Letter to the Editor

Quantum Hall–MHD equations for a non–uniform dense magnetoplasma with electron temperature anisotropy

P. K. S H U K L A[1,2]† and L. S T E N F L O[1,2]‡

[1] Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie,
Ruhr-Universität Bochum, D-44780 Bochum, Germany
(ps@tp4.rub.de)
[2] Department of Physics, Umeå University, SE-90187 Umeå, Sweden
(lennart.stenflo@physics.umu.se)

Abstract. Nonlinear quantum Hall-MHD equations for a warm dense magneto-plasma with an anisotropic electron pressure are derived. The nonlinear equations include the quantum force associated with electron tunneling effects. The newly found equations can be used to investigate the dense plasma stability, as well as different types of waves, instabilities, and nonlinear structures in a warm dense magnetoplasma.

In a dense quantum Fermi plasma, the electrons are degenerate and the corresponding electron distribution obeys Fermi–Dirac statistics. In such a dense plasma, the governing quantum kinetic equation [1, 2], deduced from the Wigner equation [3], accounts for the electron degeneracy. In the past, several authors have studied the linear properties of electrostatic waves in unmagnetized [4, 5] and magnetized [6–8] plasmas containing degenerate electrons.

Recently, there has been a renewed interest in investigating waves [9–14], instabilities [15, 16] and turbulence [17], as well as coherent nonlinear structures [18–20] at nanoscales in dense quantum plasmas. The latter are ubiquitous in astrophysical bodies [21–23] (e.g. the interior of white dwarfs, magnetars, supernovae, etc.) and in intense laser–solid density plasma interaction experiments [24–26].

In this letter, we present nonlinear quantum Hall-MHD equations for a warm dense magnetoplasma in which the degenerate electrons have a pressure anisotropy. This may arise when the electrons are heated by intense electromagnetic and neutron beams in a dense magnetoplasma.

† Also at School of Physics, University of KwaZulu-Natal, 4000 Durban, South Africa; Max-Planck-Institut für extraterrestrische Physik, D-85741 Garching, Germany; Grupo de Lasers e Plasmas/Instituto de Plasmas e Fusao Nuclear, Instituto Superior Técnico, Universidade Técnica de Lisboa, 1049-001 Lisboa, Portugal; CCLRC Centre for Fundamental Physics, Rutherford Appleton Laboratory, Chilton, Didcot, Oxon 0X11 0QX, UK; SUPA, Department of Physics, University of Strathclyde, Glasgow G4 0NG, UK; Department of Physics, CIIT, Islamabad, Pakistan.
‡ Also at Department of Physics, Linköping University, SE-58183 Linköping, Sweden.

Let us suppose that our dense Fermionic plasma is embedded in an external magnetic field $\hat{\mathbf{z}}B_0$, where $\hat{\mathbf{z}}$ is a unit vector along the z-axis, and B_0 is the strength of the magnetic field. In the presence of electromagnetic fields, the dynamics of the inertialess hot electrons is governed by

$$0 = -n_e e\left(\mathbf{E} + \frac{1}{c}\mathbf{u}_e \times \mathbf{B}\right) - \nabla \cdot \mathscr{P} + n_e \mathbf{F}_Q, \tag{1}$$

where n_e is the electron number density, $\mathbf{u}_e$ is the electron fluid velocity, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively, e is the magnitude of the electron charge, c is the speed of light in vacuum, $\mathscr{P} = P_\perp \mathbf{I} + (P_\parallel - P_\perp)\hat{\mathbf{b}}\hat{\mathbf{b}}$ is the electron pressure, $\mathbf{I}$ is the unit dyad, and $\hat{\mathbf{b}}\hat{\mathbf{b}} = \mathbf{B}\mathbf{B}/B^2$ is the dyad formed from the unit vector $\hat{\mathbf{b}}$. Here $P_\parallel$ and $P_\perp$ are the components of the electron pressure parallel and perpendicular to $\hat{\mathbf{b}}$, respectively. Furthermore, the quantum force associated with the quantum Bohm potential is [27]

$$\mathbf{F}_Q = \frac{\hbar^2}{2m_e}\nabla\left(\frac{\nabla^2 \sqrt{n_e}}{\sqrt{n_e}}\right), \tag{2}$$

where $\hbar$ is the Planck constant divided by 2π, and m_e is the electron mass. For low-phase velocity (in comparison with c) disturbances, the electromagnetic fields are determined from the Faraday law

