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SYMMETRIC SPECTRAL FACTORISATION OF SELF-ADJOINT RATIONAL MATRIX FUNCTIONS

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For a self-adjoint rational matrix function, not necessarily analytic at infinity, the existence of a right (symmetric) spectral factorisation is described in terms of a given left spectral factorisation. The formula for the right spectral factor is given in terms of the formula for the given left spectral factor. All formulas are based on a special realisation of a rational matrix function, which is different from ones that have been used before.

1. INTRODUCTION

It is well known that Wiener-Hopf and spectral factorisations of matrix and operator-valued functions have numerous applications in analysis and electrical engineering. Applications include singular integral equations (see [**6, 10, 13**]), Toeplitz operators (see [**8**]), algebraic Riccati equations (see [**14**] and [**15**]) and model reduction for linear systems (see [**1, 2**]). Moreover, in the analysis of H_∞-control problems, spectral factorisation plays an important role (see for example, [**11, 7**]). In the latter theories, the matrix functions are in general assumed to be rational, that is, their entries are quotients of polynomials.

In this paper, we use the *state space method* (see [**5**]), which depends on the notion of realisation and allows one to reduce problems concerning rational matrix functions to ones in linear algebra involving constant matrices. By a *realisation* for a rational matrix function W (which is analytic and invertible at ∞) we mean a representation for W of the form

$$W(\lambda) = I + C(\lambda - A)^{-1}B, \tag{1}$$

where A is an $n \times n$ matrix and B and C are $n \times m$ and $m \times n$ matrices, respectively. Here I denotes the $m \times m$ identity matrix.

In [**3**], Ball and Ran showed how the state space method may be applied to rational matrix functions (of the form (1)) which are analytic and invertible at infinity. We follow a similar program as in [**3**] (see also [**12**]), but with a different representation. Indeed,

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in our case, the analysis is based on the following representation of the given rational matrix function:

$$W(\lambda) = I + C(\lambda G - A)^{-1}B. \tag{2}$$

Here the matrix G is of the same order as A, the matrices B and C are as before and the pencil $\lambda G - A$ is regular on the unit circle $|\lambda| = 1$.

In this article, we extend a factorisation result of [**3**] to the general case. Given a signed antispectral factorisation

$$W(\lambda) = Y_-^*(\lambda)\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} Y_-(\lambda),$$

with respect to the unit circle, where $Y_-(\lambda)$ and its inverse are given in realisation form, the aim is to find necessary and sufficient conditions for the existence of a signed spectral factorisation

$$W(\lambda) = X_+^*(\lambda)\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} X_+(\lambda),$$

and give an explicit formula for a spectral factor $X_+(\lambda)$ and its inverse in realisation form.

The paper consists of three sections, including the introduction. The second section summarises the preliminary notions and results. Section 3 contains the main factorisation result.

2. Preliminaries

Firstly, we give terminology and notation. By a *Cauchy contour* γ, we mean the positively oriented boundary of a bounded Cauchy domain in $\mathbb{C}$. Such a contour consists of a finite number of non-intersecting closed rectifiable Jordan curves. The set of points inside γ is called the *inner domain* of γ and will be denoted by Δ_+. The *outer domain* of γ is the set $\Delta_- = \mathbb{C}_\infty \setminus \overline{\Delta_+}$. By convention $0 \in \Delta_+$ and by definition $\infty \in \Delta_-$.

Next, we consider operator pencils. Let X be a complex Banach space and let G and A be bounded linear operators on X. For $\lambda \in \mathbb{C}$, the expression $\lambda G - A$ is called a *(linear) operator pencil* on X. Given a non-empty subset Δ of the Riemann sphere $\mathbb{C}_\infty$, we say that $\lambda G - A$ is $\Delta-$ *regular* if $\lambda G - A$ (or just G if $\lambda = \infty$) is invertible for each $\lambda \in \Delta$. The *spectrum* of $\lambda G - A$, denoted by $\sigma(G, A)$, is the subset of $\mathbb{C}_\infty$ determined by the following properties: $\infty \in \sigma(G, A)$ if and only if G is not invertible, and $\sigma(G, A) \cap \mathbb{C}$ consists of all those $\lambda \in \mathbb{C}$ for which $\lambda G - A$ is not invertible. Its complement (in $\mathbb{C}_\infty$) is the *resolvent set* of $\lambda G - A$, denoted by $\rho(G, A)$.

