. We write r for the smallest j, such that $\Theta(\sigma, j) \neq 0$. (It depends on σ and the tower V_r, $r \geq 0$.) We call r the first occurrence of σ in the tower V_j, $j \geq 0$. We have $\Theta(\sigma, j) \neq 0$ for $j \geq r$ [MVW, Ch. III]. If σ is supercuspidal, then $\Theta(\sigma, j)$ is irreducible for $j \geq r$, and supercuspidal only for $j = r$. The analogous discussion is also valid for τ and the symplectic tower.

In what follows we shall assume that σ and τ are supercuspidal irreducible representations of $Sp(n)$ and $O(V_r)$, respectively, such that $\Theta(\sigma, r) = \tau$. (Hence $\Theta(\tau, n) = \sigma$.)

Theorem 2.1 *Let $\rho \in \mathrm{Irr}(GL(j, F))$ be a supercuspidal representation; if $j = 1$, then $\rho \notin \{\chi|\cdot|^{\pm(m_r - n)}, \chi|\cdot|^{\pm(m_r - n - 1)}\}$. Then $\mathrm{Ind}_{Q_j}^{Sp(n+j)}(\rho \otimes \sigma)$ reduces if and only if $\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\rho \otimes \tau)$ reduces. More precisely, we have*

(i) *Assume that ρ is not unitary and $\mathrm{Ind}_{Q_j}^{Sp(n+j)}(\rho \otimes \sigma)$ reduces. Write π_1 and π_2 for its irreducible subrepresentation and irreducible quotient, respectively. Then $\Theta(\pi_i, r+j)$ is not zero, $i = 1, 2$. Furthermore, $\Theta(\pi_1, r+j)$ and $\Theta(\pi_2, r+j)$ are the unique irreducible subrepresentation and quotient of $\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\rho \otimes \tau)$, respectively.*

(ii) *Assume that ρ is unitary and $\mathrm{Ind}_{Q_j}^{Sp(n+j)}(\rho \otimes \sigma)$ reduces. Write π_1 and π_2 for its non-equivalent irreducible subrepresentations. Then $\Theta(\pi_i, r+j)$ is not zero, $i = 1, 2$, and*

$$\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\rho \otimes \tau) \cong \Theta(\pi_1, r+j) \oplus \Theta(\pi_2, r+j).$$

(iii) *If $\mathrm{Ind}_{Q_j}^{Sp(n+j)}(\rho \otimes \sigma)$ is irreducible, then the lift is given by*

$$\Theta(\mathrm{Ind}_{Q_j}^{Sp(n+j)}(\rho \otimes \sigma), r+j) = \mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\widetilde{\rho} \otimes \tau),$$

and is also irreducible.

Let us state the following interesting corollary of Theorem 2.1.

Corollary 2.2 *Assume that $V_0 = 0$, and put $O(2j) = O(V_j)$. Let $\rho \in \mathrm{Irr}(GL(j, F))$, $j > 1$, be a supercuspidal representation. Then $\mathrm{Ind}_{Q_j}^{Sp(j)}(\rho)$ reduces if and only if $\mathrm{Ind}_{P_j}^{O(2j)}(\rho)$ reduces.*

Corollary 2.2 is just the reformulation of a part of the results obtained by Shahidi [Sh], using his theory of L-functions. (Actually, [Sh] considers $SO(2j)$ instead of $O(2j)$, *cf.* Theorem 1.3.)

