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Singular Integrals on Product Spaces Related to the Carleson Operator

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Abstract. We prove $L^p(\mathbb{T}^2)$ boundedness, $1 < p \leq 2$, of variable coefficients singular integrals that generalize the double Hilbert transform and present two phases that may be of very rough nature. These operators are involved in problems of a.e. convergence of double Fourier series, likely in the role played by the Hilbert transform in the proofs of a.e. convergence of one dimensional Fourier series. The proof due to C.Fefferman provides a basis for our method.

1 Introduction

After the initial papers [7, 8], the theory of singular integrals on product spaces $\mathbb{R}^n \times \mathbb{R}^m$ received further contributions [15, 17, 19, 20]. R. Fefferman and E. M. Stein [8], motivated by some boundary-value problems, introduced singular integrals whose kernel $K(x', y')$ cannot be written in the form $K_1(x')K_2(y')$ so that their L^p boundedness cannot be obtained immediately by an iteration argument. As an example we mention the singular integral

$$\iint_D \frac{1}{x'}\frac{1}{y'} f(x - x', y - y')\, dx'dy'$$

in case D is a subset of $\mathbb{R}^2$ symmetric with respect to the origin, but not a rectangle, as defined, for instance, by $|y'| > |x'|$. Under some smoothness conditions on $K(x', y')$ — in our example this means some regularity on the cutoff associated with the set D — they proved L^p boundedness, $1 < p < \infty$, of the singular integrals as well as maximal inequalities.

Open problems of convergence almost everywhere of double Fourier series are the motivation of this paper. Let us mention the a.e. convergence of the square partial sums S_{NN} for Walsh series [11] and of the rectangular partial sums S_{NN^2}, for Fourier series [2, 4] and Walsh series as well, acting on $L^p(\mathbb{T}^2)$ spaces, $1 < p < 2$. We will introduce operators that belong to the family of singular integrals with variable coefficients and generalize the double Hilbert transform in a radical new way. Their main feature is a variable phase possibly of very rough nature.

The a.e. convergence of the partial sums $S_N f$ for one-dimensional Fourier series of L^p functions, $1 < p < \infty$, has been obtained by proving L^p estimates for the maximal partial sums operator $\sup_N |S_N f(x)|$. This is controlled by the Carleson operator, which shows a bounded integer valued phase $n(x)$ as follows

$$Cf(x) = \int_{-\pi}^{\pi} \frac{e^{in(x)x'}}{x'} f(x - x')\, dx',$$

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provided the L^p estimates are independent on $n(x)$ and its L^∞ norm. The proofs [1, 5, 12, 16] rely on the maximal Hilbert transform, with the Hilbert transform itself defined for any real number ξ_0, as follows:

$$H_1 f(x) = \int_{-\pi}^{\pi} \frac{e^{i\xi_0 x'}}{x'} f(x - x')\, dx'.$$

This operator, being equal to $e^{i\xi_0 x} H(e^{i\xi_0 x'} f(x'))(x)$, is immediately reduced to the standard Hilbert transform H. Let us observe that H_1 is just the Carleson operator in the special case in which $n(x)$ is replaced by a constant.

We are concerned with singular integrals that in two dimensions, relative to the open problems mentioned above, appear to play the basic role that the Hilbert transform H_1 plays in the one dimensional proofs. As one might expect, the Carleson operator will be involved.

Let us consider initially the square partial sums. The operator $\sup_N |S_{NN} f(x, y)|$ leads to the following singular integral

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{iN(x,y)x'}}{x'} \frac{e^{iN(x,y)y'}}{y'} f(x - x', y - y')\, dx' dy'$$

with $N(x, y)$ any bounded integer valued function. Let us replace the phase $N(x, y)$ by a function depending on one variable only. We denote such a function by $M(x)$. If the order of integration is as follows

$$\int_{-\pi}^{\pi} \frac{e^{iM(x)y'}}{y'} \Big(\int_{-\pi}^{\pi} \frac{e^{iM(x)x'}}{x'} f(x - x', y - y')\, dx' \Big)\, dy',$$

the operator is seen to be equal to $e^{iM(x)y} H_{y'}(e^{-iM(x)y'} C_{x'} f(x, y'))(y)$. Hence it reduces to the Carleson operator C acting on the variable x' followed the Hilbert transform H acting on the variable y'. The order of integration ought to be reversed to decode the above operator, if $N(x, y)$ were replaced by a function $R(y)$.

Now we observe that it is only natural to split (smoothly) the domain of integration into two regions $|y'| > |x'|$ and $|y'| < |x'|$, giving rise to two similar operators. All this led us to study the following singular integral

$$\int_{-\pi}^{\pi} \frac{e^{iM(x)y'}}{y'} \Big(\int_{|x'|<|y'|} \frac{e^{iM(x)x'}}{x'} f(x - x', y - y')\, dx' \Big)\, dy'. \tag{1}$$

We prove its $L^p(\mathbb{T}^2)$ boundedness, $1 < p \le 2$, with norm independent on $M(x)$ and its L^∞ norm. If the phase $N(x, y)$ were replaced by $R(y)$ our proof will run similarly, with the order of integrations reversed.

We also consider the rectangular partial sums $S_{NN^2} f$. In this case the singular integral to which we are led has the phases changed accordingly and the domain of integration split differently. The natural (smooth) subdivision is still along a straight line, though its slope depends on $M(x)$ as follows

$$\int_{-\pi}^{\pi} \frac{e^{iM^2(x)y'}}{y'} \Big(\int_{|x'|<M(x)|y'|} \frac{e^{iM(x)x'}}{x'} f(x - x', y - y')\, dx' \Big)\, dy'. \tag{2}$$

We shall also prove maximal inequalities relatively to (1) and (2). In [21] these results will find a first application to the mentioned a.e. convergence problems.

Our method which makes use of the full structure of the proof of the a.e. convergence of Fourier series given in [5] requires, in turn, the boundedness of the operators obtained by replacing all phases in (1) and (2) by constants. Singular integrals with domains of integration depending on one space variable — the x variable in (2) — and constant phases were studied in [17] to be applied in [18].

2 Notations and Statement of Results

To smoothly define our kernels, we need a smooth partition of unity. We decompose the kernel $\frac{1}{x'}$ by writing $\psi_k(x') = 2^k\psi(2^k x')$, $\psi(x')$ being a C^∞ function supported on $\{|x'| \leq 2\pi\}$ such that $\frac{1}{x'} = \sum_{k=0}^\infty \psi_k(x')$ for $|x'| \leq \pi$.

For $i = 1, 2, 3$ we shall consider the operators $\mathcal{P}_i$, defined in principal value, as follows

$$\mathcal{P}_1 f(x,y) = \sum_{h=0}^\infty e^{iM(x)y'}\psi_h(y') \sum_{k\geq h} e^{iM(x)x'}\psi_k(x') * f(x,y),$$

$$\mathcal{P}_2 f(x,y) = \sum_{h=0}^\infty e^{i\alpha M(x)y'}\psi_h(y') \sum_{2^{-k}\leq(2^{-h}\alpha,1)} e^{iM(x)x'}\psi_k(x') * f(x,y),$$

$$\mathcal{P}_3 f(x,y) = \sum_{h=0}^\infty e^{iM^2(x)y'}\psi_h(y') \sum_{2^{-k}\leq(2^{-h}4M(x),1)} e^{iM(x)x'}\psi_k(x') * f(x,y),$$

where we assume $\alpha \geq 1$, $M(x) \geq 1$ and denote by $(\beta,\gamma) = \min[\beta,\gamma]$. We shall prove the following:

Theorem 1 *Let $M(x)$ be a bounded real valued function greater or equal to one. Then the operators $\mathcal{P}_i f$, $i = 1, 2, 3$ defined above, are bounded from $L^r(\mathbb{T}^2)$ to $L^p(\mathbb{T}^2)$, $1 < p < r \leq 2$, with norm independent of f, α, $M(x)$ and its L^∞ norm.*

Maximal inequalities, as well as a stronger result in L^2, hold. In [20] we proved the following theorem relative to even more general operators.

Theorem 2 *Let $M_1(x)$ and $M_2(x)$ be bounded real valued functions. Then the operator*

$$\mathcal{P}_0 f(x,y) = \sum_{h=0}^\infty e^{iM_2(x)y'}\psi_h(y') \sum_{2^{-k}\leq r(h,x)} e^{iM_1(x)x'}\psi_k(x') * f(x,y)$$

is bounded from $L^2(\mathbb{T}^2)$ to itself with norm independent of any measurable $0 < r(h,x) \leq 1$, of the phases $M_1(x), M_2(x)$ and their L^∞ norms. Moreover the maximal *operator $\tilde{\mathcal{P}}_0$ satisfies the following pointwise inequality*

$$\tilde{\mathcal{P}}_0 f(x,y) = \sup_{h_0}\Big|\sum_{h\le h_0} e^{iM_2(x)y'}\psi_h(y')\sum_{2^{-k}\le r(h,x)} e^{iM_1(x)x'}\psi_k(x') * f(x,y)\Big|$$
$$\le c\{M_{y'}\tilde{C}_{x'}f(x,y) + M_{y'}\mathcal{P}_0 f(x,y)\}.$$

Above we denote the Hardy–Littlewood maximal function acting on the variable y' by $M_{y'}$ and the maximal Carleson operator acting on x' [14, 17] by $\tilde{C}_{x'}$.

The paper is organized as follows: (a) Decomposition; (b) Admissible pairs; (c) Incomparable pairs; (d) Trees and branches B_1 and B_2; (e) Main lemmas for B_2; (f) Main lemmas for B_1; (g) Proof of Theorem 1.

We shall prove Theorem 1 for the operator $\mathcal{P} = \mathcal{P}_1$ and give indications of the changes to be made for $\mathcal{P}_2$ and $\mathcal{P}_3$. The proof proceeds by showing appropriate two dimensional analogues of Lemma 0–5 of [5], whose ending combinatorics are used unaltered. The study of trees and branches is based on the L^p boundedness, $1 < p < \infty$, of $\mathcal{P}_0$ in case of constant phases $M_1(x)$ and $M_2(x)$, proved in [17].

