ON THE GAUSS-GREEN THEOREM

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1. Introduction

In a previous paper [1], Green's theorem for line integrals in the plane was proved, for Riemann integration, assuming the integrability of $Q_x - P_y$, where $P(x, y)$ and $Q(x, y)$ are the functions involved, but *not* the integrability of the individual partial derivatives Q_x and P_y. In the present paper, this result is extended to a proof of the Gauss-Green theorem for p-space ($p \geqq 2$), for Lebesgue integration, under analogous hypotheses. The theorem is proved in the form

$$\int_\Omega \operatorname{div} g(x) d\mu_p(x) = \int_{\partial\Omega} g(x) \cdot \nu(x) d\Phi(x) \tag{1}$$

where Ω is a bounded open set in R^p (p-space), with boundary $\partial\Omega$; $g(x) = (g(x_1), \cdots, g(x_p))$ is a p-vector valued function of $x = (x_1, \cdots, x_p)$, continuous in the closure of Ω;

$$\operatorname{div} g(x) = \sum_{i=1}^{p} \frac{\partial g_i(x)}{\partial x_i};$$

$\mu_p(x)$ is p-dimensional Lebesgue measure; $\nu(x) = (\nu_1(x), \cdots, \nu_p(x))$ and $\Phi(x)$ are suitably defined unit exterior normal and surface area on the 'surface' $\partial\Omega$; and $g(x) \cdot \nu(x)$ denotes inner product of p-vectors.

In analogy with the plane case, $\operatorname{div} g(x)$ is assumed finite, except on a suitably restricted 'exceptional set', and Lebesgue integrable on Ω — but the individual partial derivatives $\partial g_i(x)/\partial x_i$ need *not* be integrable; and $\partial\Omega$ is assumed to have finite Hausdorff $(p-1)$-measure, and to satisfy a weak continuity condition. The hypothesis on Hausdorff measure, which is analogous to the requirement in [1] that the plane curve is rectifiable, is equivalent to a hypothesis on covering $\partial\Omega$ by cubes, analogous to Potts' Lemma [2] on covering a rectifiable plane curve by squares.

Other authors have assumed that the individual partial derivatives are integrable. Notably, Federer [3], [4], [5] proves the theorem, for suitable scalar $f(x)$, in the form

$$\int_\Omega \frac{\partial f(x)}{\partial x_i} d\mu_p(x) = \int_{\partial\Omega} f(x)\nu_i(x) d\Phi(x), \tag{2}$$

and Michael [6] proves (2) with a multiplicity factor inserted. Both assume, however, that $\partial f/\partial x_i$ is integrable over Ω.

The proof of (1) depends, not on the detailed definitions of $\nu(x)$ and $\Phi(x)$, but on the following properties assumed for those functions:

(I) $\nu(x)$ is a Borel-measurable function of x, which reduces to the geometric exterior normal to Ω whenever $\partial\Omega$ is differentiable at x; $\nu(x) = 0$ by convention wherever a normal is undefined.

(II) If $\nu(x)$ and $\nu^*(x)$ denote the unit exterior normals to Ω and its complement at the point $x \in \partial\Omega$, then $\nu^*(x) = -\nu(x)$.

(III) $\Phi(S)$ is a Carathéodory outer measure ([7] § 235) for subsets S of $\partial\Omega$, which equals geometric $(p-1)$-dimensional area in the neighbourhood of any point where the surface $\partial\Omega$ is differentiable. [$\Phi(x)$ denotes $\Phi(S)$ for $S = \{y : y_i \leqq x_i,\ i = 1, 2, \cdots, p\}$.]

(IV) If $\partial\Omega$ denotes the entire boundary of any bounded open set Ω, for which $\Phi(\partial\Omega) < \infty$, then

$$\int_{\partial\Omega} \nu_i(x) d\Phi(x) = 0 \qquad (i = 1, 2, \cdots, p). \tag{3}$$

Federer ([3] and [4]) defines a normal $\nu(x)$, which restricts Ω merely to be a bounded open set, and shows that this $\nu(x)$, together with $\Phi(S)$ defined as Hausdorff $(p-1)$-measure on $\partial\Omega$, satisfy (I), (II), (III), and (2). If C is any constant vector, then

$$\begin{aligned} \int_{\partial\Omega} C \cdot \nu(x) d\Phi(x) &= \sum_{i=1}^{p} C_i \int_{\partial\Omega} \nu_i(x) d\Phi(x) \\ &= 0 \quad \text{from (2)}, \end{aligned} \tag{4}$$

so that (IV) also holds. It is not obvious whether any other extensions of normal and area exist, satisfying (I) to (IV), but if they do, then Theorems 1, 2, 3 of this paper remain valid for them.