Now we will explain the assumption on ρ in Theorem 2.1. By [MVW, Théorème principal, p. 69],

$$\text{(2.1)}\quad \Theta(\sigma, r+1) \subset \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{n-m_r} \otimes \tau) \quad \text{and} \quad R_{P_1}(\Theta(\sigma, r+1)) = |\cdot|^{n-m_r} \otimes \tau.$$

Since the reducibility point $s_0 = n - m_r$ of $\mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^s \otimes \tau)$, $s \in \mathbb{R}$, is unique up to a sign [Si1, Lemma 1.2], $\mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{n-m_r+1} \otimes \tau) \cong \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{-(n-m_r+1)} \otimes \tau)$ is irreducible. Also,

$$\text{(2.2)}\quad \begin{gathered} \Theta(\tau, n+1) \subset \mathrm{Ind}_{Q_1}^{Sp(n+1)}(\chi|\cdot|^{m_r-n-1} \otimes \sigma), \\ R_{Q_1}(\Theta(\tau, n+1)) = \chi|\cdot|^{m_r-n-1} \otimes \sigma. \end{gathered}$$

Again, $\mathrm{Ind}_{Q_1}^{Sp(n+1)}(\chi|\cdot|^{m_r-n} \otimes \sigma) \cong \mathrm{Ind}_{Q_1}^{Sp(n+1)}(\chi|\cdot|^{-(m_r-n)} \otimes \sigma)$ is irreducible.

The next theorem is a complement to Theorem 2.1.

Theorem 2.3

(i) *The lift* $\Theta(\mathrm{Ind}_{Q_1}^{Sp(n+1)}(\chi|\cdot|^{m_r-n}\otimes\sigma),\ r+1)$ *has the unique proper maximal submodule. The corresponding quotient is isomorphic to* $\Theta(\sigma, r+1)$. *Furthermore,*

$$\Theta(\Theta(\sigma, r+1), n+1) = \mathrm{Ind}_{Q_1}^{Sp(n+1)}(\chi|\cdot|^{m_r-n}\otimes\sigma). \tag{2.3}$$

Finally, if π *is the other irreducible constituent of* $\mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{n-m_r}\otimes\tau)$, *then* $\Theta(\pi, n+1) = 0$.

(ii) *The lift* $\Theta(\mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{n-m_r+1}\otimes\tau), n+1)$ *has the unique proper maximal submodule. The corresponding quotient is isomorphic to* $\Theta(\tau, n+1)$. *Furthermore,*

$$\Theta(\Theta(\tau, n+1), r+1) = \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{n-m_r+1}\otimes\tau). \tag{2.4}$$

Finally, if π *is the other irreducible constituent of* $\mathrm{Ind}_{Q_1}^{Sp(n+1)}(|\cdot|^{m_r-n-1}\otimes\sigma)$, *then* $\Theta(\pi, r+1) = 0$.

Now, we are going to prove Theorems 2.1 and 2.3. Their proofs depend on the next proposition.

Proposition 2.4

(i) *If* $\rho\neq\chi|\cdot|^{m_r-n-1}$, *then* $\Theta(\chi\rho\otimes\tau, R_{P_j}(\omega_{n+j,r+j}))\cong\mathrm{Ind}_{Q_j}^{Sp(n+j)}(\widetilde{\rho}\otimes\sigma)$.

(ii) *If* $\rho\neq\chi|\cdot|^{m_r-n}$, *then* $\Theta(\rho\otimes\sigma, R_{Q_j}(\omega_{n+j,r+j}))\cong\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\widetilde{\rho}\otimes\tau)$.

We will postpone the proof of Proposition 2.4, and prove Theorems 2.1 and 2.3 first.

Proof of Theorem 2.1 We will prove the theorem in several steps. First, to simplify formulae, we write

$$\Pi=\mathrm{Ind}_{Q_j}^{Sp(n+j)}(\rho\otimes\sigma),\quad \Pi'=\mathrm{Ind}_{Q_j}^{Sp(n+j)}(\widetilde{\rho}\otimes\sigma),\quad\text{and}\quad \Upsilon=\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\rho\otimes\tau).$$

Since $\widetilde{\tau}\cong\tau$ and $\chi^2=1$, we have $\widetilde{\Upsilon}\cong\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\widetilde{\rho}\otimes\tau)$.