3 Decomposition

To a pair $p = [\omega, I]$ consisting of dyadic intervals $\omega \subseteq \mathbb{R}$ and $I \subseteq \mathbb{T}$, $|I| = 2^{-k}$ (the Lebesgue measure on $\mathbb{T}$ is normalized to $\frac{dx}{2\pi}$) and $|\omega| = |I|^{-1}$, with the associated set $E_p = \{x \in I | M(x) \in \omega\}$, in [5] there corresponds the one-dimensional operator $T_p g(x) = [e^{iM(x)x'}\psi_k(x') * g(x)]\chi_{E_p}(x)$ and in our paper the two-dimensional operator

$$S_p f(x,y) = \Big[\sum_{h\le k} e^{iM(x)y'}\psi_h(y')e^{iM(x)x'}\psi_k(x') * f(x,y)\Big]\chi_{E_p\times\mathbb{T}}(x,y).$$

It is easily seen that

$$S_p f(x,y) = e^{iM(x)y} H_{y'}(e^{-iM(x)y'}T_p f(x,y'))(y) \tag{3}$$

for every $y \in \mathbb{T}$, where $H_{y'}$ denotes a truncated Hilbert trasform. To complete the understanding of $S_p f$ recall that $|T_p g(x)| \le c(Av_{I^*}|g|)\chi_{E_p}(x)$ where I^* is the double of I and $Av_{I^*}|g| = \frac{1}{|I^*|}\int_{I^*}|g(x')|dx'$.

Then

$$\mathcal{P}f(x,y) = \sum_{p\in\mathcal{B}} S_p f(x,y),$$

where $\mathcal{B}$ denotes the set of all pairs $p = [\omega, I]$.

Pairs will be subdivided into collections $\mathcal{F}_n = \{p \in \mathcal{B} \mid 2^{-n-1} < A(p) \le 2^{-n}\}$ depending upon the number

$$A(p) = \sup_{\substack{p'=[\omega',I']\\ I\subseteq I'}} \frac{|E_{p'}|}{|I'|}\Big(\frac{\text{distance }(\omega,\omega') + |\omega|}{|\omega|}\Big)^{-2000}$$

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manifestly related to $||T_p||_2^2 \cong \frac{|E_p|}{|I|}$. Then $\mathcal{B} = \bigcup_{n=0}^{\infty} \mathcal{F}_n$.

A partial order is defined among pairs, namely $p \prec p'$ if and only if $I \subseteq I'$ and $\omega \supseteq \omega'$.

4 Admissible Pairs

For any dyadic interval $\omega \subseteq \mathbb{R}$, let $\tilde{\omega}$ be the next larger dyadic interval containing ω, and let ω^* be the double of ω. We say that ω is central if $\omega^* \subseteq \tilde{\tilde{\omega}}$ and that $[\omega, I] \in \mathcal{B}$ is admissible if ω is central.

In [5], by means of Lemma 0, it has been proved that it suffices to study

$$\sum_{\substack{p\in\mathcal{B}\\ p\text{admissible}}} T_p g(x)$$

in place of $\sum_{p\in\mathcal{B}} T_p g(x)$. Similarly we have

Lemma 4.1 *$\mathcal{P}f(x,y) = 2\lim_{N\to\infty} \frac{1}{2N}\int_{-N}^{N} \mathcal{P}_\xi f(x,y)d\xi$ where $\mathcal{P}_\xi$ is defined, with respect to a new dyadic grid G_ξ centered at ξ, as $\mathcal{P}$ is defined with respect to the dyadic grid G centered at $\xi = 0$.*

Therefore we are allowed to consider only admissible pairs p. Henceforth, we will denote by

$$\mathcal{P}f(x,y) = \sum_{\substack{p\in\mathcal{B}\\ p \text{ admissible}}} S_p f(x,y)$$

5 Incomparable Pairs

Given any collection $\mathbb{Q}$ of pairs, no two of which are comparable under $\prec$ (therefore the $T_p g(x)$'s live on two by two disjoint sets E_p) and such that $A(p) \le \delta$ for all $p \in \mathbb{Q}$, it is proved in [5] that

$$\Big\|\sum_{p\in\mathbb{Q}} T_p g(x)\Big\|_r \le c_{\eta,r}\delta^{\frac{1}{2r'}-\eta}\|g\|_r \quad (\text{any } \eta > 0) \tag{4}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$ and $1 < r \le 2$. (See Lemma 2 of [5] and Lemma 2$'$ of [6]).

Similarly we are going to prove that

Lemma 5.1 *Let $\mathbb{Q}$ be a set of pairs no two of which are comparable under $\prec$. Assume $A(p) \le \delta$ for all $p \in \mathbb{Q}$. Then for $1 < r \le 2$,*

$$\Big\|\sum_{p\in\mathbb{Q}} S_p f(x,y)\Big\|_r \le c_{\eta,r}\delta^{\frac{1}{2r'}-\eta}\|f\|_r \quad (\textit{any } \eta > 0).$$

Proof The operators $S_p f(x, y)$ live on two by two disjoint sets $E_p \times \mathbb{T}$. Therefore

$$\Big\| \sum_{p\in\mathbb{Q}} S_p f(x, y)\Big\|_r^r = \sum_{p\in\mathbb{Q}} \|S_p f(x, y)\|_r^r.$$

By (3) and the boundedness of the truncated Hilbert transform $H_{y'}$,

$$\sum_{p\in\mathbb{Q}} \|S_p f(x, y)\|^r_{L^r(dxdy)} \le c_r \sum_{p\in\mathbb{Q}} \|T_p f(x, y')\|^r_{L^r(dxdy')}$$

for almost every x fixed. Then the lemma is proved by exchanging the order of integration at the right-hand side and applying (4). ■

Remark 5.2 Lemma 5.1 holds as well for $\mathcal{P}_2 f(x, y)$. The only change has to do with the truncation of $H_{y'}$: for a fixed $p = [\omega, I]$, $|I| = 2^{-k}$, the truncation will be at $2^{-h} \ge \frac{2^{-k}}{\alpha}$ instead of $2^{-h} \ge 2^{-k}$.

Remark 5.3 Lemma 5.1 holds as well for $\mathcal{P}_3 f(x, y)$. Because of Remark 5.2 the truncation of $H_{y'}$ is fixed at $2^{-h} \ge \frac{2^{-k}}{4M(x)}$, since x is fixed due the chosen order of integration.

6 Trees and Branches

Recall that a tree P with top $p^0 = [\omega^0, I^0]$ is defined to be a set of pairs with the properties

(a) $p \prec p' \prec p''$, p' admissible and $p, p'' \in P$ imply $p' \in P$;
(b) $p \prec p^0$ for all $p \in P$.

The corresponding operator, supported on $E = \bigcup_{p\in P} E_p$, is

$$Tg(x) = \sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'} \psi_k(x') * g(x),$$

where $D = \{k \ge 0 \mid \omega(k) \text{ is central}\}$. Let us state Lemma 3 of [5] and the relevant features of its proof.

Lemma 6.1 ([5, Lemma 3]) *Let P be a tree with top $p^0 = [\omega^0, I^0]$ and suppose $A(p) \le \delta$ for all $p \in P$. Then $\|T\|_r \le c_r \delta^{\frac{1}{r}}$, $1 < r \le 2$.*

Proof The proof is based on the following decomposition of T where ξ_0, the midpoint of ω^0, is assumed to be zero without loss of generality and where

$$|e^{iM(x)x'} - 1| \le |M(x)||x'| < 2^{K_0(x)}|x'| \le 1. \tag{5}$$

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Then
(6)

$$
\begin{aligned}
|Tg(x)| &\le \Big|\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in D}} \psi_k(x')*g(x)\Big| \\
&\quad + \Big|\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in D}} (e^{iM(x)x'}-1)\psi_k(x')*g(x)\Big| \\
&\le c\Big[\sup_{\sigma\ge 2^{-K_1(x)}}\Big(\frac{1}{2\sigma}\int_{-\sigma}^{\sigma}|R*g(x+z)|\,dz\Big) \\
&\quad + \sup_{\sigma\ge 2^{-K_1(x)}}\Big(\frac{1}{2\sigma}\int_{-\sigma}^{\sigma}|g(x+z)|\,dz\Big) + 2^{K_0(x)}\chi_{|x'|\le 2^{-K_0(x)}}(x')*|g|(x)\Big] \\
&\le cM_0g(x) + cM_0(R*g)(x)
\end{aligned}
$$

where $R(x') = \sum_{k\in D}\psi_k(x')$.

Denote by $\{I_s\}$ the maximal dyadic subintervals of I^0, such that $\frac{|E(\omega_I,I)|}{|I|} > \delta$. Set $\tilde{E}_s = E(\omega_{\tilde{I}_s},\tilde{I}_s)$ and $E_s = \tilde{E}_s \cap I_s$. Then $\{I_s\}$ is a non-trivial partition of I^0, $\frac{|E_s|}{|I_s|} \le c\delta$, and if $[\omega, I] \in P, I\cap I_s \ne \varnothing$ then $\tilde{I}_s \subseteq I$ and $E_p \cap I_s \subseteq E_s$. We define, for $x \in I_s$, $M_0g(x) = \sup_{I_s\subseteq I}\frac{1}{|I|}\int_I |g(x')|dx'$ if $x\in E_s$ and $M_0g(x) = 0$ if $x \notin E_s$. Therefore $||M_0||_r \le c_r\delta^{\frac{1}{r}}$ (see also [5, (9)]). This proves the lemma. ■

Remark 6.2 Equation (6) also holds for

$$
\sum_{\substack{h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x')*g(x),
$$

any $K_0(x)\le h\le K_1(x)$, since $|M(x)||x'| \le 2^{K_0(x)}2^{-h} \le 1$. Therefore

$$
\sup_{K_0(x)\le h\le K_1(x)}\Big|\sum_{\substack{h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x')*g(x)\Big| \le cM_0g(x) + cM_0(R*g)(x).
$$