2. Boundary surface

If C is a rectifiable plane curve, of length L, then Lemma 2 of Potts [2] states that there is a covering M_δ of L by at most $4(L/\delta)+4$ closed squares, each of side δ, with disjoint interiors and sides parallel to the axes. Hence, if $K = 8L$, a constant depending only on C, M_δ consists of at most K/δ squares of side δ, whose total area $K\delta \to 0$ as $\delta \to 0$, and whose total perimeter is less than $4K$, a bound independent of δ. This fact suggests the following generalization to R^p. Let 'cube' denote 'p-dimensional hypercube with edges parallel to the axes'. A 'surface' E ($(p-1)$-dimensional manifold) in R^p will be said to satisfy the 'Potts condition' if, for a sequence of values of $\delta \downarrow 0$, E can be covered by a finite collection M_δ of closed cubes A_i with disjoint interiors, such that the edge δ_i of A_i is less than δ, for each i, and $\sum_i \delta_i^{p-1} < K$, a constant independent of δ. Denote by M_δ^* the union of the cubes of M_δ. It follows that the total p-dimensional volume of M_δ^* is less than $K\delta$, so $\to 0$ with δ, and the total $(p-1)$-dimensional surface area of the cubes of M_δ is less than $2pK$, for all δ. The 'Potts condition' is further characterized by the following two Lemmas.

LEMMA 1. *The boundary E of a bounded open set in R^p satisfies the Potts condition if and only if its Hausdorff $(p-1)$-measure, $\Phi(E)$, is finite.*

PROOF. Hausdorff measure is defined [5] as

$$\Phi(E) = 2^{-p+1}\alpha_{p-1} \lim_{r\to 0+} \left[\inf\left\{\sum_{j=1}^{\infty} (\operatorname{diam} B_j)^{p-1} : E \subset \bigcup_{j=1}^{\infty} B_j;\ \operatorname{diam} B_j < r,\ j = 1, 2, \cdots\right\}\right] \tag{5}$$

where α_{p-1} = volume of $(p-1)$-dimensional unit sphere. Let E satisfy the Potts condition. For any $r > 0$, there is a covering M_δ of E by cubes A_i of edge $< \delta$, and therefore of diameter $< \delta p^{\frac{1}{2}} < r$, by choice of δ, such that

$$\sum_i (\operatorname{diam} A_i)^{p-1} = (p^{\frac{1}{2}})^{p-1} \sum_i \delta_i^{p-1} < K(p^{\frac{1}{2}})^{p-1},$$

a constant independent of r, consequently, from (5), $\Phi(E) < \infty$.

The converse is Theorem 4.1 of Michael [8], noting that E is compact.

LEMMA 2. *Let C be a plane closed Jordan curve. Then C satisfies the Potts condition if and only if C is rectifiable.*

PROOF. If C is rectifiable, then C satisfies the Potts condition, by Potts' Lemma. Conversely, let C satisfy the Potts condition. Then C is bounded. Choose any n distinct points $P_0, P_1, \cdots, P_{n-1}$ on C, taken in order around C; denote $P_n = P_0$. Cover each P_i by a square K_i, whose edge $< 1/n$. Let C_i denote that part of the arc $P_{i-1}P_i$ which lies outside Int $(K_{i-1} \cup K_i)$. Since the C_i are disjoint compact, there is a Potts covering M of C, such that each C_i is covered by a union M_i of squares of M, and the M_i are disjoint. There are points $Q'_{i-1} \in K_{i-1} \cap \partial M_i$ and $Q''_i \in K_i \cap \partial M_i$, where ∂M_i denotes the boundary of M_i. There is an arc, of length b_i say, joining Q'_{i-1} to Q''_i, consisting of parts of edges of squares of M_i. Then, if d denotes distance, and K is the constant of the Potts hypothesis,

$$\sum_{i=1}^{n} d(P_{i-1}, P_i) \leqq \sum_{i=1}^{n} \{d(P_{i-1}, Q'_r) + b_i + d(Q''_i, P_i)\}$$
$$\leqq n \cdot \frac{\sqrt{2}}{n} + 4K + \frac{n\sqrt{2}}{n} = 2\sqrt{2} + 4K,$$

a bound independent of the P_i. So C is rectifiable.