Let $W(\lambda)$ be a rational $m \times m$ matrix function with neither a pole nor a zero on γ. Then $W(\lambda)$ admits a *(right) Wiener-Hopf factorisation* relative to γ, that is, $W(\lambda)$ factorises as

$$W(\lambda) = W_-(\lambda)D(\lambda)W_+(\lambda), \quad \lambda \in \gamma, \tag{3}$$

where W_+ and W_- are rational $m \times m$ matrix functions, W_+ has neither a pole nor a zero on $\Delta_+ \cup \gamma$ and W_- has neither a pole nor a zero on $\Delta_- \cup \gamma$ (which includes the point ∞), and

$$D(\lambda) = \operatorname{diag}(\lambda^{\kappa_j})_{j=1}^m.$$

Here $\kappa_1 \leqslant \kappa_2 \leqslant \ldots \leqslant \kappa_m$ are integers, which are uniquely determined by W (and γ), and are called the *(right) factorisation indices* of W relative to γ (see, for example, [**6**]). The factorisation is called a *(right) canonical Wiener-Hopf factorisation* if and only if the indices $\kappa_1, \ldots, \kappa_m$ are all zero. If W admits such a factorisation, then $\det W(\lambda) \neq 0$ for each $\lambda \in \gamma$. In general, this condition is only necessary but not sufficient for the existence of a canonical factorisation. We refer to a *(left) Wiener-Hopf factorisation* if in (3) the order of the factors are interchanged.

Throughout this paper, we shall consider the representation of the rational matrix function W of the form (see [**9**]):

$$W(\lambda) = I + C(\lambda G - A)^{-1}B, \quad |\lambda| = 1.$$

The square matrices G and A are both of order n say, and B and C are matrices of sizes $n \times m$ and $m \times n$, respectively. Also, we assume that the pencil $\lambda G - A$ is regular on the unit circle $\mathbb{T} = \{\lambda : |\lambda| = 1\}$.

Put $A^\times = A - BC$. Then $\det W(\lambda) \neq 0$ if and only if $\lambda G - A^\times$ is $\mathbb{T}$-regular. In this case,

$$W(\lambda)^{-1} = I - C\big(\lambda G - A^\times\big)^{-1}B, \quad |\lambda| = 1,$$

(see [**9**]).

Next, we state [**13,** Theorem 2.3]. We shall present an application of this factorisation theorem in the next section. The result involves left and right canonical factorisation of rational matrix functions. A thorough discussion of these concepts can be found in [**4**] and [**3**]. For instance, it is shown how to compute realisations for the factors W_- and W_+ of a right canonical factorisation $W = W_-W_+$, if one is given a realisation $W(\lambda) = I + C(\lambda - A)^{-1}B$ for W. Furthermore, it is known how to compute a right canonical Wiener-Hopf factorisation $W(\lambda) = W_-(\lambda)W_+(\lambda)$ in terms of a given left canonical factorisation $W(\lambda) = Y_+(\lambda)Y_-(\lambda)$ where the left factors are given by $Y_+(\lambda) = I + C_+(\lambda - A_+)^{-1}B_+$ and $Y_-(\lambda) = I + C_-(\lambda - A_-)^{-1}B_-$. However, as in [**13,** Section

2], we assume that the factors Y_+ and Y_- of a left canonical Wiener-Hopf factorisation are given by the more general realisations $Y_+(\lambda) = I + C_+(\lambda G_+ - A_+)^{-1}B_+$ and $Y_-(\lambda) = I + C_-(\lambda G_- - A_-)^{-1}B_-$. We then state a necessary and sufficient condition for the existence of a right canonical Wiener-Hopf factorisation $W(\lambda) = W_-(\lambda)W_+(\lambda)$. In this case, we provide explicit formulas for the factors W_- and W_+ in terms of the realisations of Y_+ and Y_-. Let the inner and outer domain of the unit circle $\mathbb{T}$ be denoted by $\mathbb{D}_+$ and $\mathbb{D}_-$, respectively. The result is as follows.

