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RESEARCH ARTICLE

The Ceresa class and tropical curves of hyperelliptic type

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Abstract
We define a new algebraic invariant of a graph G called the Ceresa–Zharkov class and show that it is trivial if and only if G is of hyperelliptic type, equivalently, G does not have as a minor the complete graph on four vertices or the loop of three loops. After choosing edge lengths, this class specializes to an algebraic invariant of a tropical curve with underlying graph G that is closely related to the Ceresa cycle for an algebraic curve defined over $\mathbb{C}((t))$.

Contents

1. Introduction

Given a smooth genus $g \geq 2$ algebraic curve C together with a point $p \in C$, there is a canonical null-homologous 1-cycle in its Jacobian $\mathrm{Jac}(C)$ obtained by taking the difference of the images of C under the two Abel–Jacobi maps

$$C \hookrightarrow \mathrm{Jac}(C) \quad x \mapsto [x] - [p] \quad \text{and} \quad x \mapsto [p] - [x].$$

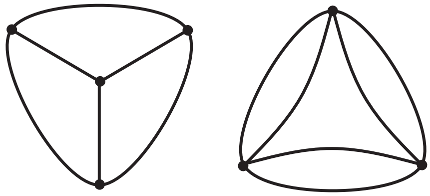

Figure 1. *The graphs K_4 (left) and L_3 (right).*

This is called the *Ceresa cycle*, and we denote it by $C - C^-$. A landmark result of Ceresa in [6] is that this cycle, for a very general curve of genus $g \geq 3$, is nontrivial in the Griffiths group of null-homologous cycles modulo algebraic equivalence. Nevertheless, it is always trivial for hyperelliptic curves, and it has long been conjectured that these are the only curves with trivial Ceresa cycle [13, Question 8.5], [4, Remark 1.2]. Although several recent results [1, 2, 4, 14] present nonhyperelliptic curves whose Ceresa cycles give rise to torsion classes in the Griffiths group, this problem remains open. The goal of this paper is to study hyperellipticity of tropical curves and graphs using cohomological invariants arising from the study of Ceresa triviality of algebraic curves defined over $\mathbb{C}((t))$.

Degeneration techniques have long been used to study complex algebraic curves, both from the topological and algebraic/arithmetic perspectives. Tropical geometry provides a systematic framework for recording the combinatorial data of a *stable* degeneration as a *tropical curve*, that is, a graph with edge lengths and (possible) vertex weights. Topologically, stable degeneration is modeled by a family $\mathcal{C} \to D$ of Riemann surfaces over a small complex disc that is holomorphic over $D \setminus x$ and the fiber over x is a stable curve. Restricting to an infinitesimal neighborhood of x, we obtain a smooth algebraic curve C over $\mathbb{C}((t))$ that has stable reduction. The tropical curve corresponding to this degeneration is the (vertex-weighted) dual graph of the special fiber and its edge lengths record the speeds to which the nodes in the special fiber are formed.

In [10], the authors define a Ceresa class in the topological and tropical contexts that agrees with the ℓ-adic Ceresa class—the image of the Ceresa cycle of a curve defined over $\mathbb{C}((t))$ under the ℓ-adic Abel-Jacobi map—after applying a suitable comparison morphism. In this paper, we define the *Ceresa–Zharkov* class $\mathbf{w}(\Gamma)$ of a tropical curve Γ; this is a particular homomorphic image of the tropical Ceresa class, see §4 for the precise formulation. Notably, nontriviality of the Ceresa–Zharkov class implies nontriviality of the tropical Ceresa class.

In an effort to emulate the notion of generic real edge lengths, we define a graph-theoretic Ceresa–Zharkov class $\mathbf{w}_\tau(G)$ (depending on a hyperelliptic quasi-involution τ of the underlying genus-g surface Σ_g, see §2.4), which lives in a module with coefficients in a polynomial ring whose variables correspond to the edges of G. Given a tropical curve Γ with underlying graph G, (a representative of) the class $\mathbf{w}(\Gamma)$ is obtained by evaluating $\mathbf{w}_\tau(G)$ at the edge lengths of Γ. We define what it means for $\mathbf{w}_\tau(G)$ to be trivial—whence we call G *Ceresa–Zharkov* trivial—so that triviality of $\mathbf{w}_\tau(G)$ implies triviality of $\mathbf{w}(\Gamma)$ for any tropical curve with underlying graph G. However, nontriviality of $\mathbf{w}_\tau(G)$ does not imply nontriviality of $\mathbf{w}(\Gamma)$ for a particular Γ with underlying graph G, rather we view this as saying that $\mathbf{w}(\Gamma)$ is *generically* nontrivial.

Our main theorem completely determines when a graph is Ceresa–Zharkov trivial, and it is closely related to hyperellipticity as we now explain. According to the tropical Torelli theorem, it is possible for nonisomorphic tropical curves to have isomorphic Jacobians as principally polarized tropical abelian varieties. A tropical curve whose Jacobian is isomorphic to that of a hyperelliptic tropical curve is said to be *of hyperelliptic type*. These tropical curves are defined in [9], where it is shown that being of hyperelliptic type depends only on the underling graph, is preserved when taking connected minors and its forbidden minors are K_4 and L_3, see Figure 1.

Theorem (Theorem 5.11). *A connected graph G of genus $g \geq 2$ is Ceresa–Zharkov trivial if and only if G is of hyperelliptic type, or equivalently, if and only if G has no K_4 or L_3 minor.*

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Our choice for the name “Ceresa–Zharkov” class comes from a related construction due to Zharkov in [17]. Independent of the tropical Ceresa class described above, Zharkov uses the tropical analogs of curves, Jacobians, the Abel-Jacobi map and algebraic equivalence, to define and study a purely tropical Ceresa cycle. The main result of his paper is that this “tropical Ceresa cycle” is not (tropically) algebraically equivalent to 0 for a generic tropical curve with a K_4 subgraph. As observed in [10, Remark 7.3], when the underlying graph of a tropical curve is K_4, Zharkov's Ceresa cycle and notion of triviality coincides with our Ceresa–Zharkov class. This suggests the following question.

Question. What is the precise relationship between the Ceresa–Zharkov class and the tropical Ceresa cycle defined by Zharkov in [17]?