Associated to a tree P we are going to consider the two-dimensional operator

$$
Bf(x,y) = \sum_{h=0}^{\infty} e^{iM(x)y'}\psi_h(y') \sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in D\\ k\ge h}} e^{iM(x)x'}\psi_k(x')*f(x,y)
$$

if $(x,y)\in E\times\mathbb{T}$; $Bf(x,y) = 0$ otherwise. We split B into two branches $B_1f(x,y)$ and $B_2f(x,y)$ that live on $E\times\mathbb{T}$ where

$$
B_1f(x,y) = \sum_{K_0(x)\le h\le K_1(x)} e^{iM(x)y'}\psi_h(y') \sum_{\substack{h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x')*f(x,y)
$$

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and

$$B_2 f(x,y) = \sum_{h \le K_0(x)} e^{iM(x)y'} \psi_h(y') \sum_{\substack{K_0(x) \le k \le K_1(x) \\ k \in D}} e^{iM(x)x'} \psi_k(x') * f(x,y).$$

B_2 is the easiest since its domain of integration is a rectangle. Clearly $B = B_1 + B_2$. We are going to prove

Lemma 6.3 *If P is a tree and $A(p) \le \delta$ for all $p \in P$ then $\|B\|_r \le c_r \delta^{\frac{1}{r}}$, $1 < r \le 2$.*

Proof We have (see also (3))

$$B_2 f(x,y) = e^{iM(x)y} H_{y'}(e^{-iM(x)y'} T f(x,y'))(y), \tag{7}$$

where $H_{y'}$ denotes here the Hilbert transform truncated at $2^{-K_0(x)}$, a fixed truncation since x is fixed. Therefore for almost every x fixed

$$\|B_2 f(x,y)\|_{L^r(dy)} \le c_r \|T f(x,y')\|_{L^r(dy')}.$$

Now integrating in dx both sides, exchanging the order of integration on the right-hand side and applying Lemma 6.1 we obtain

$$|B_2 f\|_r \le c_r \delta^{\frac{1}{r}} \|f\|_r. \tag{8}$$

We are left to prove

$$\|B_1 f\|_r \le c_r \delta^{\frac{1}{r}} \|f\|_r. \tag{9}$$

This will be done by writing $B_1 f(x,y)$ as a sum of two terms

$$B_1 f(x,y) = \text{main } f(x,y) + \text{error } f(x,y)$$

defined in (10) and (13) below. We are assuming ξ^0 midpoint of ω^0, to be equal to zero without loss of generality. Then

$$\begin{aligned} \text{main } f(x,y) &= \sum_{K_0(x) \le h \le K_1(x)} \psi_h(y') \sum_{\substack{K_0(x) \le h \le k \le K_1(x) \\ k \in D}} e^{iM(x)x'} \psi_k(x') * f(x,y) \\ &= \sum_{h \le K_1(x)} \psi_h(y') \sum_{\substack{K_0(x) \le h \le k \le K_1(x) \\ k \in D}} e^{iM(x)x'} \psi_k(x') * f(x,y) \\ &\quad - \sum_{h \le K_0(x)} \psi_h(y') \sum_{\substack{K_0(x) \le k \le K_1(x) \\ k \in D}} e^{iM(x)x'} \psi_k(x') * f(x,y). \end{aligned} \tag{10}$$

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The second term, somewhat simpler than B_2, satisfies (8). For the first term we prove that

(11)

$$\begin{aligned}
\Big|\sum_{h\le K_1(x)} \psi_h(y') * \sum_{\substack{K_0(x)\le h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x') * f(x,y)\\
- \theta_{K_1(x)}(y') * \sum_{h=0}^{\infty}\psi_h(y') * \sum_{\substack{K_0(x)\le h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x') * f(x,y)\Big|\\
\le c2^{-K_1(x)}\chi_{|y'|<2^{-K_1(x)}}(y') * \sup_{K_0(x)\le h\le K_1(x)}\Big|\sum_{\substack{h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x') * f(x,y')\Big|(y)\\
+ c2^{-K_1(x)}\chi_{|\tilde y|<2^{-K_1(x)}}(\tilde y) * \Big|\sum_{h=0}^{\infty}\psi_h(y') * \sum_{\substack{K_0(x)\le h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x') * f(x,\tilde y)\Big|(y)\\
+ c\frac{2^{-K_1(x)}}{(y')^2+2^{-2K_1(x)}} * \sup_{K_0(x)\le h\le K_1(x)}\Big|\sum_{\substack{h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x') * f(x,y')\Big|(y)
\end{aligned}$$

where $\theta(y')$ is a positive C^∞ function supported on $\{|y'|\le 1\}$ such that

$$\int_{-1}^{1}\theta(y')\,dy' = 1 \quad\text{and}\quad \theta_h(y') = 2^h\theta(2^h y').$$

For if $|y'|\le 1002^{-K_1(x)}$, then $\sum_{h\le K_1(x)}|\psi_h(y')|\le c2^{K_1(x)}$ and also $\|\check\theta_{K_1(x)}\|_1\le c2^{K_1(x)}$. If $|y'|>1002^{-K_1(x)}$, then $\sum_{h=0}^{\infty}\psi_h(y')=\sum_{h\le K_1(x)}\psi_h(y')$ and therefore

$$\begin{aligned}
\Big|\sum_{h\le K_1(x)}\psi_h(y') - \theta_{K_1(x)}(y') * \sum_{h=0}^{\infty}\psi_h(y')\Big|
&= \Big|\int\sum_{h\le K_1(x)}(\psi_h(y')-\psi_h(y'-y''))\theta_{K_1(x)}(y'')\,dy''\Big|\\
&\le c\int\sum_{h\le K_1(x)}|\psi_h'(\tilde y)|2^{-K_1(x)}\theta_{K_1(x)}(y'')\,dy''\\
&\le c\frac{2^{-K_1(x)}}{(y')^2}
\end{aligned}$$

for a suitable $\tilde y=\tilde y(y'')$. Now (11) implies the following inequality for the first term

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on the right-hand side of (10)

$$(12)\qquad \Big|\sum_{h\le K_1(x)}\psi_h(y')*\sum_{\substack{K_0(x)\le h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x')*f(x,y)\Big|$$

$$\le cM_{y'}\sup_{K_0(x)\le h\le K_1(x)}\Big|\sum_{\substack{h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x')*f(x,y')\Big|(y)$$

$$+M_{y'}\Big(\sum_{h=0}^{\infty}\psi_h(y')*\sum_{\substack{K_0(x)\le h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x')*f(x,y')\Big)(y).$$

Both terms on the right-hand side of (12) satisfy (9), as we shall prove, due to their action on the x' variable. For the first of the two, this is Remark 6.2, since

$$\sup_{K_0(x)\le h\le K_1(x)}\Big|\sum_{\substack{h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x')*f(x,y')\Big|$$

$$\le \sup_{K_0(x)\le h\le K_1(x)}\Big[cM_0f(x,y')+cM_0(R*f)(x,y')$$

$$+2^h\chi_{|x'|\le 2^{-h}}(x')*f(x,y')\Big]$$

$$\le cM_0f(x,y')+cM_0(R*f)(x,y').$$

For the second term, we set

$$H_3f(x,y)=\sum_{h=0}^{\infty}\psi_h(y')*\sum_{\substack{K_0(x)\le h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x')*f(x,y).$$

H_3 can be studied as in [17, Theorem 3]. We gain a factor of $\delta^{\frac{1}{r}}$, needed for (9), by pointing out that every time the maximal function $M_{x'}$ appears in the proof of [17, Theorem 3], we can write M_0 in the present case. We sketch this.

For almost every x fixed,

$$\|H_3f(x,y)\|_{L^r(dy)}\cong\Big\|\Big(\sum_J|S_JH_3f(x,y)|^2\Big)^{\frac12}\Big\|_{L^r(dy)}$$

by an application of the classical Littlewood–Paley S-function acting on the variable y'. With $J=[2^{\bar h},2^{\bar h+1})$, $h\lesseqgtr 0$ (and then $J=(-2^{\bar h+1},-2^{\bar h}]$),

$$|S_JH_3f(x,y)|=\Big|\int_J e^{i\eta y}\sum_h\Big\{\int_{2^{\bar h}}^{\eta}(\hat\psi_h)'(t)\,dt+\hat\psi_h(2^{\bar h})\Big\}$$

$$\times\sum_{\substack{K_0(x)\le h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x')*\hat f(x,\eta)\,d\eta\Big|$$

$$\le |S_J^{(1)}H_3f(x,y)|+|S_J^{(2)}H_3f(x,y)|$$

where

$$|S_J^{(2)} H_3 f(x,y)| \le \sum_{h=0}^{\infty} |\hat{\psi}_h(2^{\bar{h}})| \sup_{K_0(x)\le h\le K_1(x)} \Big| \sum_{\substack{h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'} \psi_k(x') * S_J f(x,y) \Big|$$
$$\le cM_0 S_J f(x,y) + M_0(C * S_J) f(x,y)$$

by exchanging the order of integration, with $Cg(x) = \sum_{k\in D} e^{iM(x)x'} \psi_k(x') * g(x)$ denoting a kind of Carleson operator [22]. We have used the inequality $\tilde{C}g(x) \le Mg(x) + MCg(x)$ for the Carleson maximal operator. By vector valued estimates for the maximal function and for the Carleson operator itself [9, Theorem 6.4, p. 519], we have

$$\Big\| \Big(\sum_J |S_J^{(2)} H_3 f(x,y)|^2 \Big)^{\frac{1}{2}} \Big\|^r_{L^r(dxdy)} \le c\delta \Big\| \Big(\sum_J |S_J f(x',y)|^2 \Big)^{\frac{1}{2}} \Big\|^r_{L^r(dx'dy)}$$
$$\le \delta \|f\|^r_{L^r(dx'dy')}.$$

Next is

$$|S_J^{(1)} H_3 f(x,y)|$$
$$\le \int_J \sum_{h=0}^{\infty} |(\hat{\psi}_h)'(t)| \sup_{K_0(x)\le h\le K_1(x)} \Big| \sum_{\substack{h\le k\le K_1(x)\\ k\in D}} e^{iM(x)x'} \psi_k(x') * S_t S_J f(x,y) \Big| \, dt$$
$$\le c \int_J \sum_{h=0}^{\infty} |(\hat{\psi}_h)'(t)| \big[M_0 S_t S_J f(x,y) + \text{similar term} \big] \, dt$$
$$\le c \Big(\int_J |M_0 S_t S_J f(x,y)|^2 \, d\gamma(t) \Big)^{\frac{1}{2}} + \text{similar term}$$

by exchanging the order of integration again. Here $d\gamma(t) = \sum_h |(\hat{\psi}_h)'(t)|$ and S_t denotes the multiplier transformation corresponding to the interval $[t, 2^{\bar{h}+1})$.