THEOREM 2.1. *Suppose that the rational $m \times m$ matrix function $W(\lambda)$ (not necessarily analytic and invertible at ∞) has a left canonical Wiener-Hopf factorisation with respect to $\mathbb{T}$, that is, $W(\lambda)$ factorises as*

$$W(\lambda) = Y_+(\lambda)Y_-(\lambda),$$

where

$$Y_+(\lambda) = I_m + C_+(\lambda G_+ - A_+)^{-1}B_+, \tag{4}$$

and

$$Y_-(\lambda) = I_m + C_-(\lambda G_- - A_-)^{-1}B_-. \tag{5}$$

Set $A_-^\times := A_- - B_-C_-$ and $A_+^\times := A_+ - B_+C_+$. Assume that $\lambda G_- - A_-$ and $\lambda G_- - A_-^\times$ are $n_- \times n_-$ matrix pencils which are $(\mathbb{D}_- \cup \mathbb{T})$-regular and that $\lambda G_+ - A_+$ and $\lambda G_+ - A_+^\times$ are $n_+ \times n_+$ matrix pencils which are $(\mathbb{D}_+ \cup \mathbb{T})$-regular. Let U and T denote the unique solutions of the Lyapunov equations

$$A_-^\times U G_+ - G_- U A_+^\times = B_- C_+ \tag{6}$$

and

$$G_+ T A_- - A_+ T G_- = B_+ C_-. \tag{7}$$

Then W has a right canonical Wiener-Hopf factorisation if and only if the $n_- \times n_-$ matrix $I_{n_-} - G_-UG_+T$ is invertible, or equivalently, if and only if the $n_- \times n_-$ matrix $I_{n_-} - UG_+TG_-$ is invertible, or equivalently, if and only if the $n_+ \times n_+$ matrix $I_{n_+} - G_+TG_-U$ is invertible, or equivalently, if and only if the $n_+ \times n_+$ matrix $I_{n_+} - TG_-UG_+$ is invertible. In this case, the factors $W_-(\lambda)$ and $W_+(\lambda)$ for a right canonical Wiener-Hopf factorisation are given by the formulas

$$W_-(\lambda) = I + [C_+TG_- + C_-](\lambda G_- - A_-)^{-1}[I_{n_-} - G_-UG_+T]^{-1}[-G_-UB_+ + B_-], \tag{8}$$

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and

$$(9)\quad W_+(\lambda) = I + [C_+ + C_- U G_+][I_{n_+} - T G_- U G_+]^{-1}(\lambda G_+ - A_+)^{-1}[B_+ - G_+ T B_-].$$

Their inverses are given by

(10)

$$W_-(\lambda)^{-1} = I - [C_+ T G_- + C_-][I_{n_-} - U G_+ T G_-]^{-1}(\lambda G_- - A_-^{\times})^{-1}[-G_- U B_+ + B_-],$$

and

$$(11)\quad W_+(\lambda)^{-1} = I - [C_+ + C_- U G_+](\lambda G_+ - A_+^{\times})^{-1}[I_{n_+} - G_+ T G_- U]^{-1}[B_+ - G_+ T B_-].$$

3. Antispectral versus spectral factorisation with respect to the unit circle

We assume that $W(\lambda)$ is a rational $m \times m$ matrix function which is analytic and invertible on the unit circle $\{\lambda : |\lambda| = 1\}$ such that $W\left(1/\overline{\lambda}\right)^* = W(\lambda)$. In general, in this section we shall use W^* to denote the function $W^*(\lambda) = W\left(1/\overline{\lambda}\right)^*$. Observe, that for a rational matrix function W, we have that $W = W^*$ if and only if $W(\lambda)$ is self-adjoint for $|\lambda| = 1$. Since by hypothesis, $W(\lambda)$ is self-adjoint and invertible on the unit circle, it follows that $W\left(e^{i\tau}\right)$ has non-zero real eigenvalues, that is, $W\left(e^{i\tau}\right)$ must have a constant number (say p) of positive eigenvalues and $q = m - p$ of negative eigenvalues for all real τ.