Here is an outline of the paper. In §2, we recall the construction of the topological and tropical Ceresa class from [10], then define the Ceresa–Zharkov class for tropical curves. Next, in §3, we define and study the polynomial algebra in which the graph-theoretic Ceresa–Zharkov class is defined. We define the Ceresa–Zharkov class for graphs in §4, as well as study its elementary properties. Finally, we investigate the relationship between Ceresa–Zharkov triviality and being of hyperelliptic type in §5, where we prove the main theorem.

2. Background

2.1. Stable graphs, tropical curves and tropical Jacobians

Given a graph G, denote by $V(G)$ and $E(G)$ its set of vertices and edges, respectively. An edge of G is a *loop* if it is adjacent to a single vertex, and a pair of nonloop edges (f, f') are *parallel* if they join the same pair of vertices. A (integral, unweighted) *tropical curve* Γ consists of a finite connected graph G, possibly with loops and parallel edges, together with a positive integer-valued function $c : E(G) \to \mathbb{Z}$ on the edge set $E(G)$. We view G as the *underlying graph* of Γ and $c(e)$ as the *length* of the edge e.[1] The *genus* of Γ, written as $g(\Gamma)$, is the first Betti number of G, that is,

$$g(\Gamma) = |E(G)| - |V(G)| + 1.$$

A tropical curve Γ is said to be *2-connected* if G has no cut-vertices; in particular, such a graph cannot have a loop or a bridge.

Let $\Gamma = (G, c)$ be a genus $g \geq 2$ tropical curve, and fix an orientation on G. The *Jacobian* of Γ is the real g-dimensional torus

$$\mathrm{Jac}(\Gamma) = H_1(G, \mathbb{R})/H_1(G, \mathbb{Z})$$

together with the positive definite symmetric quadratic form Q_Γ on $H_1(G, \mathbb{R})$ given by

$$Q_\Gamma\left(\sum_{e\in E(G)} a_e \cdot e, \sum_{e\in E(G)} b_e \cdot e\right) = \sum_{e\in E(G)} a_e b_e \cdot c(e).$$

The valence of a vertex v is the number of half edges adjacent to it; in particular a loop edge contributes 2 to the valence. A connected graph is *stable* if every vertex has valence at least 3, and a tropical curve is stable if its underlying graph is stable. Two tropical curves are *tropically equivalent* if one can be obtained from the other by a sequence of the following moves:

- adding or removing a 1-valent vertex and its incident edge, or
- adding or removing a 2-valent vertex, preserving the underlying metric space.

Every tropical curve of genus $g \geq 2$ is tropically equivalent to a unique stable tropical curve.

[1] Tropical curves are traditionally allowed to have *real* edge lengths, but we restrict to integral edge lengths since this setting allows for a simpler definition of the tropical Ceresa class.

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A tropical curve is *hyperelliptic* if there is an involution σ of Γ such that the quotient Γ/σ (in the sense of [8, §2.2]) is a tree. A tropical curve is said to be *of hyperelliptic type* if its Jacobian is isomorphic to the Jacobian of a hyperelliptic tropical curve as a principally polarized tropical abelian variety. This notion is defined and studied in [9], where it is shown that Γ is of hyperelliptic type if and only if its underlying graph has no K_4 or L_3 minor ([9, Theorem 1.1]); see Figure 1. Moreover, if Γ is a 2-connected tropical curve of hyperelliptic type, then there is a hyperelliptic tropical curve Γ' such that the underlying graph of Γ is obtained from that of Γ' by edge contractions.

2.2. Dual graphs and multitwists from tropical curves

We assume familiarity with surface topology and refer the reader to [12] for a comprehensive treatment. Given an orientable topological real surface S, possibly with boundary or punctures, denote by $\mathrm{Mod}(S)$ its mapping class group. We write Σ_g for a closed genus-g surface and Σ_g^1 a genus g surface with one boundary component. We always identify Σ_g^1 as the subsurface of Σ_g obtained by removing a small open disc. Given a curve γ on Σ_g or Σ_g^1, write T_γ for the (left-handed) Dehn twist about γ.

Let Λ be a finite collection of pairwise disjoint isotopy classes of simple closed curves on Σ_g. Its (unweighted) *dual graph* is the graph with

- a vertex v_S for every connected component S of $\Sigma_g \setminus \bigcup_{\ell\in\Lambda} \ell$, and
- an edge e_ℓ between v_S and $v_{S'}$ for each ℓ in the boundary of S and S'. The curve ℓ is said to be *dual* to e_ℓ.

Any connected genus-g graph is the dual graph of such a configuration. We are primarily interested in the case where Λ is *Lagrangian* in the sense of [10], that is, each component S of $\Sigma_g \setminus \bigcup_{\ell\in\Lambda} \ell$ has genus 0.

Let Λ_1 and Λ_2 be two Lagrangian arrangements of curves on Σ_g as above such that their dual graphs, G_1 and G_2, respectively, are stable. Then G_1 is isomorphic to G_2 if and only if there is a mapping class of Σ_g that takes Λ_1 to Λ_2; this follows from the well-known identification of the quotient of the curve complex of Σ_g by $\mathrm{Mod}(\Sigma_g)$ with the tropical moduli space of genus g tropical curves of total edge length 1; see [7, §1].

Given a genus-g tropical curve $\Gamma = (G, c)$ and Λ an arrangement of curves on Σ_g whose dual graph is G, define the multitwist

$$T_\Gamma = \prod_{e\in E(G)} T_{\ell_e}^{c(e)}.$$

(Recall that we only consider tropical curves with integer edge lengths.) By the previous paragraph and the fact that $\sigma T_\ell \sigma^{-1} = T_{\sigma(\ell)}$ for any $\sigma \in \mathrm{Mod}(\Sigma_g)$, the mapping class T_Γ is well defined up to conjugation in $\mathrm{Mod}(\Sigma_g)$. Furthermore, $T_\Gamma = T_{\Gamma'}$ where Γ' is the unique stable tropical curve tropically equivalent to Γ.