3. Admissible domains

Let Ω be a bounded open subset of R^p, whose boundary $\partial\Omega$ is a countable union of disjoint continuous images E_k of S^{p-1}, the $(p-1)$-dimensional unit sphere. Let $V = \cup V_k$, where the V_k are countably many disjoint copies of S^{p-1} in R^p. Now $E_k = f_k(V_k)$, where each f_k is continuous, so that $\partial\Omega = f(V)$, where $f|V_k = f_k$, and f is continuous. (The set V may be taken instead as a countable union of disjoint closed intervals in R.)

If $\partial\Omega$ is topologised as a subspace of R^p, then the sets

$$A = \partial\Omega \cap \{x : x_i < \alpha\} \quad \text{and} \quad B = \partial\Omega \cap \{x : x_i > \alpha\}$$

are open in $\partial\Omega$, so their inverse images $f^{-1}A$ and $f^{-1}B$ are open in V, and therefore consist of at most countably many disjoint arcwise-connected components. Consequently, if K is any open cube in R^p, $\partial\Omega \cap K$ consists of at most countably many components.

Let $L_i(\alpha) = \partial\Omega \cap \{x : x_i \leqq \alpha\}$. Since Φ is monotone, $\Psi_1(\alpha_1) = \Phi(L_1(\alpha_1))$ is a nondecreasing function of α_1, so there is a countable dense set D_1 of α_1 on which Ψ_1 is continuous. Likewise, for each $\alpha_1 \in D_1$,

$$\Psi_2(\alpha_1, \alpha_2) = \Phi(L_1(\alpha_1) \cap L_2(\alpha_2))$$

is a nondecreasing function of α_2, so there is a countable dense set D_2 of α_2 such that Ψ_2 is continuous for $\alpha_1 \in D_1$, $\alpha_2 \in D_2$; and so on. The planes $x_i = \alpha_i \in D_i$ $(i = 1, 2, \cdots, p)$ will be called *admissible planes*. Since they form a dense family, the cubes used in Potts coverings can be replaced by cuboids bounded by admissible planes, with arbitrarily little change in the bounds previously obtained; this will be assumed henceforth. If W is any open cuboid bounded by admissible planes, then any component of $W \cap \partial\Omega$ will be called an *admissible domain* in $\partial\Omega$.

LEMMA 3. *If A_i $(i = 1, 2, \cdots)$ are disjoint admissible domains in $\partial\Omega$, then*

(i) *$\Phi(\overline{A_i}) = \Phi(A_i)$, where $\overline{A_i}$ = closure of A_i in $\partial\Omega$;*

(ii) *$\Phi(A_1+A_2) = \Phi(A_1)+\Phi(A_2)$, where A_1+A_2 now denotes the interior of $\overline{A_1} \cup \overline{A_2}$; denote also $A_1+A_2+\cdots+A_n+\cdots$ = Interior of $\bigcup_1^\infty \overline{A_i}$;*

(iii) *if $A_0 = A_1+A_2+\cdots+A_n+\cdots$ is also admissible, then*

$$\Phi(A_0) = \sum_1^\infty \Phi(A_n);$$

(iv) *A_i is Φ-measurable;*

(v) *if $f(x)$ is bounded Borel-measurable, then*

$$\int_{A_1+A_2} f d\Phi = \int_{A_1} f d\Phi + \int_{A_2} f d\Phi.$$

PROOF. (i) If W is an open cuboid bounded by admissible planes, then the continuity of Φ on admissible planes implies that there is a larger cuboid W_ε, obtained by displacing outward each boundary plane of W, such that $\overline{W} \subset W_\varepsilon$, and $\Phi(W_\varepsilon \cap \partial\Omega) < \Phi(W \cap \partial\Omega)+\varepsilon$. So if A is an admissible domain, there is an admissible domain $A_\varepsilon \supset \bar{A}$ with $\Phi(A_\varepsilon) < \Phi(A)+\varepsilon$; which implies (i).