Step 1: Assume that π is an irreducible quotient of Π. Then, by Frobenius reciprocity,

$$\begin{aligned}&\mathrm{Hom}_{Sp(n+j)\times O(V_{r+j})}(\omega_{n+j,r+j},\ \pi\otimes\widetilde{\Upsilon})\\ &\qquad\cong\mathrm{Hom}_{Sp(n+j)\times GL(j,F)\times O(V_r)}(R_{P_j}(\omega_{n+j,r+j}),\pi\otimes\chi\widetilde{\rho}\otimes\tau).\end{aligned}$$

Hence, by Proposition 2.4(i),

$$\mathrm{Hom}_{Sp(n+j)\times O(V_{r+j})}(\omega_{n+j,r+j},\ \pi\otimes\widetilde{\Upsilon})\cong\mathrm{Hom}_{Sp(n+j)}(\Pi,\pi).$$

The last intertwining space is one dimensional. Hence $\Theta(\pi, r+j) \neq 0$. Finally, each irreducible subquotient of Π is a quotient of Π or Π'. So, its lift is non-zero.

Step 2: Assume that Π is reducible and ρ is not unitary. Write π_1 and π_2 for its unique irreducible subrepresentation and unique irreducible quotient, respectively. We have $R_{Q_j}(\pi_1) = \rho \otimes \sigma$ and $R_{Q_j}(\pi_2) = \widetilde{\rho} \otimes \sigma$. By Step 1, $\Theta(\pi_i, r+j) \neq 0$, $i = 1, 2$. Since $\pi_1 \otimes \Theta(\pi_1, r+j)$ is a quotient of $\omega_{n+j,r+j}$, $\rho \otimes \sigma \otimes \Theta(\pi_1, r+j)$ is a quotient of $R_{Q_j}(\omega_{n+j,r+j})$. Hence $\Theta(\pi_1, r+j)$ is a quotient of $\widetilde{\Upsilon}$. Let us prove that it is not equal to $\widetilde{\Upsilon}$. Otherwise, $R_{P_j}(\omega_{n+j,r+j})$ has a quotient $\pi_1 \otimes \chi\widetilde{\rho} \otimes \tau$. So, by Proposition 2.4, π_1 is a quotient of Π, which is a contradiction. Hence, $\Theta(\pi_1, r+j)$ is the unique irreducible quotient of $\widetilde{\Upsilon}$. Similarly, we conclude that $\Theta(\pi_2, r+j)$ is the unique irreducible quotient of Υ. Now, by Remark 1.4(ii), Theorem 2.1(i) follows.

Step3: Assume that Π is irreducible and $\rho \ncong \widetilde{\rho}$. Now, by Frobenius reciprocity,

$$\operatorname{Hom}_{Sp(n+j)}(\omega_{n+j,r+j}, \Pi) \cong \operatorname{Hom}_{GL(j,F)\times Sp(n)}(R_{Q_j}(\omega_{n+j,r+j}), \rho \otimes \sigma).$$

The usual map that gives the isomorphism above is, in fact, the isomorphism of the corresponding (not necessarily smooth) $O(V_{r+j})$-modules. Now, by the last part of Lemma 1.1 and Proposition 2.4, we get

$$\Theta(\Pi,\ r+j) \cong \Theta\big(\rho \otimes \sigma,\ R_{Q_j}(\omega_{n+j,r+j})\big) \cong \operatorname{Ind}_{P_j}^{O(V_{r+j})}(\chi\widetilde{\rho} \otimes \tau) = \widetilde{\Upsilon}.$$

Similarly, we can prove $\Theta(\Pi, r+j) = \Upsilon$. Now, $\Upsilon \cong \widetilde{\Upsilon}$. Hence, if ρ is not unitary, applying Theorem 1.3(ii), it is not difficult to see that Υ is irreducible. If ρ is unitary, then Υ is also irreducible. More precisely, $\operatorname{Ind}_{P_j^0}^{SO(V_{r+j})}(\rho \otimes \tau_0)$ is irreducible because a necessary condition for reducibility $w_0(\rho \otimes \tau_0) = \rho \otimes \tau_0$ does not hold (*cf.* [Ca] and Remark 1.4(i)). Since $\widetilde{\rho} \ncong \rho$, we can apply Theorem 1.3(iii) to see that Υ is irreducible.