2.3. The Johnson homomorphism

Let $\mathcal{I}_g^1 \leq \mathrm{Mod}(\Sigma_g^1)$, resp. $\mathcal{I}_g \leq \mathrm{Mod}(\Sigma_g)$, denote the Torelli group. Set

$$H = H_1(\Sigma_g^1, \mathbb{Z}) \cong H_1(\Sigma_g, \mathbb{Z}), \qquad \text{and} \qquad L = \wedge^3 H.$$

The intersection product on H induces a 2-form $\omega \in \wedge^2 H$, and taking the exterior product with ω yields an injection $H \hookrightarrow L$. The *Johnson homomorphism* $J : \mathcal{I}_g^1 \to L$, resp. $J : \mathcal{I}_g \to L/H$, may be characterized in the following way. By [16, Theorem 2], the Torelli group is generated by separating twists (Dehn twists about separating curves) and bounding pair maps (a product $T_\gamma T_{\gamma'}^{-1}$ where γ, γ' form a separating pair). If γ is a separating curve on Σ_g^1, then $J(T_\gamma) = 0$. Now, suppose that γ, γ' are a separating pair. The removal of $\gamma \cup \gamma'$ from Σ_g^1 separates Σ_g^1 into two surfaces S and S'; let S be

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the subsurface not containing the boundary component of Σ_g^1. The form ω restricts to the intersection 2-form on S; denote this by ω_S. Then

$$J(T_\gamma T_{\gamma'}^{-1}) = \omega_S \wedge [\gamma],$$

where $[\gamma]$ is oriented so that S appears on its right. The Johnson homomorphism on $\mathcal{I}_g$ is defined in a similar way, except that one can choose either surface S or S' in the above formula; the two possible expressions are equivalent modulo H.

2.4. The tropical Ceresa class

Next, we briefly recall the definition of the Ceresa class of Γ as was described in the Introduction; see [10] for more details. Fix a hyperelliptic quasi-involution τ, that is, a mapping class in $\mathrm{Mod}(\Sigma_g)$ that acts as $-I$ on H. Consider the map

$$\nu_\tau : \mathrm{Mod}(\Sigma_g) \to L/H \qquad \gamma \mapsto J([\gamma, \tau]),$$

where $[\gamma, \tau] = \gamma\tau\gamma^{-1}\tau^{-1}$ is the commutator. This is a 1-cocycle, and its class in $H^1(\mathrm{Mod}(\Sigma_g), L/H)$ is independent of the choice of τ [*Ibid.*, Proposition 2.1]. The *τ-Ceresa cocycle* of Γ, denoted by $\nu_\tau(\Gamma)$, is the restriction of ν_τ to $\langle T_\Gamma\rangle \cong \mathbb{Z}$. Similarly, the *Ceresa class* of Γ, denoted by $\nu(\Gamma)$, is the class of $\nu_\tau(\Gamma)$ in $H^1(\mathbb{Z}, L/H)$. Next, we will discuss several groups related to a filtration on L in which the Ceresa class lives and our computation will take place.

A subgroup Y of H is *Lagrangian* if $Y_\mathbb{Q}$ is a maximal isotropic subspace of the symplectic vector space $H_\mathbb{Q}$ (the symplectic form is induced by the intersection pairing on H) and H/Y is torsion-free. Let $Y \le H$ be the subgroup generated by the homology classes $\{[\ell] : \ell \in \Lambda\}$. This is a Lagrangian subgroup; see [*Ibid.*, §6.1]. For values $q = 0, 1, 2, 3$, define the following filtrations on L and L/H:

$$F_q L = (\wedge^q Y) \wedge (\wedge^{3-q} H), \qquad F_q(L/H) = \frac{F_q L + H}{H}, \qquad \mathrm{gr}_q^F(L/H) = \frac{F_q(L/H)}{F_{q-1}(L/H)}.$$

These filtrations and groups associated with them are discussed in detail in [*Ibid.*, §5].

Since $H \subset F_1 L$ and $F_3 L \cap H = \{0\}$, we have

$$F_0(L/H) = \frac{L}{H}, \quad F_1(L/H) = \frac{F_1 L}{H}, \quad F_2(L/H) = \frac{F_2 L + H}{H}, \quad F_3(L/H) \cong \frac{F_3 L}{F_3 L \cap H} \cong F_3 L. \tag{2.1}$$

Denote by $\delta_\Gamma : H \to H$ the pushforward map on H induced by T_Γ. Recall from [*Ibid.*, Lemma 5.1] that $\delta_\Gamma(F_q L) \subset F_{q+1} L$ and δ_Γ preserves ω. Then the map $\delta_\Gamma - I$ restricts to the graded components

$$\delta_\Gamma - I : \mathrm{gr}_q^F(L/H) \to \mathrm{gr}_{q+1}^F(L/H).$$

Define groups

$$A(\delta_\Gamma) = \mathrm{im}(H^1(\langle\delta_\Gamma\rangle, F_2(L/H)) \to H^1(\langle\delta_\Gamma\rangle, L/H)) \cong \frac{F_2 L + H}{(\delta_\Gamma - I)(F_1 L) + H}$$
$$B(\delta_\Gamma) = \mathrm{coker}(\delta_\Gamma - I : \mathrm{gr}_1^F(L/H) \to \mathrm{gr}_2^F(L/H)) \cong \frac{F_2 L + H}{(\delta_\Gamma - I)(F_1 L) + F_3 L + H}$$
$$C(\delta_\Gamma) = \mathrm{coker}\Big((\delta_\Gamma - I)^2 : \mathrm{gr}_1^F(L/H) \to \mathrm{gr}_3^F(L/H)\Big) \cong \frac{F_3 L}{(\delta_\Gamma - I)^2(F_1 L)}.$$

The isomorphisms for $A(\delta_\Gamma)$ and $B(\delta_\Gamma)$ follow from [*Ibid.*, Proposition 5.4]. These three groups are *finite* since $(\delta_\Gamma - I) : \mathrm{gr}_q^F(L/H) \to \mathrm{gr}_{q+1}^F(L/H)$ is rationally surjective for $q = 1, 2$ [*Ibid.*]. The identity

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map on $F_2L + H$ induces a homomorphism $A(\delta_\Gamma) \to B(\delta_\Gamma)$ and we have an induced homomorphism $(\delta_\Gamma - I) : B(\delta_\Gamma) \to C(\delta_\Gamma)$. By [*Ibid.*, Theorem 6.6], there is a hyperelliptic quasi-involution τ such that $\nu_\tau(\Gamma)$ lies in $F_2(L/H)$. In practice, whenever we compute the tropical Ceresa class, we always use such a τ. In particular, the Ceresa class $\nu(\Gamma)$ lies in $A(\delta_\Gamma) \subset H^1(\langle\delta_\Gamma\rangle, L/H)$. This implies the main result of [*Ibid.*], that $\nu(\Gamma)$ is torsion. Now, we fix the following notation

$$\mathbf{v}(\Gamma) = \text{image of } \nu(\Gamma) \text{ in } B(\delta_\Gamma) \qquad \mathbf{w}(\Gamma) = (\delta_\Gamma - I)(\mathbf{v}(\Gamma)) \in C(\delta_\Gamma).$$