(ii) Define distance d on $\partial\Omega$ as the restriction to $\partial\Omega$ of distance in R^p. Since $A_1 \cap A_2 = \emptyset$, $C = \overline{A_1} \cap \overline{A_2}$ *is contained in the frontiers (in* $\partial\Omega$*) of* A_1 and A_2. *By the definition of admissible domain, these frontier points* are boundary points of finitely many cuboids bounded by admissible planes. These planes may be covered by a finite union G of open cuboids, such that $\Phi(D) < \varepsilon$, where $C \subset D = G \cap \partial\Omega$. Then the sets $\overline{A_i}-D = A_i-D$ $(i = 1, 2)$ are disjoint closed sets in $\partial\Omega$; therefore $d(A_1-D, A_2-D) > 0$. Since Φ is a Carathéodory outer measure, it is additive on A_1-D and A_2-D, and the result follows.

(iii) Since $A_1+\cdots+A_n \subset \overline{A_0}$,

$$\sum_1^n \Phi(A_i) \leqq \Phi(\overline{A_0}) = \Phi(A_0) \qquad \text{by (ii) and (i);}$$

since Φ is subadditive,

$$\Phi(\overline{A_0}) \leqq \sum_1^\infty \Phi(\overline{A_i}) = \sum_1^\infty \Phi(A_i) \qquad \text{by (i).}$$

(iv) Since A_i is open in $\partial\Omega$, and Φ is a Carathéodory outer measure on $\partial\Omega$, A_i is measurable (Carathéodory [7], § 238 and § 251).

(v) From (ii) and (iv), it readily follows that, for any Borel set B (i.e. any set obtained from admissible domains by countably many unions and intersections) $\Phi(B \cap (A_1+A_2)) = \Phi(B \cap A_1)+(B \cap A_2)$; and this leads readily to (v).

LEMMA 4. *If $f(x)$ is bounded Borel-measurable; $A_1, A_2, \cdots$ are disjoint admissible domains; and $A = A_1+A_2+\cdots$ is an admissible domain, with $\Phi(A) < \infty$; then, independently of the order of summation,*

$$\int_A f d\Phi = \sum_{i=1}^\infty \int_{A_i} f d\Phi. \tag{6}$$

PROOF. Since $\Phi(A) < \infty$, $\int_A$ is finite, and by Lemma 3 (iii), so is each $\int_{A_i}$. Suppose that some sequence of partial sums of the series (6), summed in some order, converges to a limit λ, where $|\lambda-\int_A| = 3\delta > 0$. Then

$$\left|\sum_{i\in N_r} \int_{A_i} -\lambda\right| < \delta$$

for an expanding sequence of finite sets $N_r \uparrow N$, the set of all positive integers. If $F_r = A_1 + \cdots + A_r$ and $G_r = A - A_r$, then by Lemma 3 (iii)

$$\Phi(G_r) \leqq \sum_{N-N_r} \Phi(\overline{A}_r) = \sum_{N-N_r} \Phi(A_i) < \delta/\sup|f|$$

by choice of r, since $\sum \Phi(A_i) < \infty$. Since F_r, G_r are disjoint measurable sets,

$$3\delta = \left|\int_{F_r} + \int_{G_r} - \lambda\right| = \left|\sum_{N_r}\int_{A_i} + \int_{G_r} - \lambda\right| \leqq \left|\sum_{N_r}\int_{A_i} - \lambda\right| + \sup|f|\Phi(G_r) \leqq \delta + \delta,$$

so that $\delta = 0$.

4. Gauss-Green theorem

THEOREM 1. *Let Ω be a bounded open subset of R^p, whose boundary $\partial\Omega$* (i) *satisfies the Potts condition, and* (ii) *is a countable union of disjoint continuous images of S^{p-1}. Let $g : \overline{\Omega} \to R^p$ be continuous on $\overline{\Omega}$. Let div g be Lebesgue-integrable on Ω. For every cuboid $\Gamma \subset \Omega$, let the Gauss-Green theorem* (1) *hold, with Ω, $\partial\Omega$ replaced by Γ, $\partial\Omega$. Then* (1) *holds for Ω, $\partial\Omega$.*