By a *signed antispectral factorisation* of W (with respect to the unit circle) we mean a factorisation of the form:

$$W(\lambda) = Y_-^*(\lambda)\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} Y_-(\lambda),$$

where $Y_-(\lambda)$ is analytic and invertible on the exterior of the unit disc $\overline{\mathbb{D}}_- = \{\lambda : |\lambda| \geqslant 1\}$. By a *signed spectral factorisation* of W (with respect to the unit circle) we mean a factorisation of the form:

$$W(\lambda) = X_+^*(\lambda)\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} X_+(\lambda),$$

where $X_+(\lambda)$ is analytic and invertible on the closed unit disc $\overline{\mathbb{D}}_+ = \{\lambda : |\lambda| \leqslant 1\}$.

The problem which we consider here, may be viewed as a symmetrised version of the one analysed in [**13**]. Indeed, given a signed antispectral factorisation

$$W(\lambda) = Y_-^*(\lambda)\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} Y_-(\lambda),$$

find necessary and sufficient conditions for the existence of a signed spectral factorisation. In this case, give an explicit formula for a spectral factor $X_+(\lambda)$, and its inverse. In order to achieve this, we need the fact that a function W, which is self-adjoint and invertible on the unit circle, has a signed spectral factorisation if and only if it has a right canonical Wiener-Hopf factorisation with respect to the unit circle. In particular, if $W(\lambda) = W_-(\lambda)W_+(\lambda)$ is a right canonical factorisation then $W(\lambda) = W^*(\lambda) = W_+^*(\lambda)W_-^*(\lambda)$ is also one. However, it is known that such factorisations are unique up to a constant invertible factor; hence $W_-(\lambda) = W_+^*(\lambda)K$, where K is some non-singular $m \times m$ matrix, and $W(\lambda) = W_+^*(\lambda)KW_+(\lambda)$. Upon substituting $\lambda = 1$, we get that $K = W_+^*(1)^{-1}W(1)W_+(1)^{-1}$ is self-adjoint (and invertible), so as before K has p positive and q negative eigenvalues. Then K factors as:

$$K = E^* \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} E,$$

for some invertible $m \times m$ matrix E. Thus

$$W(\lambda) = X_+^*(\lambda) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} X_+(\lambda)$$

is a signed spectral factorisation, where $X_+(\lambda) = EW_+(\lambda)$. By using these observations and Theorem 2.1, we obtain an analogous result for signed spectral factorisation.

THEOREM 3.1. *Suppose that the rational $m \times m$ matrix function $W(\lambda) = W^*(\lambda)$ has a signed antispectral factorisation*

$$W(\lambda) = Y_-^*(\lambda) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} Y_-(\lambda),$$

where

$$Y_-(\lambda) = Y_-(\infty)[I_m + C_-(\lambda G_- - A_-)^{-1}B_-].$$

We may asssume that $\lambda G_- - A_-$ and $\lambda G_- - A_-^\times$ are $n_- \times n_-$ matrices with spectra in the open unit disc $\mathbb{D}_+$. Here $A_-^\times := A_- - B_-C_-$. We also assume that $Y_-(\infty)$ and $Y_-^*(\infty) = Y_-(0)^*$ are invertible, so $W(\infty)$ and $W(0) = W(\infty)^*$ are invertible. We denote by ψ the Hermitian matrix

$$\psi = Y_-(\infty)^* \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} Y_-(\infty).$$