Then by a continuous version of vector valued estimates for the maximal function we obtain

$$\Big\| \Big(\sum_J |S_J H_3 f(x,y)|^2 \Big)^{\frac{1}{2}} \Big\|^r_{L^r(dxdy)} \le c_r \Big\| \Big(\int_0^{\infty} |M_0 S_t S_J f(x,y)|^2 \, d\gamma(t) \Big)^{\frac{1}{2}} \Big\|^r_{L^r(dxdy)}$$
$$\le c_r \delta \Big\| \Big(\int_0^{\infty} |S_t S_J f(x',y)|^2 \, d\gamma(t) \Big)^{\frac{1}{2}} \Big\|^r_{L^r(dx'dy)}$$
$$\le c_r \delta \Big\| \Big(\sum_J \int_J |S_J f(x',y)|^2 \, d\gamma(t) \Big)^{\frac{1}{2}} \Big\|^r_{L^r(dx'dy)}$$

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$$\leq c_r\delta \Big\| \Big(\sum_J |S_J f(x',y)|^2 \Big)^{\frac{1}{2}} \Big\|^r_{L^r(dx'dy)}$$

$$\leq c_r \delta \|f\|^r_{L^r(dx'dy')}$$

by [23, Theorem 4′′, p. 103], by the fact $\int_J d\gamma(t) \leq c$ and by Littlewood–Paley theorem again. This concludes the proof of $\|$ main $\|_r \leq c_r\delta^{\frac{1}{r}}$.

Next is

(13)

$$\text{error } f(x,y) = \sum_{K_0(x)\leq h\leq K_1(x)} \big(e^{iM(x)y'} - 1\big)\,\psi_h(y') \sum_{\substack{h\leq k\leq K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x') * f(x,y).$$

By (5) and (6)

$$|\text{error } f(x,y)|$$

$$\leq \sum_{K_0(x)\leq h\leq K_1(x)} |e^{iM(x)y'} - 1||\psi_h(y')| \Big| \sum_{\substack{h\leq k\leq K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x') * f(x,y) \Big|$$

$$\leq c \sum_{K_0(x)\leq h\leq K_1(x)} |e^{iM(x)y'} - 1||\psi_h(y')|[M_0 f(x,y') + M_0(R*f)(x,y')]$$

$$\leq c2^{K_0(x)}\chi_{|y'|\leq 2^{-K_0(x)}}(y') * [M_0 f(x,y') + M_0(R*f)(x,y')](y)$$

$$\leq cM_{y'}M_0 f(x,y) + cM_{y'}M_0(R*f)(x,y)$$

Therefore $\|\text{error}\|_r \leq c_r\delta^{\frac{1}{r}}$ because this estimate holds for M_0. Thus (9) is proved and therefore Lemma 6.3. ■

In the above proof, the role of Carleson operator could be taken by the Hilbert transform: it suffices also to replace the other occurrence of $M(x)$ in the definition of main f, by the very same constant ξ_0 (that was assumed to be zero). More error terms correspond to this choice.

Remark 6.4 The analogue of Lemma 6.3 holds for $\mathcal{P} = \mathcal{P}_2$ as well. Clearly (7) holds the corresponding B_2, namely

$$\sum_{2^{-h}\geq \frac{2^{-K_0(x)}}{\alpha}} e^{i\alpha M(x)y'}\psi_h(y') \sum_{\substack{K_0(x)\leq k\leq K_1(x)\\ k\in D}} e^{iM(x)x'}\psi_k(x') * f(x,y)$$

and similarly for the main term of the corresponding B_1 defined in (10). For instance the first term, on the right-hand side in the analogue of (10), satisfies the corresponding estimate (12). Therefore its L^r norm is dominated by $c_r\delta^{\frac{1}{r}}$ since

$$\sup_{\frac{2^{-K_1(x)}}{\alpha}\leq 2^{-h}\leq \frac{2^{-K_0(x)}}{\alpha}} \Big| \sum_{\substack{2^{-K_1(x)}\leq 2^{-k}\leq \alpha 2^{-h}\\ k\in D}} e^{iM(x)x'}\psi_k(x') * f(x,y') \Big|$$

$$\leq cM_0 f(x,y') + cM_0(C*f)(x,y')$$

For the "error term" in the analogue of (13) it suffices to observe that

$$\sum_{\frac{2^{-K_1(x)}}{\alpha}\le 2^{-h}\le \frac{2^{-K_0(x)}}{\alpha}} |e^{i\alpha M(x)y'}-1||\psi_h(y')|$$
$$\le \alpha|M(x)|\,|y'| \sum_{\frac{2^{-K_1(x)}}{\alpha}\le 2^{-h}\le \frac{2^{-K_0(x)}}{\alpha}} |\psi_h(y')|$$
$$\le \alpha 2^{K_0(x)}\chi_{|y'|\le \frac{2^{-K_0(x)}}{\alpha}}(y')$$

Therefore the corresponding operator is dominated by $M_{y'}$.

Remark 6.5 Lemma 6.3 holds for $\mathcal{P}=\mathcal{P}_3$ as well. Regarding H_3, the summation over k now becomes $2^{-K_1(x)}\le 2^{-k}\le 2^{-h}4M(x)\le 2^{-K_0(x)}$ and the estimates involving the maximal function M_0 still hold, since M_0 is related to the unaffected lower bound $2^{-K_1(x)}$.

Regarding (13) we write the error term of the present case with ξ_0 generic and observe that the estimate now involves

$$\sum_{\frac{2^{-K_1(x)}}{4M(x)}\le 2^{-h}\le \frac{2^{-K_0(x)}}{4M(x)}} |e^{iM^2(x,y)y'}-e^{-i\xi_0^2y'}|\,|\psi_h(y')|,$$

which is still dominated by $M_{y'}$ since

$$|M^2(x)-\xi_0^2|\,|y'|<4M(x)|M(x)-\xi_0|\,|y'|\le 1.$$

7 Main Lemmas for B_2

Since B_2 has been decoded in (7), we can imediately state

Main Lemma 1 *Let $\{P_j\}$ be a family of trees with tops $[\omega_j^0,I_j^0]$. Assume that $[\omega_j^0,I_j^0]\in P_j$ for each j and that*

(a) *$A(p)<\delta$ for any $p\in P_j$;*
(b) *$p\not\le p'$ for any $p\in P_j$, $p'\in P_{j'}$, $j\ne j'$;*
(c) *no point of $[0,2\pi]$ belongs to more than $K\delta^{-20}$ of the I_j^0.*

Then there exists a set $F\subset[0,2\pi]$, $|F|\le c\frac{\delta^{100}}{K}$, with the property

$$\Big\|\sum_j B_2^{P_j}f(x,y)\Big\|_{L^2({}^cF\times\mathbb{T})}\le c_\eta(\lg K)\delta^{\frac14-\eta}\|f\|_{L^2(\mathbb{T}^2)}\quad (\textit{any }\eta>0)$$

for all $f\in L^2(\mathbb{T}^2)$ and every $K>10$.

Proof Due to property (b) the operators $T^{P_j}g(x)$ live on two by two disjoint sets E_j. Therefore

$$\Big\|\sum_j T^{P_j}g(x)\Big\|^2_{L^2(^cF)} = \sum_j \big\| T^{P_j}g(x)\big\|^2_{L^2(^cF)} \le c_\eta (\lg K)^2 \delta^{\frac{1}{2}-\eta}\|g\|_2^2$$

by [5, Main Lemma]. Therefore by (7) the $B_2^{P_j}f(x,y)$ live on two by two disjoint sets $E_j \times \mathbb{T}$ and

$$\Big\|\sum_j B_2^{P_j}f(x,y)\Big\|^2_{L^2(^cF\times\mathbb{T})} = \sum_j \big\| B_2^{P_j}f(x,y)\big\|^2_{L^2(^cF\times\mathbb{T})} \le \sum_j \big\| T^{P_j}f(x,y')\big\|^2_{L^2(^cF\times\mathbb{T})} \le c_\eta(\lg K)^2\delta^{\frac{1}{2}-\eta}\|f\|_2^2$$

by the boundedness of the Hilbert transform $H_{y'}$. ∎

Now precisely as in [5, p. 570], the Main Lemma in L^r, $1 < r < 2$, follows.

Main Lemma 2 *Let* $1 < r < 2$. *Under the above assumptions* (a), (b) *and*

(d) *no point of* $[0, 2\pi]$ *belongs to more than* $K\delta^{-1+\rho}$ *of the* I_j^0, *where* $\rho = \rho(r) > 0$ *is a small number.*

Then there exists a set $F \subseteq [0, 2\pi]$, $|F| \le c\frac{\delta^\rho}{K^M}$, *any* $M > 10$ *and* $K > K_0(r, M)$ *such that*

$$\Big\|\sum_j B_2^{P_j}f(x,y)\Big\|_{L^r(^cF\times\mathbb{T})} \le c_{r,\eta}K^{a(r)}\delta^{\sigma(r)}\|f\|_{L^r(\mathbb{T}^2)} \quad (\textit{any } \eta > 0)$$

for all $f \in L^r(\mathbb{T}^2)$, *where* $0 < a(r) < 1$ *and* $\sigma = \sigma(r) > 0$.

Remark 7.1 The above Main Lemmas hold for the operator $\mathcal{P}_2$, by Remark 5.2.

Remark 7.2 The above Main Lemmas hold for the operator $\mathcal{P}_3$, by Remark 5.3.

8 Main Lemmas for B_1

The goal here is to prove the Main Lemmas, of the preceding section, for B_1, also. Main Lemma 1 requires two-dimensional analogues of Lemma 4 and Lemma 5 of [5]. We recall the definition of *normal* tree and *separated* trees, [5, p. 562].