PROOF. Let M_δ be a Potts covering of $\partial\Omega$, consisting of closed cuboids A_i. Denote the interior of A_i by A_i^0. Let C_δ denote the union of those relatively open subsets of the boundary planes of the A_i which lie in $M_\delta^* \cap \Omega$. Then, by definition of Potts covering, $\mu_p(M_\delta^*) < K\delta$ and $\mu_{p-1}(C_\delta) < 2pK$. Let $h(x) = \operatorname{div} g(x)$ for $x \in \Omega$, $h(x) = 0$ for $x \notin \Omega$. Then

$$\int_\Omega \operatorname{div} g \, d\mu_p = \int_{R^p} h \, d\mu_p.$$

Since $h \in L(R^p)$,

$$\left|\int_{M_\delta^*} h \, d\mu_p\right| < \varepsilon$$

if $\mu_p(M_\delta^*) < \Delta(\varepsilon)$. So, if $W = \Omega - M_\delta^*$ and $\delta < K^{-1}\Delta(\varepsilon)$,

$$\left|\int_\Omega \operatorname{div} g \, d\mu_p - \int_W h \, d\mu_p\right| < \varepsilon. \tag{7}$$

The set $A_i \cap \Omega$ has boundary $\rho_i = \alpha_i \cup \sigma_i \cup \lambda_i$, where $\alpha_i = A_i^0 \cap \partial\Omega$ is the union of (at most) countably many admissible domains α_{ij}, the relatively open set $\sigma_i = \partial A_i \cap \Omega$ is the union of (at most) countably many components β_{ij} of $C_\delta - \partial W$ and γ_{ij} of ∂W, and $\lambda_i = \partial A_i \cap \partial\Omega$ satisfies $\Phi(\lambda_i) = 0$, since A_i is bounded by admissible planes. The frontiers of the open sets β_{ij} and γ_{ij}, in the relative topology of ∂A_i, are contained in λ_i. Consequently, the results of Lemmas 3 and 4 apply also to the β_{ij} and γ_{ij}; these sets will also be called 'admissible domains'.

In terms of the set composition $+$ of Lemma 3, ρ_i is the sum, over countably many indices j, of the α_{ij}, β_{ij}, γ_{ij}. The proof of Theorem 1 consists essentially in recombining the corresponding integrals in a different order; this process is validated by Lemma 4, which also shows that the frontier points (in the relative topology) of the admissible domains make no contribution.

Attach to each point $x \in \rho_i$ the unit exterior normal $\nu(x)$. For $x \in \beta_{ij}$, two normals are possible, oppositely directed, depending on which ρ_i is chosen; in the following summation, each β_{ij} contributes twice, once for each normal. With integrand $g \cdot \nu d\Phi$,

$$\begin{aligned}
\int_{\partial\Omega} &= \sum_{i,j} \int_{\alpha_{ij}} && \text{by Lemma 4} \\
(8) \qquad &= \sum_{i,j} \int_{\alpha_{ij}} + \sum_{i,j} \int_{\beta_{ij}} + \sum_{i,j} \int_{\gamma_{ij}} - \sum_{i,j} \gamma_{ij} \text{ since } \sum_{i,j} \int_{\beta_{ij}} = 0 \\
&= \sum_i \sum_j \left(\int_{\alpha_{ij}} + \int_{\beta_{ij}} + \int_{\gamma_{ij}} \right) - \sum_{i,j} \int_{\gamma_{ij}} && \text{by Lemma 4} \\
&= \sum_{i,j} \int_{\rho_{ij}} + \int_{\partial W} && \text{by Lemma 4.}
\end{aligned}$$

Since g is continuous on the compact set $\bar{\Omega}$, and $\mu_p(M_\delta) \to 0$ as $\delta \to 0$, there is δ such that the oscillation of $g(x)$ in the closure of each $A_i \cap \Omega$ is less than ε. So, for δ sufficiently small, there corresponds to each ρ_i a constant vector c_i such that, for $x \in \rho_i$,

$$g(x) = c_i + \eta_i(x) \text{ where } |\eta_i(x)| < \varepsilon.$$

Hence

$$\begin{aligned}
\int_{\rho_{ij}} g \cdot \nu d\Phi &= \int_{\rho_{ij}} c_i \cdot \nu d\Phi + \int_{\rho_{ij}} \eta_i \cdot \nu d\Phi \\
&= \int_{\rho_{ij}} \eta_i \cdot \nu d\Phi && \text{by (3)}
\end{aligned}$$

so that

$$\begin{aligned}
\left| \sum_{ij} \int_{\rho_{ij}} g \cdot \nu d\Phi \right| &\leqq \varepsilon \sum_{ij} \Phi(\rho_{ij}) \\
(9) \qquad &\leqq \varepsilon \left(2 \sum_i \Phi(\partial A_i) + \Phi(\partial\Omega) \right) && \text{by Lemma 3 (iii)} \\
&\leqq \varepsilon (4pK + \Phi(\partial\Omega))
\end{aligned}$$

where K is the constant of the family of Potts coverings.