Let U and T denote the unique solutions of the Stein equations

$$A_-^\times U G_-^{*-1}(A_-^\times)^* - G_-U = -B_-\psi^{-1}B_-^*, \tag{12}$$

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and

$$(13) \qquad A_-^* G_-^{*-1} T A_- - T G_- = C_-^* \psi C_-.$$

Then W has a signed spectral factorisation if and only if the $n_+ \times n_+$ matrix $I_{n_+} - T^*U$ is invertible, or equivalently, if and only if the $n_+ \times n_+$ matrix $I_{n_+} - TU^*$ is invertible, or equivalently, if and only if the $n_- \times n_-$ matrix $I_{n_-} - U^*T$ is invertible, or equivalently, if and only if the $n_- \times n_-$ matrix $I_{n_-} - UT^*$ is invertible, Suppose that this is the case, and let

$$Z = \left(I_{n_+} - TU^*\right)^{-1}$$

and

$$Z' = \left(I_{n_+} - T^*U\right)^{-1}.$$

Then the Hermitian matrix,

$$(14) \quad K = \psi - \psi C_- A_-^{-1} Z'^* B_- + \psi C_- A_-^{-1} G_- U Z' A_-^{*-1} C_-^* \psi + B_-^* Z' T^* G_-^{-1} B_- - B_-^* Z' A_-^{*-1} C_-^* \psi$$

is invertible and has p positive and q negative eigenvalues. Thus K has a factorisation

$$K = E^* \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} E,$$

for an invertible matrix E. Then

$$W(\lambda) = X_+^*(\lambda) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} X_+(\lambda)$$

is a signed spectral factorisation of $W(\lambda)$ where

$$(15) \quad X_+(\lambda) = E\{I_m + [-\psi^{-1} B_-^* (A_-^\times)^{*-1} + C_- U G_-^{*-1}].Z(\lambda G_-^{*-1} - A_-^{*-1})^{-1} .[A_-^{*-1} C_-^* \psi - G_-^{*-1} T B_-]\},$$

with inverse

$$(16) \quad X_+(\lambda)^{-1} = \left\{I_m - [-\psi^{-1} B_-^* (A_-^\times)^{*-1} + C_- U G_-^{*-1}][\lambda G_-^{*-1} - (A_-^\times)^{*-1}]^{-1} .Z'[A_-^{*-1} C_-^* \psi - G_-^{*-1} T B_-]\right\} E^{-1}.$$

PROOF: We have that $W(\lambda) = Y_+(\lambda) Y_-(\lambda)$, where $Y_+(\infty) = Y_-(\infty) = I_m$. We consider $W(\infty)^{-1} W(\lambda)$ and its left canonical Wiener-Hopf factorisation with respect to $\mathbb{T}$. Then

$$W(\infty)^{-1} W(\lambda) = Y_+(\lambda) Y_-(\lambda) = Y_+(\lambda)[Y_-(\infty)^{-1} Y_-(\lambda)]$$

where

$$Y_+(\lambda) = W(\infty)^{-1} Y_-^*(\lambda) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} Y_-(\infty).$$

By hypothesis,

$$Y_-(\infty)^{-1} Y_-(\lambda) = I_m + C_-(\lambda G_- - A_-)^{-1} B_-,$$

where G_-, A_-, B_- and C_- are given. Then we have that

$$\begin{aligned} Y_-^*(\lambda) &= Y_-\left(\frac{1}{\bar{\lambda}}\right)^* \\ &= [I_m + B_-^*(\lambda^{-1} G_-^* - A_-^*)^{-1} C_-^*] Y_-(\infty)^* \\ &= [I_m - B_-^* A_-^{*-1} C_-^* - B_-^* A_-^{*-1}(\lambda G_-^{*-1} - A_-^{*-1})^{-1} A_-^{*-1} C_-^*] Y_-(\infty)^*, \end{aligned}$$

and hence it is clear that

$$W(\infty) = Y_-^*(\infty) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} Y_-(\infty) = \left(I_m - B_-^* A_-^{*-1} C_-^*\right)\psi.$$

