Canad. Math. Bull. Vol. **48** (3), 2005 pp. 382–393

Uniform Estimates of Ultraspherical Polynomials of Large Order

In loving memory of mia zia, Lucia Brogi in tributi

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Abstract. In this paper we prove the sharp inequality

$$|P_n^{(s)}(x)| \le P_n^{(s)}(1)\big(|x|^n + \frac{n-1}{2s+1}(1-|x|^n)\big),$$

where $P_n^{(s)}(x)$ is the classical ultraspherical polynomial of degree n and order $s \ge n\frac{1+\sqrt{5}}{4}$. This inequality can be refined in $[0, z_n^s]$ and $[z_n^s, 1]$, where z_n^s denotes the largest zero of $P_n^{(s)}(x)$.

Introduction

The Jacobi polynomials are amongst the classical orthogonal polynomials which are most used in the applications. One of the many equivalent definition of $P_n^{(\alpha,\beta)}(x)$, the Jacobi polynomial of degree n and order (α, β), with $\alpha, \beta > -1$, is:

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\left(\frac{d^n}{dx}\right)(1-x)^{\alpha+n}(1+x)^{\beta+n}. \tag{1.1}$$

They are a complete orthogonal system in $L^2([-1,\ 1],\ (1-x)^\alpha(1+x)^\beta dx)$. When $\alpha = \beta$ the Jacobi polynomials take the name of ultraspherical, or Gegenbauer, polynomials. We will let

$$P_n^{(s)}(x) = C_n^s P_n^{(s-\frac{1}{2},\frac{1}{2})}(x), \tag{1.2}$$

where $C_n^s = \frac{\Gamma(s+\frac{1}{2})}{\Gamma(2s)}\frac{\Gamma(n+2s)}{\Gamma(n+s+\frac{1}{2})}$ and $s > -\frac{1}{2}$.

An explicit representation of $P_n^{(s)}(x)$ is the following:

$$P_n^{(s)}(x) = \sum_{m=0}^{[\frac{n}{2}]}(-1)^m\frac{\Gamma(n-m+s)}{\Gamma(s)\Gamma(m+1)\Gamma(n-2m+1)}(2x)^{n-2m}. \tag{1.3}$$

See [Sz, p. 84].

In this paper we investigate the asymptotic behavior of the ultraspherical polynomials $P_n^{(s)}(x)$ inside the orthogonality interval $[-1, 1]$ for large values of s.

Received by the editors February 1, 2003; revised October 8, 2004.
AMS subject classification: 42C05, 33C47.
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https://doi.org/10.4153/CMB-2005-035-2 Published online by Cambridge University Press

The asymptotic properties of Jacobi polynomials are very important and have been investigated for many decades, but, to the best of our knowledge, the asymptotic formulas which are in the literature do not provide sharp estimates of the remainders.

Consider for example the following well known identity, (see [Sz, p. 381]).

$$\lim_{s\to+\infty}\frac{\Gamma(n+1)}{(2s)^n}P_n^{(s)}(x)=x^n. \tag{1.4}$$

Here n fixed and $x\in[-1,1]$. To prove (1.4) we multiply both sides of (1.3) by $(2s)^{-n}$, and we let s go to infinity. We get

$$\lim_{s\to+\infty}(2s)^{-n}P_n^{(s)}(x)=\frac{x^n}{\Gamma(n+1)},$$

and hence (1.4) follows.

The proof, however, does not give any insight on the remainder

$$\rho_n^s(x)=x^n-\frac{\Gamma(n+1)}{(2s)^n}P_n^{(s)}(x).$$

In particular, we do not know how fast $\rho_n^s(x)$ goes to zero when $s\to\infty$. Observe that

$$\frac{(2s)^n}{\Gamma(n+1)}\sim P_n^{(s)}(1)=\frac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)},$$

in the sense that, by Stirling's formula,

$$\lim_{s\to\infty}P_n^{(s)}(1)\frac{\Gamma(n+1)}{(2s)^n}=\lim_{s\to\infty}\frac{\Gamma(n+2s)}{(2s)^n\Gamma(2s)}=1.$$

Therefore, (1.4) is equivalent to

$$\lim_{s\to+\infty}\frac{P_n^{(s)}(x)}{P_n^{(s)}(1)}=x^n, \tag{1.5}$$

and when n is fixed and $s\to\infty$,

$$\rho_n^s(x)\sim R_n^{(s)}(x)=x^n-\frac{P_n^{(s)}(x)}{P_n^{(s)}(1)}.$$

The asymptotic behavior of ultraspherical polynomials $P_n^{(s)}(x)$ for large values of the parameter s has been investigated by many authors. We cite for example [EL2], where the authors express $\lambda^{-\frac{n}{2}}P_n^{(\lambda)}(x\lambda^{-\frac{1}{2}})$ as a finite sum of Hermite polynomials $H_n(x)$. However, to the best of our knowledge, sharp estimates for $R_n^{(s)}(x)$ are not available in the literature.