Step 4: Assume Π is irreducible and $\rho \cong \widetilde{\rho}$. We shall show that Υ is also irreducible. Take $s \in \mathbb{R}$, and set

$$\Pi_s = \operatorname{Ind}_{Q_j}^{Sp(n+j)}(|\det|^s \rho \otimes \sigma).$$

The familiy Π_s, $s \in \mathbb{R}$, can be considered as a continuous familiy of Hermitian representations. By the usual complementary series argument, since Π_0 is irreducible, there exists $s_0 \neq 0$, such that Π_{s_0} is reducible. By Step 2, $\operatorname{Ind}_{P_j}^{O(V_{r+j})}(\chi|\det|^{s_0}\rho \otimes \tau)$ is also reducible. Considering the restriction to $SO(V_{r+j})$ (*cf.* Theorem 1.3(iii)), we see that $\operatorname{Ind}_{P_j}^{SO(V_{r+j})}(\chi|\det|^{s_0}\rho \otimes \tau_0)$ is reducible. As before, by [Si1, Lemma 1.2], $\operatorname{Ind}_{P_j}^{SO(V_{r+j})}(\chi\rho \otimes \tau_0)$ is irreducible. Clearly, we are not in the exceptional case of Theorem 1.3(iii). So, $\Upsilon = \operatorname{Ind}_{P_j}^{O(V_{r+j})}(\chi\rho \otimes \tau)$ is irreducible. Now, we can continue as in Step 3 to see $\Theta(\Pi, r+j) = \widetilde{\Upsilon}$. Steps 3 and 4 prove Theorem 2.1(iii).

Step 5: Assume Π is reducible and ρ is unitary. (Then $\rho \cong \widetilde{\rho}$.) Let us write $\Pi \cong \pi_1 \oplus \pi_2$, where π_1, π_2 are tempered mutually inequivalent irreducible representations. It follows that $R_{Q_j}(\pi_i) = \rho \otimes \sigma$, $i = 1, 2$, and

$$R_{Q_j}(\Pi) \cong R_{Q_j}(\pi_1) \oplus R_{Q_j}(\pi_2). \tag{2.5}$$

Let us prove that $\widetilde{\Upsilon}$ is also reducible. If not, then as in Step 3, we see $\Theta(\Upsilon, n+j) = \Pi$. Now, (2.5) implies that

$$\rho \otimes \sigma \otimes \widetilde{\Upsilon} \oplus \rho \otimes \sigma \otimes \widetilde{\Upsilon}$$

is a quotient of $R_{Q_j}(\omega_{n+j,r+j})$. This is a contradiction (*cf.* Proposition 2.4(ii)). Hence, by Theorem 1.3, $\widetilde{\Upsilon}$ is a direct sum of two mutually non-equivalent tempered representations.

Next, taking $\pi = \pi_i$, $i = 1, 2$, in Step 1, we see

$$\dim_{\mathbb{C}} \operatorname{Hom}_{Sp(n+j)\times GL(j,F)\times O(V_r)}(R_{P_j}(\omega_{n+j,r+j}), \pi \otimes \chi\widetilde{\rho} \otimes \tau) = 1. \tag{2.6}$$

Since, by Step 1, $\Theta(\pi_i, r+j)$ is a quotient of $\widetilde{\Upsilon}$ and $R_{P_j}(\widetilde{\Upsilon})$ is semisimple, (2.6) implies that $\Theta(\pi_i, r+j)$ is irreducible. Similarly, each irreducible subrepresentation π of $\widetilde{\Upsilon}$, satisfies $\Theta(\pi, n+j) \in \{\pi_1, \pi_2\}$. So, we must have

$$\widetilde{\Upsilon} \cong \Theta(\pi_1, r+j) \oplus \Theta(\pi_2, r+j).$$

Now Theorem 2.1(ii) follows from Remark 1.4(ii). This completes the proof of Theorem 2.1. ■

Proof of Theorem 2.3 We will prove (i). The proof of (ii) is analogous. The second part of (i) (see (2.3)) follows from (2.1) and Proposition 2.4(i). Now, to prove Theorem 2.3(i) it is enough to prove the first part. To simplify formulae, put $\Pi = \operatorname{Ind}_{Q_1}^{Sp(n+1)}(\chi|\cdot|^{m_r-n} \otimes \sigma)$.