Fix numbers $\delta > 0$ and $K > 10$. A tree P with top $[\omega^0, I^0]$ is *normal* if for $[\omega, I] \in P$ we have $|I| \le \frac{\delta^{1000}}{K}|I^0|$, $\operatorname{dist}(I, \partial I^0) > 3\frac{\delta^{100}}{K}|I^0|$ ($\partial I^0$is the boundary of I^0). Then $T^{P*}h(x)$ lives on $\{x \in I^0 \mid \operatorname{dist}(x, \partial I^0) > 2\frac{\delta^{100}}{K}\}$.

Two trees P with top $[\omega^0, I^0]$ and P' with top $[\omega^1, I^1]$ are *separated* if either $I^0 \cup I^1 = \varnothing$ or else

$$(\alpha) \qquad [\omega, I] \in P, \quad I \subseteq I^1 \text{ imply } \operatorname{dist}(\omega, \omega^1) > \delta^{-1}|\omega|,$$

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and

$$(\beta)\qquad [\omega', I'] \in P', \quad I' \subseteq I^0 \text{ imply } \operatorname{dist}(\omega', \omega^0) > \delta^{-1}|\omega'|.$$

Then the following holds:

Lemma 8.1 *Let P with top $[\omega^0, I^0]$ and P' with top $[\omega^1, I^0]$ be separated trees. Then*

$$\|B_1^{P'} B_1^{P*}\|_2 \le c_M \delta^M \quad (\text{any } M > 0).$$

Equivalently, $|(B_1^{P*} f, B_1^{P'*} g)| \le c_M \delta^M \|f\|_2 \|g\|_2$ for any $f, g \in L^2(\mathbb{T}^2)$.

Proof We are going to show that

$$B_1^{P*} f = \Phi * (B_1^{P*} f) + \mathcal{E}(f), \tag{14}$$

$$B_1^{P'*} g = \Phi' * (B_1^{P'*} g) + \mathcal{E}'(g), \tag{15}$$

where $\|\mathcal{E}\|_2 \le c_M \delta^M$ and $\|\mathcal{E}'\|_2 \le c_M \delta^M$ for suitable *bump* functions Φ and Φ'. Define $\Phi(x', y') = \varphi(x')\varphi(y')$ where φ is as in [5], that is $\varphi(x')$ is a C^∞ function on $\mathbb{R}$ satisfying

(i) φ is supported on $\{|x'| \le \delta^{\frac{1}{2}} d\}$, $\|\varphi\|_1 \le c_M$;
(ii) $|\hat{\varphi}(\xi)| \le c_M(\delta^{\frac{1}{2}} d|\xi - \xi_0|)^{-2M}$ for all ξ, in particular for $|\xi - \xi_0| > \delta^{-\frac{1}{2}} d^{-1}$($\xi_0 =$ midpoint of ω^0);
(iii) $|\hat{\varphi}(\xi) - 1| \le c_M(\delta^{\frac{1}{2}} d|\xi - \xi_0|)^{2M}$ for all ξ with $|\xi - \xi_0| \le \delta^{-\frac{1}{2}} d^{-1}$, where $d = \min\{|I| \mid [\omega, I] \in P\}$.

Then $\Phi(x', y')$ is $C^\infty(\mathbb{R}^2)$ and it satisfies

(i′) Φ is supported in $\{|x'|, |y'| \le \delta^{\frac{1}{2}} d\}$ and $\|\Phi\|_1 \le c_M$;
(ii′) $|\hat{\Phi}(\xi, \eta)| \le c_M(\delta^{\frac{1}{2}} d|\xi - \xi_0|)^{-2M}$ for all ξ with $|\xi - \xi_0| > \delta^{-\frac{1}{2}} d^{-1}$ and all η;
(iii′) $|\hat{\Phi}(\xi, \eta)| \le c_M(\delta^{\frac{1}{2}} d|\eta - \xi_0|)^{-2M}$ for all η with $|\eta - \xi_0| > \delta^{-\frac{1}{2}} d^{-1}$ and all ξ;
(iv′) $|\hat{\Phi}(\xi, \eta) - 1| \le c_M[(\delta^{\frac{1}{2}} d|\xi - \xi_0|)^{2M} + (\delta^{\frac{1}{2}} d|\eta - \xi_0|)^{2M}]$ for all ξ and η with $|\xi - \xi_0| \le \delta^{-\frac{1}{2}} d^{-1}$, $|\eta - \xi_0| \le \delta^{-\frac{1}{2}} d^{-1}$.

Similarly define $d' = \min\{|I'| \mid [\omega', I'] \in P'\}$, pick φ' corresponding to P' and consider $\Phi'(x', y')$. Then

$$\|\hat{\Phi} \cdot \hat{\Phi}'\|_{L^\infty(\mathbb{R}^2)} \le c_M \delta^M. \tag{16}$$

In fact $\|\hat{\varphi}(\xi)\widehat{\varphi'}(\xi)\|_\infty \le c_M \delta^M$ because P and P' are separated.

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To prove (14) we verify the dual statement

$$\mathcal{E}^* f(x,y) = B_1^P f(x,y) - B_1^P(\Phi * f)(x,y)$$
$$= \sum_{K_0(x)\le h\le K_1(x)} \left(e^{iM(x)y'}\psi_h(y')\right) * \sum_{\substack{h\le k\le K_1(x)\\ k\in D}} \left(e^{iM(x)x'}\psi_k(x')\right) * f(x,y)$$
$$- \sum_{K_0(x)\le h\le K_1(x)} \left(e^{iM(x)y'}\psi_h(y')\right) *$$
$$\left(\varphi * \sum_{\substack{h\le k\le K_1(x)\\ k\in D}} \left(e^{iM(x)x'}\psi_k(x')\right) * (\varphi * f)(x,y')\right)(y)$$

for $(x,y) \in E\times\mathbb{T}$, where $\mathcal{E}^*$ lives. Then, by adding and subtracting the same term, we write $\mathcal{E}^* f = \mathcal{E}_1^* f + \mathcal{E}_2^* f$ defined in (17) and (19) below.

(17)
$$\mathcal{E}_1^* f(x,y) = \sum_{K_0(x)\le h\le K_1(x)} e^{iM(x)y'}\psi_h(y') *$$
$$\sum_{\substack{h\le k\le K_1(x)\\ k\in D}} \left[e^{iM(x)x'}\psi_k(x') - \left(e^{iM(x)x'}\psi_k(x')\right) * \varphi(x')\right] * f(x,y).$$

By exchanging the order of summation we obtain

$$|\mathcal{E}_1^* f(x,y)| \le \sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in D}} \left| \sum_{K_0(x)\le h\le k} e^{iM(x)y'}\psi_h(y') * F_k(x,y')(y)\right|$$
$$\le \sum_{2^k\le d^{-1}} \left|\tilde{H}_{y'}\left(e^{-iM(x)y'}F_k(x,y')\right)(y)\right|,$$

where $\tilde{H}_{y'}$ denotes the maximal Hilbert transform and

$$F_k(x,y') = \left[e^{iM(x)x'}\psi_k(x') - \left(e^{iM(x)x'}\psi_k(x')\right) * \varphi(x')\right] * f(x,y').$$

Then obviously

(18)
$$\|\mathcal{E}_1^* f(x,y)\|_{L^2(dxdy)} \le \sum_{2^{-k}\le d^{-1}} \|\tilde{H}_{y'}\left(e^{-iM(x)y'}F_k(x,y')\right)(y)\|_{L^2(dxdy)}$$
$$\le c \sum_{2^{-k}\le d^{-1}} \|F_k(x,y')\|_{L^2(dxdy')}$$
$$\le \sum_{2^{-k}\le d^{-1}} c_M(\delta^{\frac{1}{2}}2^k d)^{2M}\|2^k\chi_{[-2^{-k},2^{-k}]}(x') * f(x,y')\|_{L^2(dxdy')}$$
$$\le c_M\delta^M\|f\|_{L^2(dx'dy')}$$

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by the boundedness of $\tilde{H}_{y'}$ and [5, (20)].

Next we consider

(19)

$$\mathcal{E}_2 f(x,y) = \sum_{K_0(x)\le h\le K_1(x)} (e^{iM(x)y'}\psi_h(y')) * \sum_{\substack{h\le k\le K_1(x)\\ k\in D}} (e^{iM(x)x'}\psi_k(x')) * \varphi(x') * f(x,y) - \sum_{K_0(x)\le h\le K_1(x)} (e^{iM(x)y'}\psi_h(y')* \Big(\varphi * \sum_{\substack{h\le k\le K_1(x)\\ k\in D}} (e^{iM(x)x'}\psi_k(x')) * (\varphi * f)(x,y')\Big)(y).$$

We clearly have

$$\begin{aligned} |\mathcal{E}_2^* f(x,y)| &\le \sum_{K_0(x)\le h\le K_1(x)} \left| e^{iM(x)y'}\psi_h(y') - (e^{iM(x)y'}\psi_h(y')) * \varphi(y')\right| * \\ &\qquad \left| \sum_{\substack{h\le k\le K_1(x)\\ k\in D}} (e^{iM(x)x'}\psi_k(x')) * (\varphi * f)(x,y')\right| (y) \\ &\le \sum_{2^h\le d^{-1}} \left| e^{iM(x)y'}\psi_h(y') - (e^{iM(x)y'}\psi_h(y')) * \varphi(y')\right| * \\ &\qquad \tilde{C}_{x'}(\varphi * f)(x,y')(y), \end{aligned}$$

where $\tilde{C}_{x'}$ denotes Carleson maximal operator [13].

Then by [5, (20)] we obtain

$$\begin{aligned} \text{(20)}\quad \|\mathcal{E}_2^* f(x,y)\|_{L^2(dxdy)} &\le \sum_{2^h\le d^{-1}} c_M(\delta^{\frac{1}{2}}2^h d)^{2M} \|2^h\chi_{[-2^{-h},2^{-h}]}(y')* \\ &\qquad \tilde{C}_{x'}(\varphi * f)(x,y')(y)\|_{L^2(dxdy)} \\ &\le c_M\delta^M \|\tilde{C}_{x'}(\varphi * f)(x,y')\|_{L^2(dxdy')} \le c_M\delta^M\|f\|_{L^2(dx'dy')} \end{aligned}$$

by the boundedness of $\tilde{C}_{x'}$ and of the convolution with $\varphi(x')$.