We prove the following:

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Theorem 1.1 *For every $-1 \le x \le 1$, $s > 0$ and $n \ge 0$,*

$$\frac{P_n^{(s)}(x)}{P_n^{(s)}(1)} = x^n - R_n^{(s)}(x), \tag{1.6}$$

where $|R_n^{(s)}(x)| \le 1 + |x|^n$, and, for $s \ge n\frac{1+\sqrt{5}}{4}$,

$$|R_n^{(s)}(x)| \le (1 - |x|^n)\frac{n-1}{2s+1}. \tag{1.7}$$

Furthermore, the inequality (1.7) *is sharp, since*

$$\lim_{x\to 1-} \frac{R_n^{(s)}(x)}{1-x^n} = \frac{n-1}{2s+1}. \tag{1.8}$$

Numerical evidence suggests that the inequality (1.7) holds for every $n \ge 0$ and every $s > 0$. However, (1.7) is interesting for ultraspherical polynomials of large order. Ultraspherical polynomials of large degree behave like Bessel functions, in the sense that

$$\lim_{n\to\infty} \frac{P_n^{(s)}\left(\cos\frac{z}{n}\right)}{P_n^{(s)}(1)} = \Gamma\left(s+\frac{1}{2}\right)\left(\frac{z}{2}\right)^{-s+\frac{1}{2}} J_{s-\frac{1}{2}}(z). \tag{1.9}$$

Then (1.9) easily follows from a well known Mehler–Heine type asymptotic formula for general Jacobi polynomials, (see [Sz, p. 167]).

The plan of the paper is the following. In Section 2 we collect together some preliminaries. We refer to [Sz] or to [AAR] for further reading. In Section 3 we prove Theorem 1.1. In Section 4 we refine the inequality proved in Section 3. We will show that we can obtain better estimates, which, in some cases, are valid for every positive integer n and for every $s > 0$, if we restrict $P_n^{(s)}(x)$ to the intervals $[0, z_n^s]$ and $[z_n^s, 1]$, where z_n^s denotes the largest zero of $P_n^{(s)}(x)$.

Section 2

In this section we collect together some preliminaries.

We have defined the ultraspherical polynomials $P_n^{(s)}(x)$ in the Introduction. They are are either even or odd functions, that is

$$P_n^{(s)}(-x) = (-1)^n P_n^{(s)}(x). \tag{2.1}$$

The ultraspherical polynomials are related to the Tchebicheff polynomials $T_n(x) = \cos(n\cos^{-1}(x))$ by the following limit relation.

$$\lim_{s\to 0} s^{-1} P_n^{(s)}(x) = \frac{2}{n} T_n(x). \tag{2.2}$$

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The derivatives of ultraspherical polynomials are constant multiples of ultraspherical polynomials. Indeed, from the definition (1.1) easily follows that

$$\frac{d}{dx}P_n^{(s)}(x) = 2sP_{n-1}^{(s+1)}(x). \tag{2.3}$$

Then $P_n^{(s)}(x)$ satisfies the following differential equation:

$$(1-x^2)y'' - (2s+1)xy' + n(n+2s)y = 0. \tag{2.4}$$

When $s > 0$ the maximum of of $P_n^{(s)}(x)$ in $[-1,1]$ can be explicitly computed. We have:

$$\sup_{-1\le x\le 1}|P_n^{(s)}(x)| = P_n^{(s)}(1) = \frac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)}, \quad s>0. \tag{2.5}$$

The L^2 norm of $P_n^{(s)}(x)$ with respect to the measure $(1-x^2)^{s-\frac{1}{2}}dx$ in $(-1,\ 1)$ can be explicitly computed as well. It is

$$\int_{-1}^{1}|P_n^{(s)}(x)|^2(1-x^2)^{s-\frac{1}{2}}\,dx = \frac{\pi 2^{1-2s}\Gamma(n+2s)}{(n+s)(\Gamma(s))^2\Gamma(n+1)}. \tag{2.6}$$

In what follows we will denote with $\widetilde{P}_n^{(s)}(x)$ the normalized ultraspherical polynomials $\frac{P_n^{(s)}(x)}{P_n^{(s)}(1)}$. We will also let $R_n^{(s)}(x) = x^n - \widetilde{P}_n^{(s)}(x)$. We state and prove a few properties of $\widetilde{P}_n^{(s)}(x)$ and $R_n^{(s)}(x)$ that we will use in the following sections.