First, if $m_r - n \neq 0$, then, as in Step 4, we see $\Theta(\Pi, r+1) \cong \operatorname{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{m_r-n} \otimes \tau)$. This completes the proof in that case.

Again, the tempered case $m_r - n = 0$ is more difficult. First,

$$\operatorname{Hom}_{Sp(n)\times O(V_{r+1})}(\omega_{n+1,r+1}, \Pi \otimes \operatorname{Ind}_{P_1}^{O(V_{r+1})}(1 \otimes \tau)) \tag{2.7}$$

$$\cong \operatorname{Hom}_{O(V_{r+1})}(\Theta(\Pi,\ r+1),\ \operatorname{Ind}_{P_1}^{O(V_{r+1})}(1 \otimes \tau)). \tag{2.8}$$

Then Proposition 2.4(i) implies that the space in (2.8) is one dimensional. Take a non-trivial map from the space in (2.7). Then the first part of the proof shows that its image is isomorphic to $\Theta(\sigma, r+1) \otimes \Pi$. Further, let φ be the corresponding map in (2.8). We claim that $V = \ker(\varphi)$ is the unique proper maximal submodule of $\Theta(\Pi, r+1)$. If not, then there is a submodule V', such that $\Theta(\Pi, r+1) = V + V'$. Hence, $V/V \cap V'$ is a quotient of $\Theta(\Pi, r+1)$. The filtration of $R_{Q_1}(\omega_{n+1,r+1})$ (*cf.* §3) and Lemma 3.3 imply that $R_{P_1}(V/V \cap V')$ has a quotient $1 \otimes \tau$. This gives a map in (2.8), which is not, up to a scalar, equal to φ. This is a contradiction, which completes the proof of Theorem 2.3. ■

3 Proof of Proposition 2.4

First we will recall filtrations of certain Jacquet modules of oscillator representations (*cf.* [Ku, Ku1] or [MVW, Ch. III, IV.5]).

(i) $R_{P_j}(\omega_{n+j,r+j})$ has filtration by I_{jk}, $0 \leq k \leq j$, where

$$\begin{aligned} I_{j0} &= |\det|^{-m_r+n+\frac{j+1}{2}} \otimes \omega_{n+j,r} && \text{(quotient)},\\ I_{jj} &= \operatorname{Ind}_{Q_j \times GL(j,F) \times O(V_r)}^{Sp(n+j)\times GL(j,F)\times O(V_r)}(\Sigma_j \otimes \omega_{n,r}) && \text{(subrepresentation)},\\ I_{jk} &= \operatorname{Ind}_{Q_k\times P'_{jk}\times O(V_r)}^{Sp(n+j)\times GL(j,F)\times O(V_r)}(\alpha_{jk} \otimes \Sigma_k \otimes \omega_{n+j-k,r}), && 0 < k < j. \end{aligned}$$

Here $P'_{j,k}$ is the standard parabolic subgroup of $GL(j,F)$ which corresponds to the partition $(j-k,k)$,

$$\alpha_{jk} = |\det|^{-m_r+n+\frac{j-k+1}{2}}$$

is a character of $GL(j-k,F)$, and Σ_k is the twist of the standard representation of $GL(k,F) \times GL(k,F)$ on smooth complex valued functions $C_c^\infty(GL(k,F))$:

$$\Sigma_k(g_1,g_2)f(h) = |\det g_1|^{-m_r-\frac{k+1}{2}}\chi(\det g_2)|\det g_2|^{m_r+\frac{k+1}{2}} f(g_1^{-1}hg_2).$$

(Here the first (resp., second) $GL(k,F)$ is a part of the Levi factor of P'_{jk}, (resp., of Q_k).)