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}, \tag{3}$$

and Ampère's law

$$\nabla \times \mathbf{B} = \frac{4\pi e}{c}(Z_i n_i \mathbf{u}_i - n_e \mathbf{u}_e), \tag{4}$$

where Z_i is the ion charge state, and n_i and $\mathbf{u}_i$ are the ion number density and the ion fluid velocity, respectively. The latter are determined from the ion continuity and momentum equations

$$\frac{dn_i}{dt} + n_i \nabla \cdot \mathbf{u}_i = 0, \tag{5}$$

and

$$\rho_i \frac{d\mathbf{u}_i}{dt} = Z_i n_i e\left(\mathbf{E} + \frac{1}{c}\mathbf{u}_i \times \mathbf{B}\right) + \rho_i \mathbf{g} + 2\rho_i \mathbf{u}_i \times \mathbf{\Omega}, \tag{6}$$

where $d/dt = (\partial/\partial t) + \mathbf{u}_i \cdot \nabla$, $\rho_i = n_i m_i$ is the ion mass density, m_i is the ion mass, $\rho_i \mathbf{g}$ is the gravity force, and $|\mathbf{\Omega}|$ is the angular rotation frequency of the ions. The quantum force acting on the ions is much smaller than that on the electrons, and therefore it is not included in (6). Since in dense quantum plasmas, the ions are much colder than the electrons, we have also neglected the anisotropic ion pressure.

We now eliminate $\mathbf{E}$ from (6) by using (1), obtaining

$$\rho_i \frac{d\mathbf{u}_i}{dt} = \frac{eZ_i n_i}{c}\left(1 - \frac{Z_i n_i}{n_e}\right)\mathbf{u}_i \times \mathbf{B}$$

$$+ \frac{Z_i n_i}{n_e}\left[\frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} - \nabla \cdot \mathscr{P}\right] + \rho_i \mathbf{g} + 2\rho_i \mathbf{u}_i \times \mathbf{\Omega} + Z_i n_i \mathbf{F}_Q, \tag{7}$$

where from (4) we used

$$\mathbf{u}_e = \frac{Z_i n_i}{n_e}\mathbf{u}_i - \frac{c\nabla \times \mathbf{B}}{4\pi e n_e}. \tag{8}$$

By using the identity

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = [(\mathbf{B} \cdot \nabla)\mathbf{B} - \tfrac{1}{2}\nabla B^2], \tag{9}$$

we can express (7) as

$$\rho_{\mathrm{i}}\frac{d\mathbf{u}_{\mathrm{i}}}{dt} = \frac{eZ_{\mathrm{i}}n_{\mathrm{i}}}{c}\left(1 - \frac{Z_{\mathrm{i}}n_{\mathrm{i}}}{n_{\mathrm{e}}}\right)\mathbf{u}_{\mathrm{i}} \times \mathbf{B} - \frac{Z_{\mathrm{i}}n_{\mathrm{i}}}{n_{\mathrm{e}}}\nabla\left(P_{\perp} + \frac{B^2}{8\pi}\right)$$

$$+ \frac{Z_{\mathrm{i}}n_{\mathrm{i}}}{n_{\mathrm{e}}}\frac{(\mathbf{B} \cdot \nabla)}{4\pi}\left[\mathbf{B} - \frac{4\pi(P_{\parallel} - P_{\perp})\mathbf{B}}{B^2}\right] + \rho_{\mathrm{i}}\mathbf{g} + 2\rho_{\mathrm{i}}\mathbf{u}_{\mathrm{i}} \times \boldsymbol{\Omega} + Z_{\mathrm{i}}n_{\mathrm{i}}\mathbf{F}_{\mathrm{Q}}, \tag{10}$$

where we have introduced $\nabla \cdot \mathscr{P} = \nabla P_{\perp} + (\mathbf{B} \cdot \nabla)[(P_{\parallel} - P_{\perp})\mathbf{B}/B^2]$.

Furthermore, by using (1) and (4) we can write (3) as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\left(\frac{Z_{\mathrm{i}}n_{\mathrm{i}}}{n_{\mathrm{e}}}\mathbf{u}_{\mathrm{i}} - \frac{c\nabla \times \mathbf{B}}{4\pi en_{\mathrm{e}}}\right) \times \mathbf{B}\right] + \frac{c}{e}\nabla \times \left(\frac{\nabla \cdot \mathscr{P}}{n_{\mathrm{e}}}\right). \tag{11}$$

Let us invoke the following Chew, Goldberger and Low equations [28] for the parallel and perpendicular components of the electron pressure

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_{\mathrm{e}} \cdot \nabla\right)P_{\parallel} + P_{\parallel}\nabla \cdot \mathbf{u}_{\mathrm{e}} + 2P_{\parallel}(\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla) \cdot \mathbf{u}_{\mathrm{e}} = 0, \tag{12}$$

and

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_{\mathrm{e}} \cdot \nabla\right)P_{\perp} + 2P_{\perp}\nabla \cdot \mathbf{u}_{\mathrm{e}} - P_{\perp}(\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla) \cdot \mathbf{u}_{\mathrm{e}} = 0, \tag{13}$$

where $\mathbf{u}_{\mathrm{e}}$ is given by (8).