Therefore, we compute that

$$\begin{aligned} Y_+(\lambda) &= W(\infty)^{-1} Y_-^*(\lambda) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} Y_-(\infty) \\ &= I_m - \psi^{-1}[I_m - B_-^* A_-^{*-1} C_-^*]^{-1} [B_-^* A_-^{*-1}(\lambda G_-^{*-1} - A_-^{*-1})^{-1} A_-^{*-1} C_-^*]\psi. \end{aligned}$$

Clearly, $W(\infty)^{-1} W(\lambda)$ has a right canonical Wiener-Hopf factorisation if and only if $W(\lambda)$ has one. By the remarks preceding this theorem, this is equivalent to the existence of a signed spectral factorisation for W.

To obtain conditions for a right canonical factorisation for $W(\infty)^{-1} W(\lambda)$ we apply Theorem 2.1 with G_-, A_-, B_- and C_- as given here, but with G_+, A_+, B_+ and C_+ given by:

$$\begin{aligned} G_+ &= G_-^{*-1}, \quad A_+ = A_-^{*-1}, \\ B_+ &= A_-^{*-1} C_-^* \psi, \quad C_+ = -\psi^{-1} B_-^* \left(A_-^\times\right)^{*-1}. \end{aligned}$$

Next, we compute the associate operator,

$$\begin{aligned} A_+^\times &= A_+ - B_+ C_+ \\ &= A_-^{*-1} + A_-^{*-1} C_-^* B_-^* \left(A_-^\times\right)^{*-1} \\ &= \left(A_-^\times\right)^{*-1}. \end{aligned}$$

By replacing G_+, $A_+^\times$ and C_+ in the Lyapunov equation,

$$A_-^\times U G_+ - G_- U A_+^\times = B_- C_+,$$

with the expressions determined above, it is clear that (12) holds. Note that the solution U of (12) is unique. Similarly, by replacing G_+, A_+ and B_+ in equation (7), we find that (13) holds and has the unique solution T. Also, from (12), we have

$$A_-^\times U G_-^{*-1} (A_-^\times)^* - G_- U = -B_- \psi^{-1} B_-^* = A_-^\times G_-^{-1} U^* (A_-^\times)^* - U^* G_-^*;$$

Then, since (12) is uniquely solvable, it follows that

$$U = G_-^{-1} U^* G_-^*.$$

Similarly, from (13), we have

$$T = G_-^* T^* G_-^{-1}.$$

Furthermore, where $Z = \left(I_{n_+} - TU^*\right)^{-1}$ and $Z' = \left(I_{n_+} - T^*U\right)^{-1}$, we conclude that

$$T^* G_-^{-1} Z'^* = Z' T^* G_-^{-1}, \quad Z'^* G_- U = G_- U Z'.$$

Thus the invertibility of $I_{n_+} - T^*U$, or equivalently, the invertibility of $I_{n_+} - TU^*$, or equivalently, the invertibility of $I_{n_-} - U^*T$, or equivalently, the invertibility of $I_{n_-} - UT^*$ is a necessary and sufficient condition for the existence of a signed spectral factorisation of $W(\lambda)$.

Then, by applying formulas (8)-(11) of Theorem 2.1 we have that

$$W(\infty)^{-1} W(\lambda) = W_-(\lambda) W_+(\lambda)$$

is a right canonical Wiener-Hopf factorisation, with

$$(17) \quad W_+(\lambda) = I_m + [-\psi^{-1} B_-^* (A_-^\times)^{*-1} + C_- U G_-^{*-1}].Z\left(\lambda G_-^{*-1} - A_-^{*-1}\right)^{-1} \cdot [A_-^{*-1} C_-^* \psi - G_-^{*-1} T B_-],$$

and its inverse

$$(18) \quad W_+(\lambda)^{-1} = I_m - [-\psi^{-1} B_-^* (A_-^\times)^{*-1} + C_- U G_-^{*-1}]\left(\lambda G_-^{*-1} - (A_-^\times)^{*-1}\right)^{-1} .Z' [A_-^{*-1} C_-^* \psi - G_-^{*-1} T B_-].$$