$$\frac{d}{dx}\widetilde{P}_n^{(s)}(x) = \frac{n(n+2s)}{1+2s}\widetilde{P}_{n-1}^{(s+1)}(x) \tag{2.7}$$

$$\frac{d^2}{d^2x}\widetilde{P}_n^{(s)}(x) = \frac{n(n-1)(n+2s)(n+2s+1)}{(1+2s)(3+2s)}\widetilde{P}_{n-2}^{(s+2)}(x), \tag{2.8}$$

$$\frac{d}{dx}R_n^{(s)}(x) = \frac{n}{1+2s}\big((n+2s)R_{n-1}^{(s+1)}(x) - (n-1)x^{n-1}\big), \tag{2.9}$$

$$\frac{d^2}{d^2x}R_n^{(s)}(x) = A_n^s R_{n-2}^{(s+2)}(x) + B_n^s x^{n-2}, \text{ where} \tag{2.10}$$

$$A_n^s = \frac{n(n-1)(n+2s+1)(n+2s)}{(1+2s)(3+2s)},$$

$$B_n^s = n(n-1) - A_n^s = -\frac{n(n-1)(n+n^2-6s+4ns-3)}{(1+2s)(3+2s)}.$$

$R_n^{(s)}(x)$ satisfies the differential equation

$$n(n+2s)y(x) - (1+2s)xy'(x) + (1-x^2)y''(x) = n(n-1)x^{n-2} \tag{2.11}$$

The proof of (2.7)–(2.11) is simple. To prove (2.7) we use (2.3) and (2.5); we gather

$$\widetilde{P}_n^{(s)}(x) = 2s\frac{P_{n-1}^{(s+1)}(x)}{P_n^{(s)}(1)} = 2s\frac{\widetilde{P}_{n-1}^{(s+1)}(x)\frac{\Gamma(n+2s+1)}{\Gamma(n)\Gamma(2s+3)}}{\frac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)}} = \frac{n(n+2s)}{1+2s}\widetilde{P}_{n-1}^{(s+1)}(x).$$

Then (2.8) follows from (2.7); (2.9) and (2.10) are a straightforward consequence of (2.7) and (2.8); (2.11) follows from (2.4)

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Section 3

In this Section we prove Theorem 1.1.

Proof of Theorem 1.1 We can prove the theorem for $0 \le x \le 1$, since by (2.1), $P_n^{(s)}(x)$ is either even or odd.

We also observe that, by (2.5), the inequality $|R_n^s(x)| \le (1 + x^n)$ is trivial because $|P_n^{(s)}(x)| \le |P_n^{(s)}(1)|$.

We prove first (1.8). We integrate both sides of (2.9) in the interval $(x, 1)$; since $R_n^{(s)}(1) = 0$, we obtain:

$$R_n^{(s)}(x) = \frac{n-1}{2s+1}(1 - x^n) + \frac{n(n+2s)}{1+2s} \int_1^x R_{n-1}^{(s+1)}(t)\, dt. \tag{3.1}$$

By the de l'Hopital rule,

$$\begin{aligned}\lim_{x \to 1^-} \frac{R_n^{(s)}(x)}{1 - x^n} &= \frac{n-1}{2s+1} + \lim_{x \to 1^-} \frac{1}{1 - x^n} \int_1^x R_{n-1}^{(s+1)}(t)\, dt \\ &= \frac{n-1}{2s+1} - \lim_{x \to 1^-} \frac{R_{n-1}^{(s+1)}(x)}{nx^{n-1}} = \frac{n-1}{2s+1},\end{aligned}$$

as required.

We prove (1.7) by induction on n. We observe first that $P_0^{(s)}(x) \equiv 1$ and $P_1^{(s)}(x) = 2sx$. Thus, $R_n^{(s)}(x) \equiv 0$ in these cases, and (1.7) is satisfied.

We shall verify (1.7) also when $n = 2$ and $n = 3$. Observe that

$$P_2^{(s)}(x) = 2s(s+1)x^2 - s, \quad \text{and} \quad R_2^{(s)}(x) = \frac{1 - x^2}{2s+1},$$

which is exactly the right-hand side of (1.7) with $n = 2$.

When $n = 3$, we get

$$P_3^{(s)}(x) = -2s(1+s)x + \frac{4s(1+s)(2+s)x^3}{3} \quad \text{and} \quad R_3^{(s)}(x) = \frac{3x(1 - x^2)}{2s+1}.$$

It is easy to verify that $3x(1 - x^2) \le 2(1 - x^3)$, and thus that $R_3^{(s)}(x)$ satisfies (1.7).

We now assume that (1.6) and (1.7) hold for every integer $1 \le m \le n-1$, with $n \ge 3$, and for every $s \ge m\frac{1+\sqrt{5}}{4}$, and we prove that the same is also true for n and for every $s \ge n\frac{1+\sqrt{5}}{4}$.