If τ satisfies $\nu_\tau(\Gamma) \in F_2(L/H)$, by [*Ibid.*, Proposition 6.7], the class $\mathbf{v}(\Gamma)$ has a particularly nice representative

$$\mathbf{v}_\tau(\Gamma) = \sum_{e \in E(G)} c(e)\, J([T_{\ell_e}, \tau]). \tag{2.2}$$

The tropical curve Γ is *Ceresa trivial* if $\nu(\Gamma) = 0$. If Γ is Ceresa trivial, then $\mathbf{v}(\Gamma) = 0$. Given the above formula, $\mathbf{v}(\Gamma)$ is easier to compute than $\nu(\Gamma)$, and therefore a good strategy to show that Γ is Ceresa nontrivial is to show that $\mathbf{v}(\Gamma) \neq 0$ in $B(\delta_\Gamma)$.

3. The action of a multitwist on some polynomial algebras

3.1. Multitwist on the homology of a surface

Throughout, given $\mathbb{Z}$-modules A and R, we set $A_R = A \otimes_{\mathbb{Z}} R$. Let Λ be a collection of pairwise disjoint isotopy classes of simple closed curves on Σ_g^1 or Σ_g. Define a polynomial ring

$$R[\Lambda] = \mathbb{Z}[x_\ell \ : \ \ell \in \Lambda]$$

which we denote by R when Λ is clear from the context. Denote by $\langle a, b\rangle$ the intersection product of homology classes $a, b \in H = H_1(\Sigma_g, \mathbb{Z})$. For any loop $\ell \in \Lambda$, let $\delta_\ell : H_R \to H_R$ be the homomorphism defined on simple tensors by

$$\delta_\ell(h \otimes a) = h \otimes a + \langle h, [\ell]\rangle [\ell] \otimes x_\ell a. \tag{3.1}$$

The map δ_ℓ is an isomorphism with inverse

$$\delta_\ell^{-1}(h \otimes a) = h \otimes a - \langle h, [\ell]\rangle [\ell] \otimes x_\ell a. \tag{3.2}$$

Because the loops in Λ are pairwise disjoint, the δ_ℓ's pairwise commute. The homomorphism δ_ℓ is related to the pushforward $T_\ell)_* : H \to H$ in the following way. Given a function $c : \Lambda \to \mathbb{Z}$, let $R \to \mathbb{Z}$ be the evaluation ring homomorphism $x_\ell \mapsto c(\ell)$. Then, under the identification $H \cong H \otimes_{\mathbb{Z}} R \otimes_R \mathbb{Z}$, the map $\delta_\ell \otimes_R \mathrm{id}_{\mathbb{Z}}$ is the pushforward of $T_\ell^{c(\ell)}$ on H by [12, Proposition 6.3].

Proposition 3.1. *Given integers* $\{a_\ell \ : \ \ell \in \Lambda\}$, *we have*

$$\left(\prod_{\ell \in \Lambda} \delta_\ell^{a_\ell}\right) - I = \sum_{\ell \in \Lambda} a_\ell(\delta_\ell - I) \quad \textit{as maps} \quad H_R \to H_R.$$

Proof. It suffices to show

$$\begin{aligned} &\delta_\ell\delta_{\ell'} - I = (\delta_\ell - I) + (\delta_{\ell'} - I), \text{ and} \\ &\delta_\ell^{-1} - I = -(\delta_\ell - I). \end{aligned}$$

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The second formula follows readily from formulas (3.1) and (3.2). For the first formula, we have

$$\begin{aligned}(\delta_\ell\delta_{\ell'} - I)(h\otimes a) &= \delta_\ell(h\otimes a + \langle h,[\ell']\rangle[\ell']\otimes x_{\ell'}a) - h\otimes a\\ &= (\delta_\ell - I)(h\otimes a) + \langle h,[\ell']\rangle\delta_\ell([\ell']\otimes x_{\ell'}a)\\ &= (\delta_\ell - I)(h\otimes a) + \langle h,[\ell']\rangle([\ell']\otimes x_{\ell'}a + \langle[\ell'],[\ell]\rangle[\ell]\otimes x_\ell x_{\ell'}a)\\ &= (\delta_\ell - I)(h\otimes a) + \langle h,[\ell']\rangle[\ell']\otimes x_{\ell'}a\\ &= (\delta_\ell - I)(h\otimes a) + (\delta_{\ell'} - I)(h\otimes a).\end{aligned}$$

□

Next, define $\mathcal{B} = \langle\delta_\ell : \ell\in\Lambda\rangle$, which is a free abelian subgroup of $\mathrm{Aut}(H_R)$ whose rank is equal to the number of nonseparating loops in Λ. Recall that $Y = \mathrm{span}_{\mathbb{Z}}\{[\ell] : \ell\in\Lambda\}$, which is a Lagrangian subspace of H. The following is a straight-forward consequence of Equation (3.1) and Proposition 3.1.

Proposition 3.2. *For any $f\in\mathcal{B}$, we have*

$$(1)\ \ (f-I)(H_R)\subset Y_R, \qquad (2)\ \ (f-I)(Y_R)=0, \qquad (3)\ \ (f-I)^2(H_R)=0.$$

Denote by $\delta_\Lambda : H_R\to H_R$ the map

$$\delta_\Lambda = \prod_{\ell\in\Lambda}\delta_\ell.$$

By Proposition 3.1, as a map $H_R\to H_R$, we have

$$\delta_\Lambda - I = \sum_{\ell\in\Lambda}(\delta_\ell - I).$$

3.2. The action of δ_Λ on L_R

The third exterior power of δ_Λ is a R-module homomorphism $\wedge^3(H_R)\to\wedge^3(H_R)$. However, the natural R-module homomorphism

$$\wedge^3(H_R)\to L_R \qquad (h_1\otimes f_1)\wedge(h_2\otimes f_2)\wedge(h_3\otimes f_3)\mapsto(h_1\wedge h_2\wedge h_3)\otimes(f_1f_2f_3).$$

is an isomorphism; thus we may view the third exterior power of δ_Λ as an R-module endomorphism of L_R which we still denote by δ_Λ. Next, we show that δ_Λ is an endomorphism of L_R/H_R, but we need the following more general setup.