In particular,

$$W(\lambda) = W(\infty) W_-(\lambda) W_+(\lambda)$$

is a right canonical Wiener-Hopf factorisation of W. Thus, the same is true for

$$W(\lambda) = W^*(\lambda) = W_+^*(\lambda) W_-^*(\lambda) W^*(\infty).$$

By the uniqueness of the right canonical factorisation, we know that there is a constant, invertible matrix K such that

$$W(\infty) W_-(\lambda) = W_+^*(\lambda) K.$$

Therefore,

$$W(\lambda) = W_+^*(\lambda) K W_+(\lambda). \tag{19}$$

By calculating both sides of (19) at a point λ on the unit circle and using the original signed antispectral factorisation for W, we get that K is invertible with p positive and q negative eigenvalues. Thus K can be factored as

$$K = E^* \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} E,$$

for some invertible matrix E. Then (19) yields

$$\begin{aligned} W(\lambda) &= W_+^*(\lambda) E^* \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} E W_+(\lambda) \\ &= X_+^*(\lambda) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} X_+(\lambda), \end{aligned}$$

which is a signed spectral factorisation of $W(\lambda)$, where $X_+(\lambda) = EW_+(\lambda)$. Employing formulas (17) and (18), we obtain the required formulas (15) and (16) for $X_+(\lambda)$ and $X_+(\lambda)^{-1}$, provided we verify that the constant K in (19) is given by the formula (14).

To evaluate K, we set $\lambda = \infty$ in (19) to obtain

$$K = W_+^*(\infty)^{-1} W(\infty) = W_+(0)^{*-1} W(\infty).$$

From (18), we see that

$$\begin{aligned} W_+(0)^{*-1} &= [W_+(0)^{-1}]^* \\ &= I_m + [\psi C_- A_-^{-1} - B_-^* T^* G_-^{-1}] Z'^*(A_-^\times).[-(A_-^\times)^{-1} B_- \psi^{-1} + G_-^{-1} U^* C_-^*]. \end{aligned}$$

We have already observed that

$$W(\infty) = [I_m - B_-^* A_-^{*-1} C_-^*] \psi.$$

To compute the product $K = W_+(0)^{*-1}W(\infty)$, we first simplify the expression

$$[-(A_-^\times)^{-1}B_-\psi^{-1} + G_-^{-1}U^*C_-^*]W(\infty)$$

as follows

$$\begin{aligned}
&[-(A_-^\times)^{-1}B_-\psi^{-1} + G_-^{-1}U^*C_-^*][\psi - B_-^*A_-^{*-1}C_-^*\psi] \\
&\quad = -(A_-^\times)^{-1}B_- - (A_-^\times)^{-1}[A_-^\times UG_-^{*-1}(A_-^\times)^* - G_-U]A_-^{*-1}C_-^*\psi \\
&\qquad + G_-^{-1}U^*C_-^*[\psi - B_-^*A_-^{*-1}C_-^*\psi] \\
&\quad = (A_-^\times)^{-1}[-B_- + G_-UA_-^{*-1}C_-^*\psi] + [G_-^{-1}U^* - UG_-^{*-1}]C_-^*[\psi - B_-^*A_-^{*-1}C_-^*\psi] \\
&\quad = (A_-^\times)^{-1}[-B_- + G_-UA_-^{*-1}C_-^*\psi]
\end{aligned}$$

Here we have used the fact that $U = G_-^{-1}U^*G_-^*$. Thus, it follows that

$$\begin{aligned}
K &= [W_+(0)^{-1}]^*W(\infty) \\
&= \psi - \psi C_-A_-^{-1}Z'^*B_- + \psi C_-A_-^{-1}G_-UZ'A_-^{*-1}C_-^*\psi \\
&\qquad + B_-^*Z'T^*G_-^{-1}B_- - B_-^*Z'A_-^{*-1}C_-^*\psi
\end{aligned}$$

which coincides with (14). This completes the proof of the theorem. ∎

The model reduction problem for discrete time systems (see [**2**]), suggests the application of Theorem 3.1 to a special form of the rational matrix function $Y_-(\lambda)$.