Now

$$\begin{aligned} (B_1^{P*} f, B_1^{P'*} g) &= (\Phi * (B_1^{P*} f), \Phi' * (B_1^{P'*} g)) \\ &\quad + (\mathcal{E}(f), \Phi' * (B_1^{P'*} g)) + (\Phi * (B_1^{P*} f), \mathcal{E}'(g)) \\ &\quad + (\mathcal{E}(f), \mathcal{E}(g)) \end{aligned}$$

The first term on the right-hand side is dominated by

$$\|\hat{\Phi}' \cdot \hat{\Phi}\|_\infty \|B_1^{P*} f\|_2 \|B_1^{P'*} g\|_2 \le c_M\delta^M \|f\|_2\|g\|_2$$

by (16) and (9) with $\delta = 1$. The remaining terms satisfy the same estimate by (18) and (20). ■

Remark 8.2 Lemma 8.1 holds for $\mathcal{P} = \mathcal{P}_2$ as well. $B_1 = B_1^\alpha$ is now defined by

$$\sum_{\frac{2^{-K_1(x)}}{\alpha} \le 2^{-h} \le \frac{2^{-K_0(x)}}{\alpha}} e^{i\alpha M(x)y'} \psi_h(y') \sum_{\substack{2^{-K_1(x)} \le 2^{-k} \le \alpha 2^{-h} \\ k \in D}} e^{iM(x)x'} \psi_k(x') * f(x, y).$$

Clearly (18) holds for the analogous $\mathcal{E}_1^*$. The analogous $\mathcal{E}_2^*$ is dominated by

$$\sum_{\frac{2^{-K_1(x)}}{\alpha} \le 2^{-h} \le \frac{2^{-K_0(x)}}{\alpha}} \left| e^{i\alpha M(x)y'} \psi_h(y') - \left(e^{i\alpha M(x)y'} \psi_h(y') \right) * \varphi^\alpha(y') \right| *$$
$$\left| \sum_{\substack{2^{-K_1(x)} \le 2^{-k} \le \alpha 2^{-h} \\ k \in D}} (e^{iM(x)x'} \psi_k(x')) * (\varphi * f)(x, y') \right| (y),$$

where $\varphi^\alpha(y')$ is supported on $\{|y'| \le \delta^{\frac{1}{2}} \frac{d}{\alpha}\}$, $\|\varphi^\alpha\|_1 \le c_M$ and $\widehat{\varphi^\alpha}(\eta)$ is concentrated around $\alpha\xi_0$, namely

(ii′)

$$|\widehat{\varphi^\alpha}(\eta)| \le c_M \Big(\frac{\delta^{\frac{1}{2}} d}{\alpha} |\eta - \alpha\xi_0| \Big)^{-2M} \text{ for } |\eta - \alpha\xi_0| > \Big(\frac{\delta^{\frac{1}{2}} d}{\alpha} \Big)^{-1};$$

(iii′)

$$|\widehat{\varphi^\alpha}(\eta) - 1| \le c_M \Big(\frac{\delta^{\frac{1}{2}} d}{\alpha} |\eta - \alpha\xi_0| \Big)^{2M} \text{ for } |\eta - \alpha\xi_0| \le \Big(\frac{\delta^{\frac{1}{2}} d}{\alpha} \Big)^{-1}.$$

Then

$$\begin{aligned} &|e^{i\alpha M(x)y'} \psi_h(y') - (e^{i\alpha M(x)y'} \psi_h(y')) * \varphi^\alpha(y')| \\ &\qquad \le \|\hat{\psi}_h(\eta - \alpha M(x))[1 - \widehat{\varphi^\alpha}(\eta)]\|_1 \chi_{|y'|_{\le 2^{-h}}}(y') \\ &\qquad \le c_M \Big(\delta^{\frac{1}{2}} \frac{d}{\alpha} 2^h \Big)^{2M} 2^h \chi_{|y'| \le 2^{-h}}(y'). \end{aligned}$$

Therefore in the analogue of (20) we shall have

$$\sum_{2^h \le \alpha d^{-1}} c_M \big(\delta^{\frac{1}{2}} \frac{d}{\alpha} 2^h \big)^{2M} \le c_M \delta^M.$$

Remark 8.3 Lemma 8.1 holds for $\mathcal{P} = \mathcal{P}_3$ as well. The ω's being central, it follows that for every pair $[\omega, I] \in P$, the interval $\omega \subset [2^\mu, 2^{\mu+1}]$ with $\mu \ge 0$ an integer. Let $\alpha = 2^\mu$. Now if

$$M(x) \in \omega = [n2^k, (n+1)2^k),$$

then

$$M^2(x) \in \omega^2 = [n^2 2^{2k}, (n+1)^2 2^{2k}).$$

The interval ω^2, not any longer dyadic as far as size and location, is comparable with $\alpha\omega = [\alpha n 2^k, \alpha(n+1)2^k)$. This is indeed the meaning of the elementary inequalities

$$2|\alpha\omega| \le |\omega^2| \le 4|\alpha\omega| \quad \text{and} \quad \operatorname{dist}(\alpha\omega, 0) \le \operatorname{dist}(\omega^2, 0) \le 2\operatorname{dist}(\alpha\omega, 0).$$

Similarly for all $[\omega', I'] \in P'$, the interval $\omega' \subseteq [2^\nu, 2^{\nu+1}]$. Let $\beta = 2^\nu$ and assume $\alpha \le \beta$. We are going to show that the collections

$$\{[\omega^2, I]\}_{[\omega,I]\in P} \quad \text{and} \quad \{[(\omega')^2, I']\}_{[\omega',I']\in P'}$$

satisfy a separation property analogous to (α) and (β). Denote by $|B-A|$ the length of the interval $[A,B]$ with $B > A > 0$. Then $|B-A| = \operatorname{dist}(\omega', \omega^0) > \delta^{-1}|\omega|$ (by the assumption (β)), so we have

$$\operatorname{dist}((\omega')^2, (\omega^0)^2) = |B^2 - A^2| > (B+A)\delta^{-1}|\omega'| > \beta\delta^{-1}|\omega'| \ge \frac{\delta^{-1}}{4}|(\omega')^2|.$$

Similarly

$$\operatorname{dist}(\omega^2, (\omega^1)^2) > \beta\delta^{-1}|\omega| \ge \alpha\delta^{-1}|\omega| \ge \frac{\delta^{-1}}{4}|\omega^2|.$$

If we choose $\varphi(y') = \varphi^{4\alpha}(y')$ supported on

$$\Big\{ |y'| \le \frac{\delta^{\frac{1}{2}} d}{4\alpha} \Big\},$$

$\|\varphi^{4\alpha}\|_1 \le c_M$ such that $\hat{\varphi}$ is concentrated around $\overline{\xi}_0$, the center of ω_0^2, then we obtain

$$\|\mathcal{E}_2^* f\|_2 \le c_M \delta^M \|.$$

Similarly with $\varphi'(y')$, replacing α by β, d by d' and $\overline{\xi}_0$ by $\overline{\xi}_0'$.

On the other hand (16) still holds.

Lemma 8.4 *Let a row $\mathcal{R}$ be a union of normal trees P_k with tops $[\omega_0^k, I_0^k]$ where the $\{I_0^k\}$ are pairwise disjoint. Let P' be a tree with top $[\omega_0', I_0']$ and suppose, for each k, that $I_0^k \subseteq I_0'$ and P_k, P' are separated. Then $\|B_1^{P'} B_1^{\mathcal{R}*}\|_2 \le c_M \delta^M$ (any $M > 10$).*

Proof We ought to prove $|\sum_k (B_1^{P'*} g, B_1^{P_k*} f)| \le c_M \delta^M \|f\|_2 \|g\|_2$. We will examine one term at the time and write

$$B_1^{P'} f = B_1^{P_k'} f + B_1^{P_k''} f + B_1^{P_k'''} f$$

following the decomposition $P' = P_k' \cup P_k'' \cup P_k'''$ of P' relative to P_k [5]. Recall that

$$P_k''' = \Big\{ [\omega', I'] \in P' \mid |I'| < \frac{\delta^{1000}}{10K}|I_k^0| \text{ and } \operatorname{dist}(I', \partial I_k^0) \le \frac{\delta^{100}}{K}|I_k^0| \text{ or } I' \cap I_k^0 = \varnothing \Big\}.$$

Since P_k is a normal tree, then

$$(21) \qquad \left(B_1^{P_k'''} g, B_1^{P_k *} f\right) = 0$$

the two factors having disjoint supports in the x variable. For, $T^{P_k} h$ applies to $h(x')$, $x' \in A = \{x' \in I_0^k \mid \operatorname{dist}(x', \partial I_0^k) > 2\frac{\delta^{100}}{K}|I_0^k|\}$ and $B_1^{P_k *} f(x, y)$ lives on $A \times \mathbb{T}$. Next recall that

$$P_k' = \left\{ [\omega', I'] \in P' \mid |I'| \le \frac{\delta^{1000}}{10K}|I_k^0|, I' \subseteq I_k^0 \text{ and } \operatorname{dist}(I', \partial I_k^0) > \frac{\delta^{100}}{K}|I_0^k| \right\}.$$

Since P_k' and P_k have the same top space I_k^0 and are separated by assumption, by Lemma 8.1 we have

$$(22) \qquad |(B_1^{P_k' *} g, B_1^{P_k *} f)| \le c_M \delta^M \|f\|_{L^2(I_k^0 \times \mathbb{T})} \|g\|_{L^2(I_k^0 \times \mathbb{T})}.$$

We are left with

$$P_k'' = \left\{ [\omega', I'] \in P' \mid |I'| > \frac{\delta^{1000}}{10K}|I_k^0| \right\}.$$

We construct Φ_k and $\mathcal{E}_k$ as in the proof of Lemma 8.1 using P_k for P, then $\|\mathcal{E}_k\|_2 \le c_M \delta^M$. In addition to (i)–(iii) above, we can assume

(iv) $\hat{\varphi}_k(\xi_0') = 0$, where ξ_0' is the midpoint of ω_0'.