To prove that $|R_n^{(s)}(x)| \le (1 - x^n)\frac{n-1}{2s+1}$ we estimate the integral on the right-hand side of (3.1), and we prove that

$$-2\frac{n-1}{2s+1}(1 - x^n) \le \frac{n(n+2s)}{1+2s} \int_1^x R_{n-1}^{(s+1)}(t)\, dt \le 0, \quad x \in [0, 1],$$

or equivalently

$$-2(n-1)(1 - x^n) \le n(n+2s) \int_1^x R_{n-1}^{(s+1)}(t)\, dt \le 0, \quad x \in [0, 1]. \tag{3.2}$$

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Let $\overline{x} \in [0,1)$ be a critical point for $I_{n-1}^{(s+1)}(x) = \int_1^x R_{n-1}^{(s+1)}(t)\,dt$. Then,

$$\frac{d}{dx} I_{n-1}^{(s+1)}(\overline{x}) = R_{n-1}^{(s+1)}(\overline{x}) = 0. \tag{3.3}$$

We integrate the differential equation (2.11), with $n-1$ in place of n and $s+1$ in place of s, in the interval $(1,\overline{x}]$; we obtain:

$$\begin{aligned}(n-1)(n+2s+1)I_{n-1}^{(s+1)}(\overline{x}) - (3+2s)\int_1^{\overline{x}} x\frac{d}{dx}R_{n-1}^{(s+1)}(x)\,dx \\ + \int_1^{\overline{x}} (1-x^2)\frac{d^2}{d^2x}R_{n-1}^{(s+1)}(x)\,dx = -(n-1)(1-\overline{x}^{n-2}).\end{aligned} \tag{3.4}$$

We integrate by parts the second integral in (3.4). We gather

$$\begin{aligned}(n-1)(n+2s+1)I_{n-1}^{(s+1)}(\overline{x}) - (3+2s)\int_1^{\overline{x}} x\frac{d}{dx}R_{n-1}^{(s+1)}(x)\,dx + (1-\overline{x}^2)\frac{d}{dx}R_{n-1}^{(s+1)}(\overline{x}) \\ + 2\int_1^{\overline{x}} x\frac{d}{dx}R_{n-1}^{(s+1)}(x)dx = -(n-1)(1-\overline{x}^{n-2}),\end{aligned}$$

that is,

$$\begin{aligned}(n-1)(n+2s+1)I_{n-1}^{(s+1)}(\overline{x}) - (2s+1)\int_1^{\overline{x}} x\frac{d}{dx}R_{n-1}^{(s+1)}(x)\,dx \\ + (1-\overline{x}^2)\frac{d}{dx}R_{n-1}^{(s+1)}(\overline{x}) = -(n-1)(1-\overline{x}^{n-2}).\end{aligned} \tag{3.5}$$

By our assumption (3.3),

$$0 = \overline{x}R_{n-1}^{(s+1)}(\overline{x}) = \int_1^{\overline{x}} \frac{d}{dx}(xR_{n-1}^{(s+1)}(x))\,dx = I_{n-1}^{(s+1)}(\overline{x}) + \int_1^{\overline{x}} x\frac{d}{dx}R_{n-1}^{(s+1)}(x)\,dx.$$

Therefore, $\int_1^{\overline{x}} x\frac{d}{dx}R_{n-1}^{(s+1)}(x)\,dx = -I_{n-1}^{(s+1)}(\overline{x})$, and from (3.5) follows that

$$n(n+2s)I_{n-1}^{(s+1)}(\overline{x}) = -(1-\overline{x}^2)\frac{d}{dx}R_{n-1}^{(s+1)}(\overline{x}) - (n-1)(1-\overline{x}^{n-2}).$$

By (2.9), $\frac{d}{dx}R_{n-1}^{(s+1)}(x) = \frac{n-1}{3+2s}((n+2s+1)R_{n-2}^{(s+2)}(x) - (n-2)x^{n-2})$. Thus,

$$\begin{aligned}n(n+2s)I_{n-1}^{(s+1)}(\overline{x}) = -(1-\overline{x}^2)\frac{n-1}{3+2s} \\ \times \left((n+2s+1)R_{n-2}^{(s+2)}(\overline{x}) - (n-2)\overline{x}^{n-2}\right) - (n-1)(1-\overline{x}^{n-2}).\end{aligned} \tag{3.6}$$

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The identity (3.6) holds at every critical point of $I_{n-1}^{(s+1)}(x)$ in $[0,1)$, and hence also at the points where $I_{n-1}^{(s+1)}(x)$ attains its maximum and minimum value. Note that (3.6) is also satisfied when $\overline{x}=1$.