Let $\iota : S\hookrightarrow\Sigma_g$ be a subsurface (possibly with boundary) of Σ_g. The homology of S splits as $H_1(S,\mathbb{Z})\cong V\oplus W$, where V is the subgroup of H generated by the boundary curves of S and W is a symplectic subspace of H whose symplectic form, which we denote by $\omega_S\in\wedge^2W$, is obtained by restricting the intersection of Σ_g to S. Let $\mathcal{B}_S$ be the subgroup of $\mathcal{B}$ generated by the δ_ℓ such that ℓ is isotopic to a curve in S.

Proposition 3.3. *If $f\in\mathcal{B}_S$, then*

1. $f(\omega_S)\equiv\omega_S \bmod (V\wedge W)_R$;
2. $(f-I)(h\wedge\omega_S) = (f-I)(h)\wedge\omega_S + f(h)\wedge\eta$ *for some* $\eta\in(V\wedge W)_R$.

Remark 3.4. As a consequence, if $S=\Sigma_g$ and $f\in\mathcal{B}$, then $(f-I)$, as an endomorphism $L_R\to L_R$, takes H_R to H_R (as an R-submodule of L_R). In particular, $\delta_\Lambda - I$ may be regarded as an R-module endomorphism of L_R/H_R.

Proof of Proposition 3.3. For (1), it suffices to show that

$$\delta_\ell(\omega_S)\equiv\omega_S \mod (V\wedge W)_R. \tag{3.3}$$

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Choose a symplectic basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of H such that $\alpha_1, \ldots, \alpha_h, \beta_1, \ldots, \beta_h$ is a symplectic basis of W. This means that

$$\omega_S = \sum_{i=1}^{h} \alpha_i \wedge \beta_i.$$

Write $[\ell] = \mu + \lambda$, where $\mu \in V$ and $\lambda \in W$. We have

$$\lambda = \sum_{i=1}^{h} (-\langle \beta_i, [\ell] \rangle \alpha_i + \langle \alpha_i, [\ell] \rangle \beta_i).$$

We compute

$$\begin{aligned} \delta_\ell(\omega_S) &= \sum_{i=1}^{h} (\alpha_i + \langle \alpha_i, [\ell] \rangle [\ell] \otimes x_\ell) \wedge (\beta_i + \langle \beta_i, [\ell] \rangle [\ell] \otimes x_\ell) \\ &= \omega_S + \sum_{i=1}^{h} (\langle \beta_i, [\ell] \rangle \alpha_i \wedge [\ell] + \langle \alpha_i, [\ell] \rangle [\ell] \wedge \beta_i) \otimes x_\ell \\ &= \omega_S + [\ell] \wedge \sum_{i=1}^{h} (-\langle \beta_i, [\ell] \rangle \alpha_i + \langle \alpha_i, [\ell] \rangle \beta_i) \otimes x_\ell \\ &= \omega_S + \mu \wedge \lambda \otimes x_\ell, \end{aligned}$$

from which formula (3.3) follows. For (2), we have

$$(f - I)(h \wedge \omega_S) = f(h) \wedge f(\omega_S) - h \wedge \omega_S = (f - I)(h) \wedge \omega_S + f(h) \wedge \eta$$

for some $\eta \in (V \wedge W)_R$. □

Similar to the integral setup in Section 2.4, we define a filtration on L_R and L_R/H_R:

$$F_q L_R := (F_q L)_R \cong (\wedge^q Y_R) \wedge (\wedge^{3-q} H_R) \qquad \text{and} \qquad F_q(L_R/H_R) := (F_q(L/H))_R$$

and denote by $\mathrm{gr}_q^F(L_R/H_R)$ the graded piece

$$\mathrm{gr}_q^F(L_R/H_R) := \frac{F_q(L_R/H_R)}{F_{q+1}(L_R/H_R)}.$$

Proposition 3.5. *Given $f \in \mathcal{B}$ and $h_1, h_2, h_3 \in H$, we have that*

$$\begin{aligned} (f - I)(h_1 \wedge h_2 \wedge h_3) \;=\; & (f - I)h_1 \wedge h_2 \wedge h_3 + h_1 \wedge (f - I)h_2 \wedge h_3 + h_1 \wedge h_2 \wedge (f - I)h_3 \\ & + (f - I)h_1 \wedge (f - I)h_2 \wedge h_3 + (f - I)h_1 \wedge h_2 \wedge (f - I)h_3 \\ & + h_1 \wedge (f - I)h_2 \wedge (f - I)h_3 \\ & + (f - I)h_1 \wedge (f - I)h_2 \wedge (f - I)h_3. \end{aligned}$$

In particular, we have that $(f - I)(F_q(L_R)) \subset F_{q+1}(L_R)$.

As a consequence of this proposition, the map $\delta_\wedge - I$ on L_R/H_R takes $F_q(L_R/H_R)$ to $F_{q+1}(L_R/H_R)$ and hence induces a map on graded components:

$$\delta_\wedge - I : \mathrm{gr}_q^F(L_R/H_R) \to \mathrm{gr}_{q+1}^F(L_R/H_R).$$

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Proposition 3.6. *As maps* $\mathrm{gr}^F_q(L_R/H_R) \to \mathrm{gr}^F_{q+1}(L_R/H_R)$*, we have*

$$(\delta_\Lambda - I)(\alpha) = \sum_{\ell\in\Lambda}(\delta_\ell - I)(\alpha).$$

Proof. By Proposition 3.5, we have that

$$(\delta_\Lambda - I)(h_1\wedge h_2\wedge h_3) \equiv (\delta_\Lambda - I)h_1\wedge h_2\wedge h_3 + h_1\wedge(\delta_\Lambda - I)h_2\wedge h_3 + h_1\wedge h_2\wedge(\delta_\Lambda - I)h_3 \mod F_q(L_R/H_R).$$

Now, apply Proposition 3.1. □

In summary, the maps $\delta_\Lambda - I$ and $\sum_{\ell\in\Lambda}(\delta_\ell - I)$ coincide when viewed as maps on H_R or the graded components $\mathrm{gr}^F_q(L_R/H_R)$ but are different when viewed on the whole L_R/H_R. This difference is important in the proof of Proposition 4.2.