Thus

$$(23) \qquad (B_1^{P_k'' *} g, B_1^{P_k *} f) = (B_1^{P' *} g, \mathcal{E}_k(f)) - (B_1^{P_k' *} g, \mathcal{E}_k(f)) - (B_1^{P_k''' *} g, \mathcal{E}_k(f)) + (B_1^{P_k'' *} g, \Phi_k * B_1^{P_k *} f).$$

The first three terms are easy. We have

$$(24) \qquad \left|(B_1^{P' *} g, \mathcal{E}_k(f)\right| \le c_M \delta^M \|B_1^{P' *} g\|_{L^2(I_k^0 \times \mathbb{T})} \|f\|_{L^2(I_k^0 \times \mathbb{T})},$$

since $\mathcal{E}_k f(x, y)$ lives on $I_k^0 \times \mathbb{T}$, being P_k a normal tree. Since in addition P_k' is a normal tree we obtain

$$(25) \qquad \left|(B_1^{P_k' *} g, \mathcal{E}_k(f))\right| \le c_M \delta^M \|g\|_{L^2(I_k^0 \times \mathbb{T})} \|f\|_{L^2(I_k^0 \times \mathbb{T})}.$$

Finally since the two factors have disjoint supports, we have

$$(26) \qquad (B_1^{P_k''' *} g, \mathcal{E}_k(f)) = 0.$$

We are reduced to estimate

$$(27) \qquad \left(B_1^{P_k'' *} g, \Phi_k * B_1^{P_k *} f\right) = \left(g, B_1^{P_k''}(\Phi_k * F_k)\right)$$

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where $F_k = B_1^{P_k *} f$. Let us write explicitly

$$B_1^{P_k''}(\Phi_k * F_k)(x,y) = \sum_{K_0(x) \le h \le K_1(x)} e^{iM(x)y'} \psi_h(y') \sum_{\substack{h \le j \le K_1(x) \\ j \in D}} (e^{iM(x)x'} \psi_j(x')) * \Phi_k * F_k(x,y)$$

for suitable $K_0(x)$ and $K_1(x)$. By definition of P_k'' we have $2^{-K_1(x)} > \frac{\delta^{1000}}{10K} |I_k^0|$. Also recall that $\Phi_k(x',y') = \varphi_k(x')\varphi_k(y')$ with $\varphi_k(x')$ supported on

$$\{|x'| \le \delta^{\frac{1}{2}} d_k\}, \quad d_k = \min_{[\omega,I] \in P_k} |I|$$

and so $d_k < \frac{\delta^{1000}}{K} |I_k^0|$, being P_k a normal tree. With d_k'' similarly defined with respect to P_k'' we then observe that

$$\left| B_1^{P_k''}(\Phi_k * F_k)(x,y) \right| \le \sum_{2^j, 2^h \le (d_k'')^{-1}} \left| (e^{iM(x)y'} \psi_h(y')) * \varphi_k(y') \right| * \left| (e^{iM(x)x'} \psi_j(x')) * \varphi_k(x') \right| * |F_k|(x,y).$$

Now [5, (24)] states

$$\sum_{2^j \le (d_k'')^{-1}} \left| (e^{iM(x)x'} \psi_j(x')) * \varphi_k(z) \right| \le c_M \delta^M \sum_{2^j \le (d_k'')^{-1}} d_k 2^{2j} \chi_{|z| < 2^{-j}}(z) \le c_M \delta^M \frac{d_k''}{|z|^2 + (d_k'')^2} = c_M \delta^M U_k(z).$$

The above estimate is based an the following facts

$$|M(x) - \xi_0'| \le |I'|^{-1} = 2^j \le (d_k'')^{-1},$$

$$\left| \frac{\partial}{\partial_\xi} \hat{\varphi}_k(\xi_0') \right| \le c_M (\delta^{\frac{1}{2}} d_k)(\delta^{\frac{1}{2}} d_k |\xi_0' - \xi_0|)^{-2M} \le c_M (\delta^{\frac{1}{2}} d_k) \delta^M,$$

$$\|(e^{iM(x)x'} \psi_j(x')) * \varphi_k(z)\|_\infty \le \|\hat{\psi}_j(\xi - M(x)) \hat{\varphi}_k(\xi)\|_1 \le c_M (\delta^{\frac{1}{2}} d_k) \delta^M 2^{2j},$$

$$\operatorname{supp}((e^{iM(x)x'} \psi_j(x')) * \varphi_k(z)) \subseteq \{|z| \le 2^{-j}\}.$$

Since the index h spans the same set of the index j, it is also true that

$$\sum_{2^h \le (d_k'')^{-1}} \left| (e^{iM(x)y'} \psi_h(y')) * \varphi_k(w) \right| \le c_M \delta^M U_k(w).$$

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Thus, going back to (27), we have

$$\begin{aligned}\Big|\Big(B_1^{P_k''*}g,\Phi_k * B_1^{P_k*}f\Big)\Big| &\le \Big(|g|,\big|B_1^{P_k''}(\Phi_k * F_k)\big|\Big)\\ &\le c_M\delta^{2M}(|g|,U_k(w)U_k(z)*|F_k|)\\ &\le c_M\delta^{2M}(U_k(z)U_k(w)*|g|,|F_k|)\\ &\le c_M\delta^{2M}\|g^*\|_{L^2(I_k^0\times\mathbb{T})}\|f\|_{L^2(I_k^0\times\mathbb{T})},\end{aligned}$$

where g^* denotes the two dimensional maximal function of g. Then, as in [5], Lemma 8.4 follows by an application of Schwartz's inequality and Lemma 6.3 with $\delta = 1$. ■

Remark 8.5 Lemma 8.4 holds for $\mathcal{P} = \mathcal{P}_2$ as well. Relatively to (27), only the summation over h changes. Specifically, assuming $\widehat{\varphi_k^\alpha}(\alpha\xi_0') = 0$, we have

$$\begin{aligned}\sum_{2^h\le\alpha(d_k'')^{-1}}\big|e^{i\alpha M(x)y'}\psi_h(y')*\varphi_k^\alpha(w)\big| &\le \sum_{2^h\le\alpha(d_k'')^{-1}} c_M\delta^M\frac{d_k''}{\alpha}2^{2h}\chi_{|w|<2^{-h}}(w)\\ &\le c_M\delta^M\frac{d_k''/\alpha}{w^2+(d_k''/\alpha)^2}.\end{aligned}$$

Remark 8.6 Lemma 8.4 holds for $\mathcal{P} = \mathcal{P}_3$ as well. Relatively to (27) what has now to be estimated is

$$Q(w) = \sum_{2^h\le 4\beta(d_k'')^{-1}}\big|e^{iM^2(x)y'}\psi_h(y')*\varphi^{\alpha_k}(w)\big|,$$

where $\varphi^{\alpha_k}(y')$ is supported on $\big\{|y'|\le\frac{\delta^{\frac12}d_k}{4\alpha_k}\big\}$ and $\alpha_k = M(x)$ relatively to P_k, similarly $\beta = M(x)$ relatively to P' as in Remark 8.3.

First assume $d_k''/\beta \ge \delta^{\frac12}d_k/\alpha_k$. Denoting by $\overline{\xi}_0'$ the midpoint of $(\omega_0')^2$, we may assume $\frac{d}{d\eta}\widehat{\varphi^{\alpha_k}}(\eta)_{|\eta=\overline{\xi}_0'} = 0$. Then similarly to the proof of Lemma 8.4 we have

$$\operatorname{supp}\{e^{iM^2(x)y'}\psi_h(y')*\varphi^{\alpha_k}(w)\}\subseteq\{|w|\le 2^{-h}\},$$

and

$$\big\|\hat{\psi}_h(\eta-M^2(x))\cdot\widehat{\varphi^{\alpha_k}}(\eta)\big\|_1\le c_M\delta^M\Big(\frac{\delta^{\frac12}d_k}{4\alpha_k}\Big)2^{2h},$$

since the main contribution to the above integral comes from the size of $\widehat{\varphi^{\alpha_k}}$ on the essential support of $\hat{\psi}_h$. Therefore

$$Q(w)\le c_M\delta^M\frac{d_k''/4\beta}{w^2+(d_k''/4\beta)^2}.$$

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If instead $d_k''/\beta < \delta^{\frac{1}{2}} d_k/\alpha_k$, then it remains to estimate

$$\overline{Q}(w) = \sum_{(\delta^{\frac{1}{2}} d_k)^{-1} 4\alpha_k \le 2^h \le (d_k'')^{-1} 4\beta} \left| e^{iM^2(x)y'} \psi_h(y') * \varphi^{\alpha_k}(w) \right|.$$

Now

$$\operatorname{supp}\{e^{iM^2(x)y'} \psi_h(y') * \varphi^{\alpha_k}(w)\} \subseteq \left\{ |w| \le \frac{\delta^{\frac{1}{2}} d_k}{4\alpha_k} \right\}$$

and

$$\sum_{(\delta^{\frac{1}{2}} d_k)^{-1} 4\alpha_k \le 2^h \le (d_k'')^{-1} 4\beta} \|\hat{\psi}_h(\eta - M^2(x)) \cdot \widehat{\varphi^{\alpha_k}}(\eta)\|_1 \le c_M \delta^M \left(\frac{\delta^{\frac{1}{2}} d_k}{4\alpha_k} \right)^{-1},$$

since, due to the separation property, the main contribution to the above summation comes from the term corresponding to $2^h = (d_k'')^{-1} 4\beta$ and is determined by the size of $\hat{\psi}_h$ on the essential support of $\widehat{\varphi^{\alpha_k}}$.

Therefore the factor $c_M \delta^M$ is gained and, aside from that, $\overline{Q}(w)$ is dominated operatorwise by an average.

From now on the proof is the same as in [5] for $\mathcal{P} = \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$. We sketch it.