By assumption, $R_{n-2}^{(s+2)}(\overline{x}) \geq -\frac{n-3}{2s+5}(1-\overline{x}^{n-2})$. Hence,

$$n(n+2s)I_{n-1}^{(s+1)}(\overline{x}) \leq (1-\overline{x}^2)(1-\overline{x}^{n-2})\frac{(n-1)(n-3)(n+2s+1)}{(3+2s)(5+2s)} + \frac{(n-2)(n-1)}{3+2s}\overline{x}^{n-2}(1-\overline{x}^2)-(n-1)(1-\overline{x}^{n-2}). \tag{3.7}$$

We prove that the right-hand side of (3.7) is ≤ 0 for every $\overline{x}\leq 1$, which is equivalent to proving that

$$(1-x^2)\frac{(n-3)(n+2s+1)}{(5+2s)(3+2s)}+\frac{n-2}{3+2s}\frac{x^{n-2}(1-x^2)}{1-x^{n-2}} \leq 1. \tag{3.8}$$

We prove first that if $n\geq 4$, then $\frac{x^{n-2}(1-x^2)}{1-x^{n-2}} \leq \frac{2}{n-2}x^2$, or

$$\frac{x^{n-4}(1-x^2)}{1-x^{n-2}} \leq \frac{2}{n-2}. \tag{3.9}$$

Let $f(x)$ be the function on the left-hand side of (3.9). When $n=4$ then $f(x)\equiv 1$, which is the right-hand side of (3.6). When $n>4$ it is easy to verify that $f(x)$ is increasing in $[0,\ 1]$. Thus,

$$f(x) \leq \lim_{x\to 1^-}\frac{x^{n-4}(1-x^2)}{1-x^{n-2}} = \frac{2}{n-2}$$

as required. Thus, (3.8) follows if we prove that

$$(1-x^2)\frac{(n-3)(n+2s+1)}{(3+2s)(5+2s)}+\frac{2}{3+2s}x^2 \leq 1,$$

or equivalently that

$$\max\left\{\frac{2}{3+2s},\frac{(n-3)(n+2s+1)}{(3+2s)(5+2s)}\right\} = \frac{(n-3)(n+2s+1)}{(3+2s)(5+2s)} \leq 1. \tag{3.10}$$

It is easy to verify that (3.10) is satisfied if $s \geq \dfrac{n-11+\sqrt{49-30\,n+5\,n^2}}{4}$, and also that

$$\frac{s}{n} \leq \lim_{n\to\infty}\frac{n-11+\sqrt{49-30\,n+5\,n^2}}{4n} = \frac{1+\sqrt{5}}{4},$$

as required.

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We are left to prove that $I_{n-1}^{(s+1)}(\overline{x}) \geq -2\frac{n-1}{n(n+2s)}(1-x^n)$. We go back to (3.6), and we use $R_{n-2}^{(s+2)}(\overline{x}) \leq \frac{n-3}{2s+5}(1-\overline{x}^{n-2})$. We obtain

$$\begin{aligned}(3.11)\quad n(n+2s)I_{n-1}^{(s+1)}(\overline{x}) \geq -(1-\overline{x}^2)(1-\overline{x}^{n-2})\frac{(n-1)(n-3)(n+2s+1)}{(3+2s)(5+2s)}\\ + \frac{(n-1)(n-2)}{3+2s}\overline{x}^{n-2}(1-\overline{x}^2) - (n-1)(1-\overline{x}^{n-2}).\end{aligned}$$

We prove that the right-hand side of the inequality (3.11) is $\geq -2(n-1)(1-x^n)$ for every $x \in [0,1]$, or equivalently, we prove that

$$\begin{aligned}(1-x^2)\frac{(n-1)(n-3)(n+2s+1)}{(5+2s)(3+2s)} - \frac{(n-1)(n-2)}{3+2s}\Big(\frac{1-x^2}{1-x^{n-2}}\Big)x^{n-2} + n - 1\\ \leq 2(n-1)\frac{1-x^n}{1-x^{n-2}}.\end{aligned}$$

Since $1-x^2 \leq 1$, $(\frac{1-x^2}{1-x^{n-2}})x^{n-2} \geq 0$, and $\frac{1-x^n}{1-x^{n-2}} \geq 1$ whenever $0 \leq x \leq 1$, it is sufficient to prove that

$$\frac{(n-1)(n-3)(n+2s+1)}{(5+2s)(3+2s)} + n - 1 \leq 2(n-1),$$

or

$$\frac{(n-3)(n+2s+1)}{(5+2s)(3+2s)} \leq 1,$$

which is true whenever $s \geq n\frac{1+\sqrt{5}}{4}$.

The proof of Theorem 1.1 is complete. ■

Section 4

The following upper bound for the largest zero of $P_n^{(s)}(x)$ is an easy consequence of Theorem 1.1.