3.3. *Configuration of loops from graphs*

Let G be a connected genus-g stable graph and $\Lambda = \{\ell_e : e \in E(G)\}$ a collection of pairwise disjoint isotopy classes of simple closed curves on Σ_g whose dual graph is G, as in §2.2. The polynomial ring

$$R[G] = \mathbb{Z}[x_e : e \in E(G)]$$

is identified with $R[\Lambda]$ by the relabeling $x_e = x_{\ell_e}$. We simply write R when the graph G is clear from context. Define the endomorphism δ_G on H_R (respectively, L_R and L_R/H_R) by $\delta_G = \delta_\Lambda$. The map δ_G is closely related to the polarization of the Jacobian of a tropical curve, as we see in the following construction.

Define the symmetric quadratic form Q_G on $H_1(G,\mathbb{Z})\otimes R$

$$Q_G\left(\sum_{e\in E(G)} a_e\cdot e, \sum_{e\in E(G)} b_e\cdot e\right) = \sum_{e\in E(G)} a_e b_e \otimes x_e.$$

If $\Gamma = (G,c)$ is a tropical curve, then Q_Γ is obtained from Q_G by the substitution $x_e \mapsto c(e)$. Let us give an explicit description of Q_G. Enumerate the edges of G by $e_1,\dots,e_n$ so that $e_{g+1},\dots,e_n$ are the edges of a spanning tree T. Fix an arbitrary orientation on G. For $j = 1,\dots,g$, the graph $T\cup\{e_j\}$ contains a unique cycle γ_j; orient γ_j in the direction of e_j. The cycles $\gamma_1,\dots,\gamma_g$ form a basis of $H_1(G,\mathbb{Z})$. Let $x_i = x_{e_i}$. The (i,j)–entry of the matrix of Q_G with respect to this basis is a linear form in the x_k's: x_k appears with coefficient 1 if γ_i and γ_j traverse e_k in the same direction, -1 if they traverse in the opposite direction and 0 if they do not meet at e_k. For details and examples, see [5, §4].

Now, we give the promised comparison to δ_G. Let $\ell_k = \ell_{e_k}$. Orient ℓ_k so that the subsurface corresponding to the target of e_k lies to the left of ℓ_k. Viewing G as a one-dimensional CW-complex, there is an embedding $\iota : G \hookrightarrow \Sigma_g$ so that $\iota(e_j)\cap\ell_j$ is a point. For $j = 1,\dots,g$, denote by α_j and β_j the homology classes $-[\iota(\gamma_j)]$ and $[\ell_j]$, respectively; the orientations are chosen so that the signed intersection number of α_i with β_i is 1. Then $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$ is a symplectic basis for H.

Since the $\ell\in\Lambda$ are pairwise nonintersecting, we have that $\delta_G(\beta_j) = \beta_j$. For the α_j's, we have

$$\delta_G(\alpha_j) = \alpha_j + \sum_{k=1}^n \langle\alpha_j,[\ell_k]\rangle[\ell_k]\otimes x_k.$$

The β_i–coefficient of $\delta_G(\alpha_j)$ is

$$\langle\alpha_i,\delta_G(\alpha_j)\rangle = \sum_{k=1}^n \langle\alpha_i,[\ell_k]\rangle\langle\alpha_j,[\ell_k]\rangle\otimes x_k.$$

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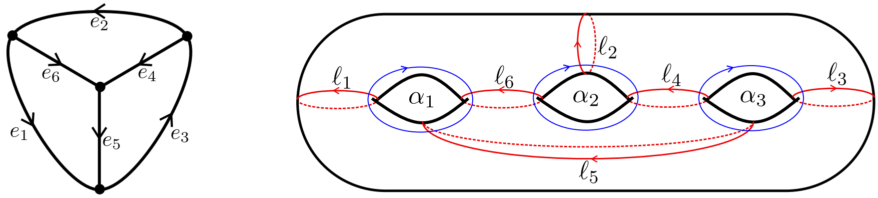

Figure 2. *An arrangement of curves on Σ_3 with dual graph K_4.*

This is a linear form in x_k: x_k appears with coefficient 1 if α_i and α_j have the same intersection pairing with ℓ_k, -1 if they have opposite intersection pairing and 0 otherwise. Thus, the matrix of $\delta_G : H_R \to H_R$ with respect to the ordered basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ is given by

$$\delta_G = \begin{pmatrix} I & 0 \\ Q_G & I \end{pmatrix}. \tag{3.4}$$

This allows us to view Q_G as a map $Y_R^\perp \to Y_R$, where

$$Y = \mathrm{span}_{\mathbb{Z}}\{[\ell] \,:\, \ell \in \Lambda\} = \mathrm{span}_{\mathbb{Z}}\{\beta_1, \ldots, \beta_g\}, \quad \text{and} \quad Y^\perp = \mathrm{span}_{\mathbb{Z}}\{\alpha_1, \ldots, \alpha_g\}.$$

Example 3.7. Consider $G = K_4$, the complete graph on 4 vertices, with the orientation as in Figure 2. To its right is an arrangement of curves on Σ_3 whose dual graph is K_4. Observe that each ℓ_i is oriented so that the subsurface corresponding to the target of e_i lies to the left of ℓ_i. Also, observe that α_i is oriented so that the signed intersection product of α_i and $\beta_i := [\ell_i]$ is 1. The matrix Q_G is

$$Q_G = \begin{pmatrix} x_1 + x_5 + x_6 & -x_6 & -x_5 \\ -x_6 & x_2 + x_4 + x_6 & -x_4 \\ -x_5 & -x_4 & x_3 + x_4 + x_5 \end{pmatrix}. \tag{3.5}$$

Combining the above connection between δ_G and Q_G with Proposition 3.5, we deduce the following identities:

$$\begin{aligned} (\delta_G - I)(\alpha_i \wedge \alpha_j \wedge \beta_k) &= Q_G(\alpha_i) \wedge \alpha_j \wedge \beta_k + \alpha_i \wedge Q_G(\alpha_j) \wedge \beta_k + Q_G(\alpha_i) \wedge Q_G(\alpha_j) \wedge \beta_k, \\ (\delta_G - I)(\alpha_i \wedge \beta_j \wedge \beta_k) &= Q_G(\alpha_i) \wedge \beta_j \wedge \beta_k, \\ (\delta_G - I)^2(\alpha_i \wedge \alpha_j \wedge \beta_k) &= 2\, Q_G(\alpha_i) \wedge Q_G(\alpha_j) \wedge \beta_k. \end{aligned} \tag{3.6}$$

Remark 3.8. Denote by $R_d \subset R$ the subspace of homogeneous degree d-forms in R. Because the entries of Q_G are linear forms in R, the map $\delta_G - I$ may be viewed as a map $H_{R_d} \to H_{R_{d+1}}$.

