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Remainders of metric completions

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Certain topological spaces X may bear various uniform structures compatible with the topology of X ; to each uniform structure there corresponds a *completion* of X , that is, a complete space Z containing X as a dense subspace. For *compact* completions, there has been extensive study of the relationship between X and the possible *remainders* $Z \backslash X$. This paper begins a study of the more general, and apparently easier, problem of the relationship between X and its not necessarily compact remainders. We find that for spaces X admitting a complete metric, every space Y which satisfies certain conditions obviously necessary for Y to be the remainder of a completion of X in fact occurs as such a completion.

A uniformisable topological space X may bear various proximity structures, each of which corresponds to a compactification of X , and various uniformities, each of which corresponds to a completion of X [1]. We regard X itself as a subspace of each of its completions or compactifications. If Z is any completion (compact or not) of X , we shall refer to $Z \backslash X$ as the *remainder* of Z . There has been considerable investigation of how many different compactifications are admitted by each uniformisable space. Various conditions have been found [5, 6] on spaces X and Y for Y to be the remainder of some compactification of X . Also the lattice of all compactifications of X [1] has been studied, and characterised as a lattice [3]. In this paper we contribute towards an answer to the question: which spaces Y can be the remainder of some completion of a space X ? We consider the case when X and Y are

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metrisable, and use only elementary calculations with the metrics. Our main result (Theorem 5) is that if X is any non-compact space admitting a complete metric, and if Y is any complete separable metric space, then there is a metric ρ on X such that the remainder of the completion of (X, ρ) is isometric to Y.

We denote the set of natural numbers by N, and the set of real numbers by R. The sign $\uplus$ denotes the disjoint union of two sets. We write $\{a_1, a_2, a_3, \ldots\}_{\neq}$ for a family $\{a_1, a_2, a_3, \ldots\}$ all of whose members are distinct.

THEOREM 1. *Let the metric ζ on N be topologically discrete, and let the completion of (N, ζ) be $(N \uplus Y, \eta)$. Then (Y, η) is complete and separable.*

Proof. We first show that Y is closed in $N \uplus Y$. Let $n \in N$, and let U be an open neighbourhood of n in $N \uplus Y$ such that $U \cap N = \{n\}$. Suppose that $y \in U \cap Y$. Then $U \cap B_\eta(y, \eta(y, n))$ would be an open neighbourhood of y containing no point of N, so that y could not lie in the completion of N. Thus in fact no such y can exist, and U does not meet Y. Hence Y is closed in the complete space $N \uplus Y$, and so Y is itself complete.

Furthermore, $N \uplus Y$ is separable and metric, and so second-countable. Hence its subspace Y is second-countable, and thus separable. □

THEOREM 2. *Let (Y, η) be a complete separable metric space. Then there is a topologically discrete metric ζ on N, such that the remainder of the completion of (N, ζ) is isometric to (Y, η).*

Proof. We construct, on the set $N \uplus Y$, a complete metric ζ which is equal to η on Y, which is topologically discrete on N, and which makes N dense in $N \uplus Y$.

Suppose D is countable and dense in Y, where $D = \{d_1, d_2, d_3, \ldots\}_{\neq}$. We divide N into the disjoint union of sequences q_i^k $(k = 1, 2, 3, \ldots)$, one for each point d_i. Let $N \cup Y = Z$, and let $\zeta : Z \times Z \to R$ be defined as follows:

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$$\zeta | Y \times Y = \eta ,$$

$$\zeta\left(y, q_i^k\right) = \zeta\left(q_i^k, y\right) = 2^{-k} + \eta(d_i, y) , \quad (y \in Y) ,$$

$$\zeta\left(q_i^k, q_i^l\right) = |2^{-k}-2^{-l}| ,$$

$$\zeta\left(q_i^k, q_j^l\right) = 2^{-k} + \eta(d_i, d_j) + 2^{-l} , \quad (i \neq j) .$$

It is clear that ζ is a metric on Z, that (N, ζ) is topologically discrete, and (by an easy diagonal construction) that N is dense in $N \uplus Y$. We show that (Z, ζ) is complete.