Corollary 4.1 *Let z_n^s be the largest zero of $P_n^{(s)}(x)$. Let $s \geq n\frac{1+\sqrt{5}}{4}$. Then*

$$z_n^s \leq \Big(\frac{n-1}{n+2s}\Big)^{\frac{1}{n}}. \tag{4.1}$$

Proof The proof is simple. Indeed, if $z = z_n^s$, then $P_n^{(s)}(z) = z^n - R_n^{(s)}(z) = 0$. By (1.7),

$$z^n = |R_n^{(s)}(z)| \leq \frac{n-1}{2s+1}(1-z^n)$$

from which (4.1) follows.

Our upper bound is far from being optimal. There is a lot of literature concerning the zeros of Jacobi polynomials and good asymptotic estimates are known. See [I, IS1, IS2, EL1, EL2, E], just to cite a few.

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To the best of our knowledge, the best available upper bound for z_n^s is in [ADGR].

$$z_n^s < \sqrt{\frac{(n-1)(n+2s-2)}{(n+s-2)(n+s-1)}}\cos\Big(\frac{\pi}{n+1}\Big), \quad n \geq 1. \tag{4.2}$$

The inequality (4.2) improves the following inequality due to Elbert [E].

$$z_n^s < \frac{\sqrt{(n-1)(n+2s+1)}}{n+s}. \tag{4.3}$$

Let us recall the following inequality, which holds for every $s > 0$ and $n \geq 2$.

$$z_n^s > z_{n-1}^{s+1}. \tag{4.4}$$

Indeed, by (2.7) $\frac{d}{dx}\widetilde{P}_n^{(s)}(x) = \frac{n(n+2s)}{1+2s}\widetilde{P}_{n-1}^{(s+1)}(x)$. Since the zeros of ultraspherical polynomials are real and lie in the interval $[-1,\ 1]$, (see [Sz]), and between any two zeros of $\widetilde{P}_n^{(s)}(x)$ there is at least a zero of its derivative, then the largest zero of $\widetilde{P}_n^{(s)}(x)$ is larger than the largest zero of $\widetilde{P}_{n-1}^{(s+1)}(x)$. ■

We prove that the asymptotic formula (1.6) can be refined in the intervals $[0, z_n^s]$ and $[z_n^s, 1]$.

Theorem 4.2

(a) *Let $n \geq 1$ and $s > 0$. For every $z_n^s \leq x \leq 1$ the following sharp inequality holds.*

$$0 \leq \widetilde{P}_n^{(s)}(x) \leq x^n. \tag{4.5}$$

(b) *For every $0 \leq x \leq z_n^s$ and $n \geq 3$,*

$$|\widetilde{P}_n^{(s)}(x)| \leq \frac{(n-1)(n+2s+1)}{(2s+1)(2s+3)}(1-(z_{n-1}^{s+1})^2)(z_{n-1}^{s+1})^{n-2}. \tag{4.6}$$

When $n = 2$ and $0 \leq x \leq z_2^s = \frac{1}{\sqrt{2(s+1)}}$, then

$$|\widetilde{P}_2^{(s)}(x)| \leq \frac{1}{2s+1}. \tag{4.7}$$

(c) *For every $0 \leq x \leq z_n^s$, $n \geq 2$ and $s \geq n\frac{1+\sqrt{5}}{4}$,*

$$|\widetilde{P}_n^{(s)}(x)| \leq \frac{n-1}{2s+1}(1-x^n). \tag{4.8}$$

Remark The estimates (4.5) and (4.6) are valid without any restriction on n and s. When $0 \leq x \leq z_{n-1}^{s+1}$, $s \geq n\frac{1+\sqrt{5}}{4}$ and $n \geq 3$, we can also prove the following easy refinement of (4.6):

$$|\widetilde{P}_n^{(s)}(x)| \leq (1-x^n)(z_{n-1}^{s+1})^{n-2}. \tag{4.9}$$

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Indeed, $\frac{(n-1)(n+2s+1)}{(2s+1)(2s+3)} < 1$ whenever $s \geq n\frac{1+\sqrt{5}}{4}$, and $1-(z_{n-1}^{s+1})^2 \leq 1-x^2 \leq 1-x^n$ if $0 \leq x \leq z_{n-1}^{s+1}$.

By (4.2) or (4.3), the inequality (4.9) implies (4.8) when $0 \leq x \leq z_{n-1}^{s+1}$ and $n \geq 5$, but in the interval $[0, z_n^s]$ a different proof is needed.

Proof of Theorem 4.2 We prove (4.5) first. Since $R_n^{(s)}(x) = x^n - \widetilde{P}_n^{(s)}(x)$, (4.5) is equivalent to proving that

$$0 \leq R_n^{(s)}(x) \leq 2x^n \tag{4.10}$$

whenever $x \geq z_n^s$.