4. The Ceresa–Zharkov class

4.1. Ceresa–Zharkov triviality

Recall that we view Σ_g^1 as a subsurface of Σ_g obtained by removing a small open disc D from Σ_g. Recall that a hyperelliptic quasi-involution is an element of $\mathrm{Mod}(\Sigma_g^1)$ or $\mathrm{Mod}(\Sigma_g)$ whose action on $H = H_1(\Sigma_g^1, \mathbb{Z}) \cong H_1(\Sigma_g, \mathbb{Z})$ is $-I$. Let τ' be a hyperelliptic quasi-involution on Σ_g^1 and τ its image under the natural map $\mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_g)$. Let Λ' be a finite collection of pairwise disjoint isotopy classes of simple closed curves on Σ_g^1 and Λ the image of this collection under the inclusion $\Sigma_g^1 \hookrightarrow \Sigma_g$; note that any finite collection of simple closed curves can be isotoped away from the disc D, so Λ may represent any finite collection of isotopy classes of simple closed curves on Σ_g.

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The τ'-*Ceresa cocycle* of Λ', respectively the τ-*Ceresa cocycle* of Λ, is defined as

$$\boldsymbol{\mu}_{\tau'}(\Lambda') = \sum_{\ell \in \Lambda'} J([T_\ell, \tau']) \otimes x_\ell \in L_R, \quad \text{respectively,} \quad \mathbf{v}_\tau(\Lambda) = \sum_{\ell \in \Lambda} J([T_\ell, \tau]) \otimes x_\ell \in L_R/H_R,$$

where J is the Johnson homomorphism; see §2.3. Define the τ-*Ceresa–Zharkov cocycle* as

$$\mathbf{w}_\tau(\Lambda) = (\delta_\Lambda - I)(\mathbf{v}_\tau(\Lambda)).$$

When G is a genus-g graph, we write $\mathbf{v}_\tau(G) = \mathbf{v}_\tau(\Lambda)$ and $\mathbf{w}_\tau(G) = \mathbf{w}_\tau(\Lambda)$, where Λ is an arrangement of curves whose dual graph is G. We say that G is *Ceresa–Zharkov trivial* if there is a τ such that $\mathbf{w}_\tau(G) = 0$. In Lemma 4.3, we show that this notion is a well-defined property of the graph G. While our primary interest is in $\mathbf{w}_\tau(G)$, we occasionally need the class $\boldsymbol{\mu}_{\tau'}(\Lambda')$ in §5. Compare these to the definitions of the Ceresa–Zharkov class and Ceresa–Zharkov triviality of a tropical curve defined in §2.4. In practice, when we compute the tropical Ceresa class for a tropical curve Γ, we use a hyperelliptic involution τ such that $\nu_\tau(\Gamma)$ lies in $F_2(L/H)$ as in §2.4. We do something similar in the graph-theoretic case. As before, $R_d \subset R$ denotes the linear subspace of degree d forms.

Proposition 4.1. *There exists a hyperelliptic quasi-involution τ' of Σ_g^1 such that*

$$\boldsymbol{\mu}_{\tau'}(\Lambda') \in F_2 L \otimes R_1.$$

In particular, $\mathbf{v}_\tau(G)$ lies in $F_2(L/H) \otimes R_1$ and $\mathbf{w}_\tau(G)$ lies in $F_3 L \otimes R_2$, where τ is the image of τ' under $\mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_g)$.

Proof. By [10, Theorem 6.6], there is a hyperelliptic quasi-involution τ' such that each $J([T_\ell, \tau'])$ lies in $F_2 L$. The proposition now follows from Remark 3.8. □

Here is a characterization for Ceresa–Zharkov triviality when we use such a τ.

Proposition 4.2. *If G is Ceresa–Zharkov trivial and $\mathbf{v}_\tau(G) \in F_2(L/H) \otimes R_1$, then*

$$\mathbf{w}_\tau(G) \in (\delta_G - I)^2 (F_1(L/H)). \tag{4.1}$$

Conversely, if there exists a τ such that Equation (4.1) holds, then G is Ceresa–Zharkov trivial.

To prove this proposition, we first derive a characterization of Ceresa–Zharkov triviality that works for *any* τ but is more cumbersome to state due to the fact that the endomorphism $\delta_G - I$ of L_R/H_R does not split as the sum $\sum_{e \in E(G)} (\delta_e - I)$ as discussed in §3.2.

Denote by $\psi_G : L_R/H_R \to L_R/H_R$ the composition

$$\psi_G = (\delta_G - I) \circ \left(\sum_{e \in E(G)} (\delta_e - 1) \right).$$

Lemma 4.3. *The following are equivalent.*

1. *The graph G is Ceresa–Zharkov trivial;*
2. *for every hyperelliptic quasi-involution τ, we have $\mathbf{w}_\tau(G) \in \psi_G(L/H)$;*
3. *there is a hyperelliptic quasi-involution τ such that $\mathbf{w}_\tau(G) \in \psi_G(L/H)$.*

Proof. Given hyperelliptic quasi-involutions $\tau, \tilde{\tau}$, we have

$$(J([T_{\ell_e}, \tilde{\tau}]) - J([T_{\ell_e}, \tau])) \otimes x_e = (T_{\ell_e} - I)_* (J(\tilde{\tau}^{-1}\tau)) \otimes x_e = (\delta_e - I)(J(\tilde{\tau}^{-1}\tau))$$

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and therefore

$$\mathbf{v}_{\tilde{\tau}}(G) - \mathbf{v}_{\tau}(G) = \sum_{e \in E(G)} (\delta_e - I)(J(\tilde{\tau}^{-1}\tau)). \tag{4.2}$$