(ii) $R_{Q_j}(\omega_{n+j,r+j})$ has filtration by J_{jk}, $0 \leq k \leq j$, where

$$\begin{aligned} J_{j0} &= \chi|\det|^{m_r-n+\frac{j-1}{2}} \otimes \omega_{n,r+j} && \text{(quotient)},\\ J_{jj} &= \operatorname{Ind}_{Sp(n)\times GL(j,F)\times P_j}^{Sp(n)\times GL(j,F)\times O(V_{r+j})}(\Sigma'_j \otimes \omega_{n,r}) && \text{(subrepresentation)},\\ J_{jk} &= \operatorname{Ind}_{Sp(n)\times Q'_{jk}\times P_k}^{Sp(n)\times GL(j,F)\times O(V_{r+j})}(\beta_{jk} \otimes \Sigma'_k \otimes \omega_{n,r+j-k}), && 0 < k < j. \end{aligned}$$

Here $Q'_{j,k}$ is the standard parabolic subgroup of $GL(j,F)$ which corresponds to the partition $(j-k,k)$, $\beta_{jk} = \chi|\det|^{m_r-n+\frac{j-k-1}{2}}$ is a character of $GL(j-k,F)$, and Σ_k is the twist of the standard representation of $GL(k,F) \times GL(k,F)$ on smooth complex valued functions $C_c^\infty(GL(k,F))$:

$$\Sigma'_k(g_1,g_2)f(h) = |\det g_1|^{m_r+j-\frac{k+1}{2}}\chi(\det g_1)|\det g_2|^{-m_r-j+\frac{k+1}{2}} f(g_1^{-1}hg_2).$$

(Here the first (resp., the second) $GL(k,F)$, is a part of the Levi factor of Q'_{jk}, (resp., of P_k).)

Remark 3.1 If $\pi \in \operatorname{Irr}(GL(k,F))$, then $\Theta(\pi,\Sigma_k) \cong \chi\widetilde{\pi}$. The same applies for Σ'_k. Here we may consider π as a representation of $GL(k,F) \times 1$ or $1 \times GL(k,F)$.

We will prove only Proposition 2.4(ii). The proof of (i) is analogous. To simplify notation, we will write

$$M'_j = GL(j,F) \times Sp(n) \quad \text{and} \quad \rho' = \rho \otimes \sigma.$$

For $s \in \mathbb{C}$, we put $\rho'_s = |\det|^s \rho \otimes \sigma$. Now, the filtration of $R_{Q_j}(\omega_{n+j,r+j})$ given in (ii) immediately implies

Lemma 3.2 *Assume that $j > 1$, and $s \in \mathbb{C}$. Then*

$$\mathrm{Hom}_{M_j'}(R_{Q_j}(\omega_{n+j,r+j})/J_{jj}, \rho_s') = 0.$$

Now, let us finish the proof of Proposition 2.4(ii), assuming the following lemma.

Lemma 3.3 *For each supercuspidal representation $\rho' = \rho \otimes \sigma \in \mathrm{Irr}(M_j')$,*

$$\Theta(\rho',\ J_{jj}) \cong \mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\widetilde{\rho} \otimes \tau).$$

First, recall that if $V \in \mathcal{A}(M_j')$, then we can decompose it into the direct sum of smooth modules [Be1, Ch. II, Proposition 26]

$$V \cong V(\rho') \oplus V_{\rho'},$$

such that each irreducible subquotient of $V(\rho')$ is of the form ρ_s', for some $s \in \mathbb{C}$, and $V_{\rho'}$ does not have an irreducible subquotient of the form ρ_s'. If $V(\rho')$ is not zero, it has an irreducible quotient.