Let (z_m) be a Cauchy sequence in Z. If (z_m) has a subsequence in Y, then as $\zeta | Y \times Y = \eta$ and (Y, η) is complete, this subsequence converges and thus so does (z_m). If however (z_m) has no subsequence in Y, it must have a subsequence in N; and so it is enough for us to show that every Cauchy sequence in (N, ζ) converges in Z.

Suppose then that $(z_m : m = 1, 2, 3, \ldots)$ is Cauchy, where $z_m = q_{i(m)}^{k(m)}$. Then either $\{i(m) : m = 1, 2, 3, \ldots\}$ is finite, or there is a subsequence of $i(m)$ consisting of distinct values.

If $\{i(m) : m = 1, 2, 3, \ldots\}$ is finite, then not more than one value of $i(m)$ can be repeated infinitely often, since (z_m) is Cauchy. Thus for all large m we have $z_m = q_i^{k(m)}$, for some i independent of m. But then $k(m) \to \infty$ as $m \to \infty$, again since (z_m) is Cauchy, and so $z_m \to d_i$ as $m \to \infty$.

There remains the possibility that for some sequence of values of m, we have $z_m = q_{i(m)}^{k(m)}$, where all values $i(m)$ are distinct. Then, for m and m' in this sequence we have

$$\zeta(z_m, z_{m'}) = 2^{-k(m)} + \eta\left(d_{i(m)}, d_{i(m')}\right) + 2^{-k(m')} , \quad (m \neq m') ,$$

and this quantity can be made arbitrarily small by choosing m and m' large enough. Hence, as m runs through the sequence in question,

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$(d_{i(m)})$ is a Cauchy sequence, and also $k(m) \to \infty$. But the Cauchy sequence $(d_{i(m)})$ in Y converges in Y, to y say: and then $\zeta(z_m, y) = 2^{-k(m)} + \eta(d_{i(m)}, y)$, which tends to 0 as m increases through our sequence. Thus (z_m) has a subsequence tending to y , and hence the whole sequence (z_m) has limit y . □

We seek to extend Theorem 2 to spaces other than N , with topologies other than the discrete topology. We consider first some circumstances under which the completion of a space is determined by a closed subset of the space.

THEOREM 3. *Let* (X, ξ) *be a complete metric space and let* F *be a closed subset of* X . *Let* ρ *be a metric on* X *, topologically equivalent to* ξ *, with* $\rho \le \xi$ *and such that if* $\rho(x, z) < \rho(x, F)$ *then* $\rho(x, z) = \xi(x, z)$. *Then the remainder of the completion of* (X, ρ) *is isometric to the remainder of the completion of* (F, ρ) .

Proof. The remainder of the completion of (X, ρ) is represented by ρ-Cauchy sequences in X with no limit in X . Two such sequences (x_n) and (z_n) represent the same point of the remainder if and only if $\rho(x_n, z_n) \to 0$, and, more generally, the distance between the points represented by (x_n) and by (z_n) is $\lim\rho(x_n, z_n)$. We need then a mapping H with the following properties:

(i) $H(x_n)$ is defined whenever (x_n) is Cauchy and free in (X, ρ) ;

(ii) $H(x_n)$ is a sequence in F ;

(iii) $H(x_n)$ is ρ-Cauchy;

(iv) $H(x_n)$ is ρ-free;

(v) $\lim_{r\to\infty} \left[(H(x_n))_r, (H(z_n))_r\right] = \lim_{n\to\infty} (x_n, z_n)$.