Now the Gauss-Green theorem applies, by hypothesis, to W, which is a finite union of cuboids $\subset \Omega$. Combining this with (7) and (9),

$$(10) \qquad \left| \int_\Omega \operatorname{div} g d\mu_p - \int_{\partial\Omega} g \cdot \nu d\Phi \right| \leqq B \cdot \varepsilon$$

for constant B; which proves the theorem.

LEMMA 5. (*Saks* [9], *page* 198.) *Let w be a real function of one variable, such that $w'(x)$ exists p.p. in $[a, b]$; let F be a closed non-empty subset of $[a, b]$; let N be a finite constant such that*

$$|w(x_2)-w(x_1)| \leqq N|x_2-x_1| \text{ whenever } x_1 \in F \text{ and } x_2 \in [a, b].$$

Then

$$|w(b)-w(a)-\int_F w'(x)dx| \leqq N(b-a-\mu_1(F)).$$

PROOF. (Saks) Let $u(x) = w(x)$ on $F \cup \{a, b\}$, and linear on the complementary intervals. Then $u(x)$ is Lipschitz, therefore absolutely continuous. Hence

$$w(b)-w(a) = u(b)-u(a) = \int_a^b u'(x)dx.$$

But $u'(x) = w'(x)$ p.p. in F, and $|u'(x)| \leqq N$ at each $x \in' F$, which proves the result.

THEOREM 2. *Let W be an open cuboid in R^p; let K be an open cuboid containing $\overline{W}$. Let $g(x)$ be continuous on K; let div $g(x)$ be finite for all $x \in K$ and Lebesgue integrable on W. Then the Gauss-Green theorem* (1) *holds for W, ∂W.*

PROOF. A point $x \in \overline{W}$ will be called *admissible* if it has an open neighbourhood $N(x) \subset K$, such that for every cuboid $C \subset N(x)$, (1) holds for C, ∂C. Let F denote the complement, with respect to $\overline{W}$, of the set of admissible points. From its construction, F is closed. Suppose that F is not empty; this will lead to a contradiction.

For $n = 1, 2, \cdots$, denote by F_n the set of points x for which

$$(11) \qquad \max_{i=1,2,\cdots,p} |g(x_1, \cdots, x_{i-1}, x_i+h, x_{i+1}, \cdots, x_p) \\ -g(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_p)| \leqq n|h| \text{ for } |h| < n^{-1}.$$

Since $\partial g_i(x)/\partial x_i$ is finite for all x, $\overline{W} \subset \cup_n F_n$. Then, according to Baire's category theorem ([9] page 55) there is an open cuboid I such that $F \cap F_N$ is dense in $F \cap I$ for some integer N. Since also F and F_N are closed, $\emptyset \neq I \cap F \subset \bar{I} \cap (F \cap F_N) \subset F_N$. Let $x_0 \in I \cap F$. Let Q be any closed cuboid of diameter $\leqq N^{-1}$, where $x_0 \in Q \subset I$.

Given $\delta > 0$, there is a countable covering of $E = F \cap Q$ by open cuboids G, such that

$$\sum_1^\infty \mu_p(G_j) < \mu_p(F \cap Q)+\delta.$$

Since $F \cap Q$ is compact, a finite subset of the G_j covers $F \cap Q$. Since also $\mu_p(\bar{G}_j) = \mu_p(G_j)$, there is a finite covering of $F \cap Q$ by closed cuboids S_j