Equations (5) and (9)–(13), which form a closed system under the quasi-neutrality condition [29] $e(Z_{\mathrm{i}}n_{\mathrm{i}} - n_{\mathrm{e}}) = -q_{\mathrm{d}}n_{\mathrm{d}}$, are the desired quantum Hall-MHD equations. Here $q_{\mathrm{d}}n_{\mathrm{d}}$ is the charge density of dust grains (if present in a non-uniform dense magnetoplasma). In the absence of charged dust grains the condition $Z_{\mathrm{i}}n_{\mathrm{i}} = n_{\mathrm{e}} \equiv n$ would suffice.

In a quasi-neutral unmagnetized isotropic plasma without dust grains, we have from (10)

$$\frac{d\mathbf{u}_{\mathrm{i}}}{dt} = -\frac{\nabla P}{\rho} + \mathbf{g} + 2\mathbf{u}_{\mathrm{i}} \times \boldsymbol{\Omega} + \frac{Z_{\mathrm{i}}\hbar^2}{2m_{\mathrm{e}}m_{\mathrm{i}}}\nabla\left(\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right), \tag{14}$$

where P is the isotropic electron pressure and $\rho = nm_{\mathrm{i}}$. In an isotropic dense plasma, we also have the adiabatic equation of state

$$\frac{d}{dt}\left(\frac{P}{\rho^{\gamma}}\right) = 0, \tag{15}$$

where γ is the adiabatic index.

Equations (14) and (15) are similar in form to the equations describing solitary atmospheric waves [30]. Thus, linearizing (14) and (15) around the equilibrium values (namely ρ_0 and P_0), we obtain the dispersion relation for a non-uniform plasma

$$\omega^2 = \omega_{\mathrm{g}}^2\frac{k_{\perp}^2}{k^2} + \frac{4}{k^2}(\mathbf{k} \cdot \boldsymbol{\Omega})^2, \tag{16}$$

where ω is the frequency, $\omega_{\mathrm{g}} = [(d_z\rho_0/\rho_0 - d_zP_0/\gamma P_0)d_zP_0/\rho_0]^{1/2}$, and $k_{\perp}^2 = k^2 - k_z^2$. Here $k_{\perp}$ and k_z are the wave numbers across and along the density and pressure inhomogeneities, namely $d_z\rho_0$ and d_zP_0. The equilibrium, without ion flow,

is determined from

$$g = -\frac{d_z P_0}{\rho_0} + \frac{Z_i \hbar^2}{2 m_e m_i} \frac{d}{dz} \left(\frac{d_z^2 \sqrt{\rho_0}}{\sqrt{\rho_0}} \right), \tag{17}$$

where $\mathbf{g} = -g\hat{\mathbf{z}}$ and $\hat{\mathbf{z}}$ is the unit vector along the z-axis in a Cartesian coordinate system.

Following Stenflo [31], the dynamical equations for nonlinearly interacting modes in our inhomogeneous dense unmagnetized plasma are

$$\frac{\partial \nabla^2 u_z}{\partial t} - \hat{\mathbf{z}} \cdot (\nabla \times \nabla \times \mathbf{u}_i \cdot \nabla \mathbf{u}_i) + \nabla_\perp^2 \chi + R = 0, \tag{18}$$

and

$$\frac{d\chi}{dt} - \omega_g^2 u_z = 0, \tag{19}$$

where $u_z = \hat{\mathbf{z}} \cdot \mathbf{u}_i$, $\chi = (1/\rho) - (1/\rho_0)$, and $R = 2\hat{\mathbf{z}} \cdot (\nabla \times \nabla \times (\mathbf{u}_i \times \boldsymbol{\Omega}))$. Here χ has been normalized by the factor $d_z P_0$. In two-space dimensions, the coupled equations (18) and (19) with $R = 0$ (namely no ion rotation) admit a stationary dipolar vortex [31, 32] (a modon) whose profile is presented in [31].