Now let (x_n) be a Cauchy sequence in (X, ρ) , and let us suppose

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first that $\rho(x_n, F)$ is bounded away from zero. That is, we have $\delta > 0$ such that $\delta \le \rho(x_n, F)$ for all n. Let n_0 be chosen so that for all m and n greater than n_0, $\rho(x_m, x_n) < \delta$. Then for such m and n, $\rho(x_m, x_n) < \rho(x_m, F)$, so that $\rho(x_m, x_n) = \xi(x_m, x_n)$. Hence (x_n) is a ξ-Cauchy sequence, so it is convergent in ξ and hence in ρ. Thus, if $\rho(x_n, F)$ is bounded away from zero, (x_n) cannot be free.

Now let (x_n) be a free Cauchy sequence in (X, ρ). Then $\rho(x_n, F)$ is not bounded away from zero, and so for each positive integer r there is $n(r)$ so that $\rho(x_{n(r)}, F) < \frac{1}{r}$. Hence there is a point f_r in F such that $\rho(x_{n(r)}, f_r) < \frac{1}{r}$. We let $H(x_n) = (f_r)$; then the properties (i) to (v) above are all easy to check. □

We shall use Theorem 3 by first modifying the metric of a given space on a closed subset in such a way as will produce a certain remainder, and then extending the modification to the whole space. The chief tool used to modify a metric while preserving the topology will be the following lemma.

LEMMA. *Let X be a set and let the function $\sigma : X \times X \to R$ have the following properties:*

$$\sigma(x, y) \ge 0 \quad (\text{all } x \text{ and } y),$$

$$\sigma(x, x) = 0 \quad (\text{all } x),$$

and

$$\sigma(x, y) = \sigma(y, x) \quad (\text{all } x \text{ and } y).$$

Moreover let $\rho : X \times X \to R$ be defined as follows:

$$\rho(x, y) = \inf\{\sigma(x, a_1) + \sigma(a_1, a_2) + \ldots + \sigma(a_m, y) : m \ge 0,\ a_j \in X \ (1 \le j \le m)\},$$

where if $m = 0$ the sum is understood to be $\sigma(x, y)$. Then ρ is a pseudometric on X.

Proof. Clearly, ρ is symmetric and non-negative, and $\rho(x, x) = 0$. The sums $\sigma(x, a_1) + \ldots + \sigma(a_m, y) + \sigma(y, b_1) + \ldots + \sigma(b_k, z)$, whose

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infinmum is $\rho(x, y) + \rho(y, z)$, all lie in the class of sums whose infimum is $\rho(x, z)$. Thus $\rho(x, z) \le \rho(x, y) + \rho(y, z)$. □

We shall call $x, a_1, a_2, \ldots, a_m, y$ a *string* of points from x to y . In the application of this lemma, we shall several times have occasion to remark that certain strings $x, a_1, a_2, \ldots, a_m, y$ may be omitted from the calculation of the infimum for $\rho(x, y)$, as the value of $\sigma(x, a_1) + \ldots + \sigma(a_m, y)$ is not less than the value of another such sum. We shall call such strings *ignorable*. When we consider a string $x, a_1, a_2, \ldots, a_m, y$ from x to y we shall, without specific comment, write $a_0 = x$ and $a_{m+1} = y$.

THEOREM 4. *Let* (X, ξ) *be a complete metric space, and let* F *be a closed subset of* X . *Let* φ *be a metric on* F , *topologically equivalent to* $\xi|F \times F$, *and with* $\varphi \le \xi$, *and let the remainder of the completion of* (F, φ) *be* (Y, η) . *Then there is a metric* ρ *on* X , *topologically equivalent to* ξ , *with* $\rho|F \times F = \varphi$, *and such that the remainder of the completion of* (X, ρ) *is isometric to* (Y, η) .

Proof. Let $\sigma : X \times X \to R$ be defined by

$$\sigma(x, x') = \begin{cases} \varphi(x, x') & \text{if } \{x, x'\} \subseteq F , \\ \xi(x, x') & \text{if } \{x, x'\} \not\subseteq F . \end{cases}$$

Let ρ be the pseudometric constructed from σ as in the lemma. As $\varphi \le \xi$, it is clear that $\rho|F \times F = \varphi$. We prove that ρ is a metric and that it satisfies the hypotheses of Theorem 3.