When $x > z_n^s$, $\widetilde{P}_n^{(s)}(x)$ does not vanish, and hence it is either positive or negative. Since $\widetilde{P}_n^{(s)}(1) = 1 > 0$, then $\widetilde{P}_n^{(s)}(x) \geq 0$ whenever $x \geq z_n^s$. Therefore, $R_n^{(s)}(x) = x^n - \widetilde{P}_n^{(s)}(x) \leq x^n$, which is better than the inequality on the right-hand side of (4.10).

We are left to prove that $R_n^{(s)}(x) \geq 0$ whenever $x \geq z_n^s$. To this aim we use induction on n. It is trivial to verify the cases $n = 1$ and $n = 2$, (see the previous Section). We now assume that $R_m^{(s)}(x) \geq 0$ whenever $x \geq z_m^{(s)}$ and $m \geq n-1$, with $m \geq 3$, and we prove that the same holds also for n.

It is convenient to use the identity (3.1). Since

$$R_n^{(s)}(x) = \frac{n-1}{2s+1}(1-x^n) + \frac{n(n+2s)}{1+2s} I_{n-1}^{(s+1)}(x),$$

where $I_{n-1}^{(s+1)}(x) = \int_1^x R_{n-1}^{(s+1)}(t)\,dt$, proving that $R_n^{(s)}(x) \geq 0$ is equivalent to proving that

$$I_{n-1}^{(s+1)}(x) = \int_1^x R_{n-1}^{(s+1)}(t)\,dt \geq -\frac{n-1}{n(n+2s)}(1-x^n). \tag{4.11}$$

It is sufficient to prove that (4.11) holds at the critical points of $I_{n-1}^{(s+1)}(x)$ in $[z_n^s, 1]$. We use the identity (3.6), which is satisfied at every critical point of $I_{n-1}^{(s+1)}(x)$ in $[0,1]$.

$$\begin{aligned}\frac{n(n+2s)}{1-\overline{x}^2} I_{n-1}^{(s+1)}(\overline{x}) = &-\frac{(n-1)(n+2s+1)}{3+2s} R_{n-2}^{(s+2)}(\overline{x}) \\ &+ \frac{(n-2)(n-1)}{3+2s}\overline{x}^{n-2} - (n-1)\frac{1-\overline{x}^{n-2}}{1-\overline{x}^2}.\end{aligned}$$

Observe that the largest zero of $P_n^{(s)}(x)$ is larger than the largest zero of $P_{n-2}^{(s+2)}$. By assumption, $R_{n-2}^{(s+2)}(\overline{x}) \leq \overline{x}^{n-2}$ for every $\overline{x} > z_n^s$, and

$$n(n+2s) I_{n-1}^{(s+1)}(\overline{x}) \geq -(n-1)\overline{x}^{n-2}(1-\overline{x}^2) - (n-1)(1-\overline{x}^{n-2}) = -(n-1)(1-\overline{x}^n).$$

Thus,

$$n(n+2s) I_{n-1}^{(s+1)}(\overline{x}) \geq -(n-1)(1-\overline{x}^n),$$

as required.

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We now prove (4.6). It is well known, (see *e.g.*, [Sz]), that the local maxima of $|P_n^{(s)}(x)|$ are increasing. The critical points of $P_n^{(s)}(x)$ are the zeros of $P_{n-1}^{(s+1)}(x)$, and hence $|P_n^{(s)}(x)|$, restricted to the interval $[0,\ z_n^s]$, attains its maximum at z_{n-1}^{s+1}.

To estimate $\widetilde{P}_n^{(s)}(z_{n-1}^{s+1})$ we use the differential equation (2.4). Since $\widetilde{P}_{n-1}^{(s+1)}(z_{n-1}^{s+1}) = 0$, we obtain

$$n(n+2s)\widetilde{P}_n^{(s)}(z_{n-1}^{s+1}) = -(1-(z_{n-1}^{s+1})^2)\frac{d^2}{d^2x}\widetilde{P}_n^{(s)}(z_{n-1}^{s+1}).$$

By (2.8), $\frac{d^2}{d^2x}\widetilde{P}_n^{(s)}(z_{n-1}^{s+1}) = \frac{n(n-1)(n+2s)(n+2s+1)}{(1+2s)(3+2s)}\widetilde{P}_{n-2}^{(s+2)}(z_{n-1}^{s+1})$, and

$$\widetilde{P}_n^{(s)}(z_{n-1}^{s+1}) = -(1-(z_{n-1}^{s+1})^2)\frac{(n-1)(n+2s+1)}{(2s+n+1)(3+2s)}\widetilde{P}_{n-2}^{(s+2)}(z_{n-1}^{s+1}).$$

Since $z_{n-1}^{s+1} > z_{n-2}^{s+2}$, by (4.5) we gather

$$|\widetilde{P}_n^{(s)}(z_{n-1}^{s+1})| \le (1-(z_{n-1}^{s+1})^2)\frac{(n-1)(n+2s+1)}{(1+2s)(3+2s)}(z_{n-1}^{s+1})^{n-2}, \tag{4.12}$$

from which (4.6) follows.