If $j > 1$, then Lemma 3.2 implies $(R_{Q_j}(\omega_{n+j,r+j})/J_{jj})(\rho') = 0$. Now it is not difficult to see

$$R_{Q_j}(\omega_{n+j,r+j})(\rho') = J_{jj}(\rho'),$$

considering $J_{jj} \subset R_{Q_j}(\omega_{n+j,r+j})$. Hence, the natural map

$$\mathrm{Hom}_{M_j'}(R_{Q_j}(\omega_{n+j,r+j}), \rho') \to \mathrm{Hom}_{M_j'}(J_{jj}, \rho') \tag{3.1}$$

is an isomorphism of vector spaces. But, the map is also $O(V_{r+j})$-equivariant. Hence, we have (*cf.* Lemma 1.1)

$$\begin{aligned}\Theta(R_{Q_j}(\omega_{n+j,r+j}), \rho')^{\sim} &\cong \mathrm{Hom}_{M_j'}(R_{Q_j}(\omega_{n+j,r+j}), \rho')_\infty\\ &\cong \mathrm{Hom}_{M_j'}(J_{jj}, \rho')_\infty \cong \widetilde{\Theta(J_{jj}, \rho')}.\end{aligned}$$

Now, Lemma 3.3 completes the proof of Proposition 2.4(ii) in that case.

If $j = 1$, then ρ is a character of $F^\times$, which is by the assumption of Proposition 2.4(ii) different from the character that appears in J_{10} (*cf.* (b)). Now, $R_{Q_1}(\omega_{n+1,r+1})$ has an obvious quotient

$$\rho' \otimes \Theta(\rho', J_{11}) \oplus J_{10}.$$

Hence, one can see that the natural map (3.1) is an isomorphism. Finally, the proof of Proposition 2.4(ii) can be completed as before.

It remains to prove Lemma 3.3. To achieve that, we need the following simple extension of a result from Bernstein [Be]:

Lemma 3.4 *Assume that an l-group G' is the semidirect product $G \rtimes \mathbb{Z}/2\mathbb{Z}$, where G is a connected reductive F-group. Let $P = MN$ be a parabolic subgroup of G, and let $\overline{P} = M\overline{N}$ be the opposite parabolic subgroup of P. Assume that $\mathbb{Z}/2\mathbb{Z}$ normalizes M, N and $\overline{N}$. Put $M' = M \rtimes \mathbb{Z}/2\mathbb{Z}$, $P = M'N$, and $\overline{P}' = M'\overline{N}$. If $\pi \in \mathcal{A}(M')$ and $\Pi \in \mathcal{A}(G')$, then we have an isomorphism $\phi \mapsto \phi_0$*

$$\mathrm{Hom}_{G'}(\mathrm{Ind}_{P'}^{G'}(\pi), \Pi) \cong \mathrm{Hom}_{M'}(\pi, R_{\overline{P}'}(\Pi)),$$

where ϕ_0 is given by the composition of the natural inclusion (through a part of filtration that corresponds to an open orbit $P'\overline{P}'$ in $P' \setminus G'$)

$$\pi \hookrightarrow R_{\overline{P}'}(\mathrm{Ind}_{P'}^{G'}(\pi)),$$

and the natural map $\phi\colon R_{\overline{P}'}(\mathrm{Ind}_{P'}^{G'}(\pi)) \longrightarrow R_{\overline{P}'}(\Pi)$.

Proof If G' is connected, then Bernstein has shown that the map $\phi \mapsto \phi_0$ is an isomorphism. Now, Lemma 3.4 follows, considering the restriction of all representations in question to G. ■

Proof of Lemma 3.3 Set $\Pi = \Theta(\rho', J_{jj})$ and $\overline{P}_j = M_j\overline{N}_j^0$. Clearly, $\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\widetilde{\rho}\otimes\tau)$ is a quotient of Π. To prove the lemma, it is enough to see that Π is a quotient of that induced representation.