(i) To show that ρ is a metric, we obtain a formula for it rather simpler than that given in the lemma. Now, for the calculation of ρ , any string $x, a_1, \ldots, a_m, x'$ with three consecutive terms a_{i-1}, a_i, a_{i+1} in F is ignorable, since

$$\varphi(a_{i-1}, a_i) + \varphi(a_i, a_{i+1}) \ge \varphi(a_{i-1}, a_{i+1}) .$$

Moreover, if two consecutive terms of the string $x, a_1, \ldots, a_m, x'$ are in $X \backslash F$, and if $m \ge 1$, then the string is ignorable since

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$$\xi(a_{i-1}, a_i) + \xi(a_i, a_{i+1}) \geq \xi(a_{i-1}, a_{i+1}) .$$

Finally, if $\{a_{i-1}, a_{i+1}\} \subseteq F$ and $a_i \in X \backslash F$ then

$$\begin{aligned}\sigma(a_{i-1}, a_i) + \sigma(a_i, a_{i+1}) &= \xi(a_{i-1}, a_i) + \xi(a_i, a_{i+1}) \\ &\geq \xi(a_{i-1}, a_{i+1}) \\ &\geq \varphi(a_{i-1}, a_{i+1}) = \sigma(a_{i-1}, a_{i+1}) ,\end{aligned}$$

so again such a string is ignorable. Thus to calculate ρ it is enough to consider strings of the following forms:

$$\begin{array}{ll} x, x' & \{x, x'\} \subseteq X \backslash F ; \\ x, a_1, a_2, x' & \{x, x'\} \subseteq X \backslash F , \quad \{a_1, a_2\} \subseteq F ; \\ a, a' & \{a, a'\} \subseteq F ; \\ a, a_1, x & \{a, a_1\} \subseteq F , \quad x \in X \backslash F ; \\ a, x & a \in F , \quad x \in X \backslash F . \end{array}$$

Thus, if $\{x, x'\} \subseteq X \backslash F$ and $\{a, a'\} \subseteq F$, we have

$$\rho(x, x') = \min\left\{\xi(x, x'), \inf\{\xi(x, a_1)+\varphi(a_1, a_2)+\xi(a_2, x') : \{a_1, a_2\} \subseteq F\}\right\} ,$$

and

$$\rho(a, a') = \varphi(a, a') ,$$

and

$$\rho(a, x) = \inf\{\varphi(a, a_1)+\xi(a_1, x) : a_1 \in F\} .$$

We observe that the first of these formulae includes the other two, if x and x' are allowed to range over the whole of X.

We now suppose that $\rho(x, x') = 0$. Then either $\xi(x, x') = 0$, and so $x = x'$, or else

$$\inf\left\{\xi(x, a_1)+\varphi(a_1, a_2)+\xi(a_2, x') : \{a_1, a_2\} \subseteq F\right\} = 0 .$$

If the last equation holds, let (b_n) and (c_n) be sequences in F such that as $n \to \infty$,

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$$\xi(x, b_n) + \varphi(b_n, c_n) + \xi(c_n, x') \to 0 .$$

Then $\xi(x, b_n) \to 0$, and so, as F is closed, we have $x \in F$. Similarly $x' \in F$. But then $\rho(x, x') = \varphi(x, x')$ and hence again $x = x'$. Thus ρ is a metric.