In summary, we have presented the nonlinear quantum Hall-MHD equations in a non-uniform dense magnetoplasma with degenerate electrons having an anisotropic pressure. These equations can be used to investigate different kinds of waves, anisotropic electron pressure driven instabilities, as well as multiscale turbulence and structures in non-uniform dense magnetoplasmas. As an illustration, we have presented an example of linear and nonlinear electrostatic waves in non-uniform unmagnetized plasmas containing electron density and electron pressure gradients. It turns out that the quantum effects appear through the equilibrium state in the analyses of linear and nonlinear electrostatic modes in a non-uniform dense Fermi plasma.

References

[1] Klimontovich, Yu. L. and Silin, V. P. 1952 *ZhETF* **23**, 151.
[2] Kon'kov, V. L., Gorshenko, V. N. and Bogdanov, E. P. 1967 *Izvestiya VUZ Fizika* **10**, 71.
[3] Wigner, E. 1932 *Phys. Rev.* **40**, 749.
[4] Bohm, D. and Pines, D. 1953 *Phys. Rev.* **92**, 609.
[5] Pines, D. 1961 *Plasma Phys.* **2**, 2.
[6] Kelley, D. C. 1964 *Phys. Rev.* **134**, A641.
[7] Oberman, C. and Ron, A. 1963 *Phys. Rev.* **130**, 1291.
[8] Kuzelev, M. V. and Rukhadze, A. A. 1999 *Phys. Usp.* **42**, 603.
[9] Maafa, N. 1993 *Phys. Scr.* **48**, 351.
[10] Melrose, D. B. and Weise, J. I. 2002 *Phys. Plasmas* **9**, 4473.
[11] Shukla, P. K. 2007 *Phys. Lett.* A **369**, 312.
[12] Ren, H., Wu, Z. and Chu, P. K. 2007 *Phys. Plasmas* **14**, 062102.
[13] Marklund, M. and Brodin, G. 2007 *Phys. Rev. Lett.* **98**, 025001.
[14] Brodin, G. and Marklund, M. 2007 *Phys. Rev.* E **76**, 055403.
[15] Manfredi, G. 2005 *Fields Inst. Comm.* **46**, 263.
[16] Shukla, P. K. and Stenflo, L. 2006 *Phys. Plasmas* **13**, 044505.
[17] Shaikh, D. and Shukla, P. K. 2007 *Phys. Rev. Lett.* **99**, 125002.
[18] Shukla, P. K. and Eliasson, B. 2006 *Phys. Rev. Lett.* **96**, 245001.

[19] Shukla, P. K. and Eliasson, B. 2007 *Phys. Rev. Lett.* **99**, 096401.

[20] Brodin, G. and Marklund, M. 2007 *Phys. Plasmas* **14**, 112107.

[21] Chabrier, G., Douchin, F. and Potekhin, A. Y. 2002 *J. Phys.: Condens. Matter* **14**, 9133.

[22] Chabrier, G., Saumon, D. and Potekhin, A. Y. 2006 *J. Phys. A: Math. Gen.* **39**, 4411.

[23] Harding, A. K. and Lai, D. 2006 *Rep. Prog. Phys.* **69**, 2631.

[24] Marklund, M. and Shukla, P. K. 2006 *Rev. Mod. Phys.* **78**, 591.

[25] Salamin, Y. I., Hu, S. X., Hatsagortsyan, K. Z. and Keitel, C. H. 2006 *Phys. Rep.* **427**, 41.

[26] Glenzer, S. H. et al. 2007 *Phys. Rev. Lett.* **98**, 065002.

[27] Gardner, C. L. and Ringhofer, C. 1996 *Phys. Rev.* E **53**, 157.

[28] Bittencourt, J. A. 1986 *Fundamentals of Plasma Physics*. New York: Pergamon.

[29] Bharuthram, R., Saleem, H. and Shukla, P. K. 1992 *Phys. Scr.* **45**, 512.
Shukla, P. K. and Rao, N. N. 1996 *Phys. Plasmas* **3**, 1770.
Birk, G. T., Kopp, A. and Shukla, P. K. 1996 *Phys. Plasmas* **3**, 3564.
Mamun, A. A. and Shukla, P. K. 2001 *Phys. Lett.* A **290**, 173.
Shukla, P. K. and Mamun, A. A. 2003 *New J. Phys.* **5**, 17.

[30] Stenflo, L. 1991 *Phys. Scr.* **43**, 599.

[31] Stenflo, L. 1987 *Phys. Fluids* **30**, 3297.

[32] Shukla, P. K., Birk, G. T. and Bingham, R. 1995 *Geophys. Res. Lett.* **22**, 671.