Let φ be the natural ephimorphism of $M_j' \times O(V_{r+j})$-modules $J_{jj} \to \rho' \otimes \Pi$. Then, as in Lemma 3.4, we can define a morphism

$$\varphi_0\colon \Sigma_j' \otimes \omega_{n,r} \longrightarrow \rho' \otimes R_{\overline{P}_j}(\Pi).$$

Remark 3.1 implies that the image of φ_0 is isomorphic to

$$\rho' \otimes \chi\widetilde{\rho} \otimes \tau. \tag{3.2}$$

So, we can factor $\varphi_0 = \varphi''\cdot\varphi'$, where φ' is the natural projection from $\Sigma_j'\otimes\omega_{n,r}$ to the module given by (3.2), and φ'' is an inclusion. Write $\mathrm{Ind}(\varphi')$ for the morphism of the corresponding induced modules. Let φ_1 be the morphism from Lemma 3.4, such that $(\varphi_1)_0 = \varphi''$. It is not difficult to see $(\varphi_1 \cdot \mathrm{Ind}(\varphi'))_0 = \varphi_0$. Hence, by Lemma 3.4, $\varphi = \varphi_1 \cdot \mathrm{Ind}(\varphi')$. This implies that the image of φ is isomorphic to a quotient of $\rho' \otimes \mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi\widetilde{\rho} \otimes \tau)$. This completes the proof of Lemma 3.3. ■

4 An Example

In this section we will assume that the characteristic of F is zero, because we will apply the reducibility results obtained by Shahidi [Sh].

We continue by recalling some results from [Sh]. Assume that ρ is a unitary supercuspidal representation of $GL(j, F)$, $j > 1$. Then, following [Sh], we call ρ a representation of symplectic type if $\mathrm{Ind}_{Q_j}^{Sp(j)}(|\det|^{1/2}\rho)$ is reducible, and a representation of orthogonal type if $\mathrm{Ind}_{Q_j}^{Sp(j)}(\rho)$ is reducible. In both cases $\rho \cong \widetilde{\rho}$. Also, Shahidi [Sh, Lemma 3.6] implies that every selfcontragredient supercuspidal representation is exactly of one of the above types. (For more details see [Sh], and for later interpretation in terms of K-types we refer to [MR].) One would like to describe all reducibilities of induced representations induced in terms of reducibility discussed above for split classical groups or their inner forms. The theorem below gives new examples of reducibilities.

Theorem 4.1 *Assume that V_0 is a four dimensional anisotropic space. (Then $\chi = 1$.) Let $\rho \in \mathrm{Irr}(GL(j,F))$ be a supercuspidal unitary representation, and let $s \in \mathbb{R}$. Put $I(s) = \mathrm{Ind}_{P_j^0}^{SO(V_j)}(|\det|^s\rho \otimes 1_{SO(V_0)})$. Then we have*

- (i) *If $\rho \not\cong \widetilde{\rho}$, then $I(s)$ is irreducible, for all $s \in \mathbb{R}$.*
- (ii) *If ρ has orthogonal type (hence $j > 1$), then $I(0)$ is reducible, and $I(s)$ is irreducible, for $s \neq 0$.*
- (iii) *If ρ has symplectic type (hence $j > 1$), then $I(s)$ is irreducible, for $s \neq \pm 1/2$, and $I(\pm 1/2)$ is reducible.*
- (iv) *If ρ is the trivial character of $GL(1,F)$, then $I(s)$ is irreducible for $s \neq \pm 2$ and $I(\pm 2)$ is reducible. If ρ is a nontrivial quadratic character of $GL(1,F)$, then $I(0)$ is reducible and $I(s)$ is irreducible for $s \neq 0$.*

Proof Note that $\Theta(1_{O(V_0)}, 0) = 1_{Sp(0)}$. Now, the theorem is a consequence of Theorems 1.3, 2.1, and 2.3, well-known reducibilities of induced representations in $SL(2) = Sp(1)$, and the fact that we can have at most one point of reducibility point for $s \geq 0$ (*cf.* [Si1]). ■

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