($j = 1, \cdots, r$) which may be assumed to have disjoint interiors, and to lie within Q, such that

$$\sum_1^r \mu_p(S_j) < \mu_p(F \cap Q)+\delta. \tag{12}$$

Let S_j be the cuboid $a_j \leqq x_j \leqq b_j$ ($j = 1, \cdots, p$). Let the line specified by fixed values of $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_p$ intersect $F \cap S_j$ in the set $T_i = T_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_p)$, whose linear measure is $\mu_1(T_i)$. Then, from Lemma 5,

$$\begin{aligned}\psi(x_1, \cdots, x_{i-1}; x_{i+1}, \cdots, x_p)&\\ \equiv |g_i(x_1, \cdots, x_{i-1}, b_i, x_{i+1}, \cdots, x_p)&-g_i(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_p)\\ -\int_{T_i} \frac{\partial g_i}{\partial x_i}&(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_p)dx_i)|\\ \leqq N \cdot (b_i-a_i-\mu_1(T_i)).&\end{aligned}$$

So, integrating with respect to $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_p$ over $a_j \leqq x_j \leqq b_j$,

$$\begin{aligned}&\left|\int_{\partial S_j} g_i(x)\nu_i(x)d\Phi(x)-\int_{S_j\cap F}\frac{\partial g_i(x)}{\partial x_i}\,d\mu_p(x)\right|\\ &\qquad = \int \psi(x_1, \cdots, x_{i-1}; x_{i+1}, \cdots, x_p)dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_p, \\ &\qquad\qquad\text{since } S_j \text{ is a cuboid}\\ &\qquad \leqq N\int (b_i-a_i-\mu_1(T_i))dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_p\\ &\qquad = N(\mu_p(S_j)-\mu_p(S_j \cap F)) \qquad \text{by Fubini's theorem.}\end{aligned} \tag{13}$$

Define the set function $H(S)$ on closed cuboids S by

$$pH(S) = \int_{\partial S} g(x) \cdot \nu(x)d\Phi(x)-\int_S \operatorname{div} g(x)d\mu_p(x). \tag{14}$$

Then $H(S)$ is additive on cuboids whose interiors are disjoint, and, from the definition of F,

$$H(S) = 0 \text{ if } F \cap S = \emptyset. \tag{15}$$

Now since H is additive,

$$\begin{aligned}|H(\bigcup_1^r S_j)| &\leqq \sum_1^r |H(S_j)|\\ &\leqq N\sum_{j=1}^r [\mu_p(S_j)-\mu_p(S_j \cap F)+\int_{S_j-F} |\operatorname{div} g(x)|d\mu_p(x)] \text{ by (13)}\\ &\leqq N\mu_p((\bigcup_1^r S_j)-F \cap Q)+N\int_{(\cup S_j)-(F\cap Q)} |\operatorname{div} g(x)|d\mu_p(x).\end{aligned} \tag{16}$$

Since g is integrable over W, the integral in (16) can be made less than $\varepsilon/(2N)$ by choosing $\mu_p((\cup S_j)-(F \cap Q)) < \Delta(\varepsilon/2N)$, say. From (12)

$$\mu_p((\cup S_j)-(F \cap Q) < \min(\varepsilon/2N, \Delta(\varepsilon/2N)) \tag{17}$$

if δ is chosen less than the quantity on the right of (17). Hence $|H(\cup S_j)| < \varepsilon$.

Now

$$\begin{aligned} |H(Q)| &= |H(Q-\cup S_j)+H(\cup S_j)| \\ &\leqq |H(Q-\cup S_j)|+|H(\cup S_j)| \\ &< 0+\varepsilon, \end{aligned}$$

since $Q-S_j \subset Q-F$. Since ε is arbitrary, $H(Q) = 0$. Since this is true for every sufficiently small cuboid Q containing x_0, the assumption $x_0 \in F$ is contradicted. Hence F is empty.

THEOREM 3. *Let Ω be a bounded open subset of R^p, whose boundary $\partial\Omega$ satisfies the Potts condition (or equivalently, by Lemma 1, has $\Phi(\partial\Omega) < \infty$), and is a countable union of disjoint continuous images of S^{p-1}. Let E be a subset of Ω which satisfies the same hypotheses as $\partial\Omega$. Let the function $g : \bar{\Omega} \to R^p$ be continuous; let div g exist (with finite value) at all points of $\Omega-E$, and be integrable on Ω. Then the Gauss-Green theorem (1) holds for Ω, $\partial\Omega$.*

REMARKS. The topological hypothesis on $\partial\Omega$ is an analog of the hypothesis, in Green's theorem for two dimensions, that the boundary is a closed Jordan curve.