(ii) To show that ρ and ξ are topologically equivalent, we consider first the ball, centre x and radius ε, in the two metrics. By construction $\rho \le \xi$, and so $B_\rho(x, \varepsilon) \supseteq B_\xi(x, \varepsilon)$ for all x and all ε. On the other hand, if $x \in X \backslash F$ and if $\delta = \min\{\varepsilon, \xi(x, F)\}$, then it is easy to see that $B_\rho(x, \delta) \subseteq B_\xi(x, \varepsilon)$. We complete the proof by showing that if $x \in F$ and if $\rho(x, z_n) \to 0$ then also $\xi(x, z_n) \to 0$. For, since

$$\rho(x, z_n) = \inf\{\varphi(x, a_1)+\xi(a_1, z_n) : a_1 \in F\} ,$$

we can select b_n in F such that

$$\varphi(x, b_n) + \xi(b_n, z_n) \to 0 .$$

But φ is topologically equivalent to ξ, and hence also

$$\xi(x, b_n) + \xi(b_n, z_n) \to 0 ,$$

so that $\xi(x, z_n) \to 0$, as required.

(iii) We have already observed that $\rho \le \xi$, since $\varphi \le \xi$.

(iv) If $\rho(x, z) \ne \xi(x, z)$ then

$$\begin{aligned} \rho(x, z) &= \inf\{\xi(x, a_1)+\varphi(a_1, a_2)+\xi(a_2, z) : \{a_1, a_2\} \subseteq F\} \\ &\ge \inf\{\xi(x, a_1) : a_1 \in F\} \\ &= \xi(x, F) \\ &\ge \rho(x, F) , \end{aligned}$$

so that if $\rho(x, z) < \rho(x, F)$ then $\rho(x, z) = \xi(x, z)$. Thus all the hypotheses of Theorem 3 hold, and so the remainder of the completion of (X, ρ) is isometric to the remainder of the completion of (F, ρ), which since $\rho|F \times F = \varphi$, is (Y, η). □

Finally we put together Theorems 2 and 4.

THEOREM 5. *Let* (X, κ) *be a complete non-compact metric space and*

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let (Y, η) *be a complete separable metric space whose metric is bounded. Then there is a metric* ρ *on* X *, topologically equivalent to* κ *, such that the remainder of* (X, ρ) *is isometric to* (Y, η) .

Proof. Since X is complete but not compact, it is not totally bounded. That is, $\exists \varepsilon > 0$ such that for every finite set E in X there is an x in X with $\kappa(x, E) \geq \varepsilon$. We choose a_1 arbitrarily in X , and then inductively choose a_n such that $\kappa(a_n, \{a_1, a_2, \ldots, a_{n-1}\}) \geq \varepsilon$. Let $F = \{a_1, a_2, a_3, \ldots\}$. Then F is clearly closed in X , and every pair of distinct members a, a' of F has $\kappa(a, a') \geq \varepsilon$. Now let L be a bound for the metric η ; we define a new metric ξ for X by writing

$$\xi(x, x') = \varepsilon^{-1}(1+L)\kappa(x, x') .$$

Then ξ is clearly uniformly equivalent to κ , and $\xi(a, a') \geq 1 + L$ if $\{a, a'\} \subseteq F$. We take the space (Y, η) , and construct on N the corresponding metric ζ as in Theorem 2, with the construction given in the proof of that theorem. We observe that, with this construction, ζ is bounded above by $1 + L$. Let the metric φ be defined on F by

$$\varphi(a_m, a_n) = \zeta(m, n) .$$

Then for all a and a' in F ,

$$\varphi(a, a') \leq 1 + L \leq \xi(a, a') .$$

Thus (X, ξ), F and φ satisfy the hypotheses of Theorem 4, and the proof of Theorem 5 is complete. □

COROLLARY 1. *Every completely metrizable topological space either is compact or has uncountably many different uniformities compatible with its topology.*

Gál and Doss [*4*, *2*] considered spaces which, though not compact, have unique compatible uniformity. All the explicit examples seem to be constructed from uncountable ordinals, and so are certainly not metrizable. Corollary 1 shows that such a space can never bear a complete metric.

Note added in proof. Theorem 5 of this paper overlaps considerably with a result (Theorem 1) announced by V.K. Bel'nov in "On metric extensions", *Soviet Math. Dokl.* 13 (1972), 220-224 = *Dokl. Akad. Nauk SSSR* 202 (1972), 991-994.

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