Corollary 8.7 *Let $\mathcal{R} = P_1 \cup P_2 \cup \cdots$ and $\mathcal{R}' = P_1' \cup P_2' \cup \cdots$ be rows with tops $[\omega_k^0, I_k^0]$ for P_k and $[\omega_{k'}^1, I_{k'}^1]$ for P_k'. Suppose that each I_k^0 is contained in an $I_{k'}^1$ with P_k and $P_{k'}'$ separated. Then*

$$\left\| B_1^{\mathcal{R}'} B_1^{R*} \right\|_2 \le c_M \delta^M \quad (\textit{any } M > 10).$$

Also observe that if $A(p) \le \delta$ for $p \in \mathcal{R}$ above, then $\|B_1^{\mathcal{R}}\|_r \le c_r \delta^{\frac{1}{r}}$ by Lemma 6.3.

Main Lemma 3 *Let $\{P_j\}$ be a family of trees with tops $[\omega_j^0, I_j^0]$. Assume that $[\omega_j^0, I_j^0] \in P_j$ for each j and that*

(a) *$A(p) < \delta$ for any $p \in P_j$;*

(b) *$p \not\le p'$ for any $p \in P_j$, $p' \in P_{j'}$, $j \ne j'$;*

(c) *no point of $[0, 2\pi]$ belongs to more than $K\delta^{-20}$ of the I_j^0.*

Then there exists a set $F \subset [0, 2\pi]$, $|F| \le c\frac{\delta^{100}}{K}$, with the property

$$\left\| \sum_j B_1^{P_j} f(x, y) \right\|_{L^2({}^cF \times \mathbb{T})} \le c_\eta (\lg K) \delta^{\frac{1}{4} - \eta} \|f\|_{L^2(\mathbb{T}^2)} \quad (\textit{any } \eta > 0)$$

for all $f \in L^2(\mathbb{T}^2)$ and every $K > 10$.

Proof The proof of [5, Main Lemma] follows from Lemmas 2, 3, 4 and 5. Of those lemmas we proved analogues in Lemmas 5.1, 6.3, 8.1, and 8.4. ∎

Main Lemma 4 *Let $1 < r < 2$. Assume* (a) *and* (b) *of Main Lemma 3 and also*

(d) *no point of $[0, 2\pi]$ belongs to more than $K\delta^{-1+\rho}$ of the I_j^0, where $\rho = \rho(r) > 0$ is a small number.*

Then there exists a set $F \subseteq [0, 2\pi]$, $|F| \le c\frac{\delta^\rho}{K^M}$, any $M > 10$ and $K > K_0(r, M)$ such that

$$\Big\| \sum_j B_1^{P_j} f(x, y) \Big\|_{L^r({}^cF\times\mathbb{T})} \le c_{r,\eta} K^{a(r)} \delta^{\sigma(r)} \|f\|_{L^r(\mathbb{T}^2)} \quad (\text{any } \eta > 0)$$

for all $f \in L^r(\mathbb{T}^2)$, where $0 < a(r) < 1$ and $\sigma = \sigma(r) > 0$.

Proof Increasing the length of the chains, to be skimmed off from the top $\mathcal{C}^+$ and from the bottom $\mathcal{C}^-$, to

$$\lg(e^{K^a}\delta^{-10000}) \le c_\varepsilon \delta^{-\varepsilon} K^a, \ 0 < a < 1, \quad (\text{any } \varepsilon > 0).$$

by Lemma 5.1 we have for $1 < r \le 2$

$$\|B_1^{\mathcal{C}^+}\|_r, \|B_1^{\mathcal{C}^-}\|_r \le c_{r,\varepsilon,\eta} K^a \delta^{\frac{1}{2r'} - \eta - \varepsilon}. \tag{28}$$

The remaining pairs $[\omega, I]$ of the trimmed trees P_j^0 satisfy

$$|I| \le \frac{\delta^{10000}}{e^{K^a}} |I_j^0| \le \frac{\delta^{10000}}{K^{M+1}} |I_j^0|, \quad K \ge K_0(a, M)$$

and the P_j^0's are separated, with δ in (α) and (β) replaced by $\delta' = K^{-(M+1)}\delta^{1000}$, $K \ge K_0(a, M)$. Defining the exceptional sets

$$F_j = \big\{ x \in I_j^0 \mid \operatorname{dist}(x, \partial I_j^0) \le 10 \frac{\delta^{200}}{K^{M+1}} |I_j^0| \big\}$$

and disregarding, as we may, the pairs in $P_j^b = \{[\omega, I] \in P_j^0 \mid I \subseteq F_j\}$, we are left with normal trees $P_j^\# = P_j^0 \backslash P_j^b$. Therefore for every row $\mathcal{R}$, it holds $\|B_1^{\mathcal{R}}\|_r \le c_r \delta^{\frac{1}{r}}$ and so

$$\Big\| \sum_{j=1}^{K\delta^{-1+\rho}} B_1^{\mathcal{R}_j} f \Big\|_{L^r({}^cF\times\mathbb{T})} \le c_r \delta^{\frac{1}{r} - 1 + \rho} K \|f\|_r, \quad 1 < r < 2.$$

For $r = 1 + \epsilon$ we choose ρ such that $\frac{1}{r} - 1 + \rho = 0$. Moreover,

$$\Big\| \sum_{j=1}^{K\delta^{-1+\rho}} B_1^{\mathcal{R}_j} f \Big\|_{L^2({}^cF\times\mathbb{T})} \le c\delta^{\frac{1}{2}} \|f\|_2$$

by Cotlar's lemma [5, p. 567]. Thus by interpolation

$$\Big\| \sum_{j=1}^{K\delta^{-1+\rho}} B_1^{\mathcal{R}_j} \Big\|_{L^r({}^cF\times\mathbb{T})} \le c_r \delta^{\sigma(r)} K^{a(r)} \|f\|_r \tag{29}$$

for $1 < r < 2$, where $\sigma(r) > 0$ and $0 < a(r) < 1$. Now (28) and (29) imply the lemma. ∎

9 Proof of Theorem 1

As a consequence of the Main Lemma 1–4,

Corollary 9.1 *Let $\mathcal{F}$ be a set of pairs. Assume*

(a) *$A(p) \le \delta$ for all $p \in \mathcal{F}$;*
(b) *if p, p'' belong to $\mathcal{F}$ and $p < p' < p''$ then $p' \in \mathcal{F}$;*
(c) *if $p, p', p'' \in \mathcal{F}$ and $p < p', p < p''$ then either $p' < p''$ or $p'' < p'$;*
(d) *for any point $x \in [0, 2\pi]$ there are at most $K\delta^{-1+\rho(r)}$ mutually incomparable $[\omega_i, I_i] \in \mathcal{F}$ with $x \in I_i$.*

Then there exists $F \subset [0, 2\pi]$ with $|F| < c\frac{\delta^{\rho(r)}}{K^M}, K > K_0(r, M)($ any $M > 10)$ such that

$$\left\| B^{\mathcal{F}} f(x, y)\right\|_{L^r({}^cF\times\mathbb{T})} \le c_{r,\eta} K^{a(r)} \delta^{\sigma(r)-\eta} \|f\|_{L^r(\mathbb{T}^2)}$$

for all $f \in L^r(\mathbb{T}^2)$, with $0 < a(r) < 1$ and $\sigma(r) > 0$.

A set $\mathcal{F}$ satisfying Corollary 9.1(a)–(d) above is called a forest. In [5] the collection $\mathcal{F}_n = \{p \in \mathcal{B} \mid 2^{-n-1} < A(p) \le 2^{-n}\}$ has been skimmed off at the top by removing ascending chains of length $(n+2)$. By Lemma 5.1 the corresponding estimate in L^r is $c_{\eta,r}(n+1)2^{-n(\frac{1}{2r'}-\eta)}$.

Then considering $\{\bar{p}_j\}, \bar{p}_j = [\bar{\omega}_j, \bar{I}_j]$, the set of maximal pairs such that $\frac{|E(\omega,I)|}{|I|} \ge 2^{-n-1}$, another exceptional set is introduced, namely

$$G_n = \{x \in [0, 2\pi] \mid x \text{ is contained in more than } (K/2)2^{2n} \text{of the } \bar{I}_j\}$$

such that $|G_n| \le c\frac{2^{-n}}{K^M}$ for $K > K_0(M)$, any $M > 10$ ([5, pp.568–570]). Removing all pairs $p = [\omega, I] \in \mathcal{F}_n$ such that $I \subseteq G_n$, we are left with a collection, denoted by $\mathcal{F}_n^{\#}$, that decomposes as a disjoint union of at most $2n(\lg K) + 1$ forests $\{\mathcal{F}_{n,s}\}_s$. Therefore by the Corollary 9.1

$$\left\| B^{\mathcal{F}_n^{\#}} f\right\|_{L^r({}^cF_n\times\mathbb{T})} \le c_{r,\eta}(n+1)K^{a(r)}(\lg K)2^{-n[\sigma(r)-\eta]}\|f\|_r$$

with $|F_n| \le |\cup_s F_{n,s}| \le c(n+1)\frac{2^{-n\rho(r)}}{K^M}\lg K$.

Finally defining $E_n = F_n \cup G_n$ and $E = \bigcup_n E_n$ we obtain

$$\|\mathcal{P}f\|_{L^r({}^cE\times\mathbb{T})} \le c_r K^{a(r)} \lg K \|f\|_r$$

with $|E| \le c_r\frac{\lg K}{K^M}$ for $K > K_0(r, M)$ and any $M > 10$. Therefore

$$\begin{aligned}\left|\{(x, y) \mid |\mathcal{P}f(x, y)| > \alpha\}\right| &\le \frac{\|\mathcal{P}f\|^r_{L^r({}^cE\times\mathbb{T})}}{\alpha^r} + |E| \\ &\le c_r(\lg K)^r K^{ra(r)}\frac{\|f\|^r}{\alpha^r} + c_r\frac{\lg K}{K^M} \le c_{r,\varepsilon}\left(\frac{\|f\|_r}{\alpha}\right)^{r-\varepsilon}\end{aligned}$$

for any $\varepsilon > 0$, having chosen K so to minimize the right-hand side. Thus $\|\mathcal{P}f\|_p \le c_{p,r}\|f\|_r$ for any $p < r$.

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