The subset E may consist, e.g., of countably many points, or lines, etc., within Ω, on which one or more derivatives $\partial g_i/\partial x_i$ fail to exist; since $\mu_p(E) = 0$ (from the Potts condition), div g is defined a.e. on Ω.

The Looman-Menchoff theorem (Saks [9]) states that if $f(z) = u+iv$ is a continuous function of complex z on domain Ω, and u and v have their first partial derivatives finite in Ω except on a countable set E, and satisfy the Cauchy-Riemann equations a.e. in Ω, then $\oint_C f(z)dz = 0$ for each closed rectangle C in Ω. Theorem 3 of this paper shows that this exceptional set E can be considerably enlarged.

PROOF. Let M be a closed Potts covering of E, with parameter δ. The hypotheses of Theorem 2, and consequently the Gauss-Green theorem, hold for each cuboid $K \subset \Omega-M$. Therefore, by Theorem 1, the Gauss-Green theorem holds also for $\Omega-M$ and its boundary.

Since E satisfies the same hypotheses as $\partial\Omega$, the arguments which lead to (7) and (9) in the proof of Theorem 1 show also that, for sufficiently small δ,

$$\left|\int_\Omega \operatorname{div} g\, d\mu_p - \int_{\Omega-M} \operatorname{div} g\, d\mu_p\right| < \varepsilon$$

$$\left|\int_{\partial\Omega} - \int_{\partial(\Omega-M)} g\cdot\nu\, d\Phi\right| < k\cdot\varepsilon$$

where k is constant. Since ε is arbitrary, these results combine to prove the Gauss-Green theorem for Ω, $\partial\Omega$.

5. Examples

(I) Theorem 3, or even the two-dimensional Riemann-integral version in [1], is a non-trivial extension of the usual Gauss-Green theorem. An example in two dimensions is as follows.

Let Ω denote the interior of the unit circle $x_1^2+x_2^2 = 1$. Let

$$g_1(x_1, x_2) = x_2 r^2 \sin \pi/r^4$$
$$g_2(x_1, x_2) = -x_1 r^2 \sin \pi/r^4$$

where $r^2 = x_1^2+x_2^2$. Then g_1 and g_2 are continuous, and even differentiable, at all points in Ω, since for $r \neq 0$,

$$\frac{\partial g_1}{\partial x_1} = -2x_1x_2 \sin\frac{\pi}{r^4} + \frac{4\pi x_1 x_2}{r^4}\cos\frac{\pi}{r^4} = -\frac{\partial g_2}{\partial x_2},$$

and $|[g_1(x_1, x_2)-g_1(0, 0)]/r| < r$ (and similarly for g_2).

Thus $\operatorname{div} g(x) = 0$ in Ω, so is integrable, and Green's theorem holds for these functions. But if $\partial g_1/\partial x_1$ were integrable on Ω, it would follow (since $2x_1x_2 \sin \pi/r^4$ is continuous) that

$$\iint \left|\frac{x_1x_2}{r^4}\cos\frac{\pi}{r^4}\right| dx_1 dx_2 < \infty,$$

hence in polar coordinates,

$$\int_0^1 \left|\cos\frac{\pi}{r^4}\right| \frac{dr}{r} < \infty$$

or (with $r = S^{-\frac{1}{4}}$)

$$\int_0^1 |\cos \pi S|\, \frac{dS}{S} < \infty.$$

Since this integral diverges, $\partial g_1/\partial x_1$ is *not* integrable on Ω, consequently the usual forms of Green's theorem do not apply.

(II) Theorem 3 is untrue if the exceptional set E, on which $\operatorname{div} g$ fails to exist, is increased to an arbitrary null set (i.e. $\mu_p(E) = 0$). A counter-example for $p = 2$ is given by Ω = unit square $(0 \leqq x_1 \leqq 1,\ 0 \leqq x_2 \leqq 1)$, $g_2(x) = 0$, $g_1(x) = \phi(x_1)\phi(x_2)$, where $\phi(x)$ is Cantor's monotonic function for which $\phi'(x) = 0$ except on a null set N, but $\phi(1)-\phi(0) = 1$. Then $\operatorname{div} g = 0$ except on the null set $E = N \times N$, so that

$$\int_{\Omega} \operatorname{div} g \, d\mu_2 = 0, \text{ but } \int_{\partial\Omega} g \cdot \nu \, d\Phi \neq 0.$$

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