Corollary 3.2. *Suppose $L(\lambda) = C(\lambda G - A)^{-1}B$ is a $p \times q$ rational matrix function of Mcmillan degree n such that all the poles of L are in the open unit disk $\mathbb{D}_+$. Thus we may assume that $\sigma(G, A) \subset \mathbb{D}_+$. For μ a positive real number, define the matrix function $W(\lambda)$ by*

$$W(\lambda) = \begin{pmatrix} I_p & 0 \\ L^*(\lambda) & \mu I_q \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \begin{pmatrix} I_p & L(\lambda) \\ 0 & \mu I_q \end{pmatrix}$$

and let U and T be the unique solutions of the Stein equations

$$A(\mu^2U)G^{*-1}A^* - G(\mu^2U) = BB^*$$

and

$$A^*G^{*-1}TA - TG = C^*C.$$

Then W has a signed spectral factorisation if and only if the matrix $I_n - T^*U$ is invertible, or equivalently, if and only if the matrix $I_n - TU^*$ is invertible, or equivalently, if and only if the matrix $I_n - U^*T$ is invertible, or equivalently, if and only if the matrix

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$I_n - UT^*$ is invertible. When this is the case, the factor $X_+(\lambda)$ for a signed spectral factorisation

$$W(\lambda) = X_+^*(\lambda) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} X_+(\lambda)$$

is computed as follows. Set

$$Z = (I_n - TU^*)^{-1}$$

and

$$Z' = (I_n - T^*U)^{-1},$$

and let K be the $(p+q) \times (p+q)$ matrix

$$K = \begin{pmatrix} I + CA^{-1}GUZ'A^{*-1}C^* & -CA^{-1}Z'^*B \\ -B^*Z'A^{*-1}C^* & -\mu^2 I_q + B^*Z'T^*G^{-1}B \end{pmatrix}.$$

Then K is Hermitian with p positive and q negative eigenvalues, and so has the factorisation

$$K = E^* \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} E,$$

for an invertible $(p+q) \times (p+q)$ matrix E. Then the spectral factor $X_+(\lambda)$ for $W(\lambda)$ in this case is given by

$$X_+(\lambda) = E\left\{\begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix} + \begin{pmatrix} CUG^{*-1} \\ \mu^{-2}B^*A^{*-1} \end{pmatrix} Z\left(\lambda G_-^{*-1} - A_-^{*-1}\right)^{-1} \left[A^{*-1}C^* \quad -G^{*-1}TB\right]\right\}$$

with inverse given by

$$X_+(\lambda)^{-1} = \left\{\begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix} - \begin{pmatrix} CUG^{*-1} \\ \mu^{-2}B^*A^{*-1} \end{pmatrix} \left(\lambda G_-^{*-1} - A_-^{*-1}\right)^{-1} Z' \left[A^{*-1}C^* \quad -G^{*-1}TB\right]\right\} E^{-1}.$$

PROOF: The result follows from Theorem 3.1 above, if one takes

$$\begin{aligned} Y_-(\lambda) &= \begin{pmatrix} I_p & 0 \\ 0 & \mu I_q \end{pmatrix} \begin{pmatrix} I_p & L(\lambda) \\ 0 & I_q \end{pmatrix} \\ &= \begin{pmatrix} I_p & 0 \\ 0 & \mu I_q \end{pmatrix} \left[\begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix} + \begin{pmatrix} C \\ 0 \end{pmatrix} (\lambda G - A)^{-1} (0 \ B)\right]. \end{aligned}$$

Note that then both $Y_-(\lambda)$ and

$$Y_-(\lambda)^{-1} = \begin{pmatrix} I_p & -L(\lambda) \\ 0 & I_q \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & \mu^{-1} I_q \end{pmatrix}$$

are analytic on the complement of the unit disk $\mathbb{D}_+$ since all the poles of $L(\lambda)$ are assumed to be in $\mathbb{D}_+$. ∎

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