Let us prove (4.8) for $n = 2$. Recall that, $\widetilde{P}_2^{(s)}(x) = -\frac{1-2(1+s)x^2}{2s+1}$, and $z_2^s = \frac{1}{\sqrt{2(s+1)}}$. The maximum of $|\widetilde{P}_2^{(s)}(x)|$ in the interval $[0,\ z_2^s]$ is at $x = 0$. Therefore, $|\widetilde{P}_2^{(s)}(x)|$ is decreasing in $[0,\ z_2^s]$, and one can see that $h(x) = (1-x^2)^{-1}|\widetilde{P}_2^{(s)}(x)|$ is decreasing too. The maximum of $h(x)$ in $[0, z_2^s]$ is then $h(0) = (2s+1)^{-1}$. Consequently, $|\widetilde{P}_2^{(s)}(x)| \le \frac{1-x^2}{2s+1}$, as required.

To prove (4.8) when $n \ge 3$ we start from (4.6), and we prove that

$$\frac{n+2s+1}{2s+3}(1-(z_{n-1}^{s+1})^2)(z_{n-1}^{s+1})^{n-2} \le 1-(z_n^s)^n. \tag{4.13}$$

This is sufficient to prove (4.8).

It is easy to prove that the function $f(x) = x^{n-2}(1-x^2)$ is increasing whenever $0 \le x \le \sqrt{\frac{n-2}{n}}$ and is decreasing when $x \ge \sqrt{\frac{n-2}{n}}$. Therefore, $f(x) \le f(\sqrt{\frac{n-2}{n}}) = \frac{2}{n-2}(1-\frac{2}{n})^{\frac{n}{2}}$, and we can reduce matters to proving that

$$g(n,\ s) = \frac{2(n+2s+1)}{(n-2)(2s+3)}\Big(1-\frac{2}{n}\Big)^{\frac{n}{2}} \le 1-(z_n^s)^n.$$

We prove that $g(n,\ s)$ is a decreasing function of n, and hence that $g(n,s) \le g(3,\ s) = \frac{4(2+s)}{3\sqrt{3}(3+2s)}$; indeed,

$$\frac{\partial}{\partial n}g(n,s) = g(n,s)\Big(\frac{1}{1+n+2\,s}+\frac{1}{2}\log\Big(\frac{n-2}{n}\Big)\Big).$$

It is easy to see that $\frac{\partial}{\partial n}g(n,\ s)$ is negative for every $s \ge 0$ and $n \ge 3$, which implies that $n \longrightarrow g(n,\ s)$ is decreasing; (4.13) follows if we prove that

$$g(3,\ s) = \frac{4(2+s)}{3\sqrt{3}(3+2s)} \le 1-(z_n^s)^n. \tag{4.14}$$

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By (4.2), $z_n^s < \frac{\sqrt{n^2+2ns-s-1}}{n+s} = \sqrt{1 - \frac{s^2+s+1}{(n+s)^2}}$.

Let $h(n,s) = \left(1 - \frac{s^2+s+1}{(n+s)^2}\right)^{\frac{n}{2}}$. Clearly, $h(n,s)$ is a decreasing function of s, and hence

$$h(n,s) \le h\Big(n, n\frac{1+\sqrt{5}}{4}\Big) = \Big(\frac{2}{5+\sqrt{5}}\Big)^n \Big(\frac{2(3+\sqrt{5})n^2 - 4 - (1+\sqrt{5})n}{n^2}\Big)^{\frac{n}{2}}. \tag{4.15}$$

It is not too difficult to see that function on the right-hand side of (4.15) is a decreasing function of n. Thus,

$$h(n, n\frac{1+\sqrt{5}}{4}) \le h(3, 3\frac{1+\sqrt{5}}{4}) = \frac{8(47+15\sqrt{5})^{\frac{3}{2}}}{27(5+\sqrt{5})^3}.$$

Therefore, (4.14) follows if we prove that

$$g(3,\ s) + h\Big(3, 3\frac{1+\sqrt{5}}{4}\Big) = \frac{4(2+s)}{3\sqrt{3}(3+2s)} + \frac{8(47+15\sqrt{5})^{\frac{3}{2}}}{27(5+\sqrt{5})^3} \le 1,$$

which is certainly true for every $s \ge 3\frac{1+\sqrt{5}}{4}$.

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https://doi.org/10.4153/CMB-2005-035-2 Published online by Cambridge University Press