Suppose G is Ceresa–Zharkov trivial, say $\mathbf{w}_\tau(G) = 0$ for some τ. Applying $\delta_G - I$ to formula (4.2), we see that (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is trivial. Now, suppose (3) is true, say $\mathbf{w}_\tau(G) = \psi_G(\mathbf{u})$ for some $\mathbf{u} \in L/H$. By surjectivity of the Johnson homomorphism, there is a $t \in \mathcal{I}_g$ such that $J(t) = \mathbf{u}$. The function $\tilde{\tau} = \tau t$ is a hyperelliptic quasi-involution, and by the above equation, we have

$$\mathbf{v}_{\tilde{\tau}}(G) = \mathbf{v}_\tau(G) - \sum_{e \in E(G)} (\delta_e - I)(\mathbf{u}).$$

Applying $\delta_G - I$, we get

$$\mathbf{w}_{\tilde{\tau}}(G) = \mathbf{w}_\tau(G) - \psi_G(\mathbf{u}) = 0,$$

as required. □

Proof of Proposition 4.2. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be the symplectic basis of H from §3.3. We first record the image of the simple wedges under the map ψ_G using Proposition 3.5:

$$\begin{aligned}
\psi_G(\alpha_i \wedge \alpha_j \wedge \alpha_k) &= 2(Q_G\alpha_i \wedge Q_G\alpha_j \wedge \alpha_k + Q_G\alpha_i \wedge \alpha_j \wedge Q_G\alpha_k + \alpha_i \wedge Q_G\alpha_j \wedge Q_G\alpha_k) \\
&\quad + 3\, Q_G\alpha_i \wedge Q_G\alpha_j \wedge Q_G\alpha_k \\
\psi_G(\alpha_i \wedge \alpha_j \wedge \beta_k) &= 2\, Q_G\alpha_i \wedge Q_G\alpha_j \wedge \beta_k \\
\psi_G(\alpha_i \wedge \beta_j \wedge \beta_k) &= \psi_G(\beta_i \wedge \beta_j \wedge \beta_k) = 0.
\end{aligned}$$

By this computation, formula (3.6) and the fact that $(\delta_G - I)^2|_{F_2(L/H)} = 0$, we deduce that $(\delta_G - I)^2|_{F_1(L/H)} = \psi_G|_{F_1(L/H)}$. Therefore,

$$(\delta_G - I)^2(F_1(L/H)) = \psi_G(F_1(L/H)) = \psi_G(L/H) \cap (F_3(L/H) \otimes R_2). \tag{4.3}$$

Suppose G is Ceresa–Zharkov trivial and that $\mathbf{v}_\tau(G) \in F_2(L/H) \otimes R_1$. Then the class $\mathbf{w}_\tau(G)$ lies in $\psi_G(L/H) \cap (F_3(L/H) \otimes R_2)$ by Lemma 4.3. So formula (4.1) follows from Equation (4.3), as required. Conversely, if there exists a τ such that formula (4.1) holds, then G is Ceresa–Zharkov trivial by Lemma 4.3 and formula (4.3). □

4.2. Edge-contraction

In this section, we prove that Ceresa–Zharkov triviality is preserved under edge contraction. We prove in §5 that Ceresa–Zharkov triviality is a minor-closed property.

Suppose G is a connected graph, $f \in E(G)$ is a nonloop edge and G/f the graph obtained from G by contracting f. Denote by $\zeta_f : R[G] \to R[G/f]$ the map that evaluates x_f to 0; this is a retract of the natural inclusion $R[G/f] \subset R[G]$.

Proposition 4.4. *With G and f as above, we have an equality of maps* $(L/H)_{R[G]} \to (L/H)_{R[G/f]}$*:*

$$(1 \otimes \zeta_f) \circ (\delta_G - I) = (\delta_{G/f} - I) \circ (1 \otimes \zeta_f). \tag{4.4}$$

Also,

$$(1 \otimes \zeta_f)(\mathbf{v}_\tau(G)) = \mathbf{v}_\tau(G/f).$$

In particular, if G is Ceresa–Zharkov trivial, then so is G/f.

Note that G and G/f have the same genus g and both can be embedded into Σ_g as described in §3.3. So the same $\tau \in \mathrm{Mod}(\Sigma_g)$ gives Ceresa cocycles of G and G/f.

Proof. First, consider the maps $1 \otimes \zeta_f$ and $\delta_e - I$ on H_R. We have

$$(1 \otimes \zeta_f)(\delta_e - I) = (\delta_e - I)(1 \otimes \zeta_f) \quad \text{when } e \neq f, \quad \text{and } (1 \otimes \zeta_f)(\delta_f - I) = 0.$$

As a map $H_R \to H_R$, by Proposition 3.1, we have

$$(1 \otimes \zeta_f)(\delta_G - I) = (1 \otimes \zeta_f) \sum_{e \in E(G)} (\delta_e - I) = \sum_{e \in E(G/f)} (\delta_e - I)(1 \otimes \zeta_f) = (\delta_{G/f} - I)(1 \otimes \zeta_f).$$

Thus, the identity in formula (4.4) holds as maps $H_{R[G]} \to H_{R[G/f]}$, and one readily extends this as an identity of morphisms $(L/H)_{R[G]} \to (L/H)_{R[G\backslash f]}$ using Proposition 3.5. Next, consider what happens for the Ceresa class. We have

$$(1 \otimes \zeta_f)(\mathbf{v}_\tau(G)) = (1 \otimes \zeta_f) \sum_{e \in E(G)} J([T_{\ell_e}, \tau]) \otimes x_e = \sum_{e \in E(G/f)} J([T_{\ell_e}, \tau]) \otimes x_e = \mathbf{v}_\tau(G/f).$$

Finally, suppose G is Ceresa–Zharkov trivial, say $\mathbf{w}_\tau(G) = 0$ for some τ. Then,

$$(\delta_{G/f} - I)(\mathbf{v}_\tau(G/f)) = (\delta_{G/f} - I) \circ (1 \otimes \zeta_f)(\mathbf{v}_\tau(G)) = (1 \otimes \zeta_f) \circ (\delta_G - I)(\mathbf{v}_\tau(G)) = 0,$$

and therefore G/f is Ceresa–Zharkov trivial. □

4.3. Tropical equivalence

Similar to the setting of tropical curves, two graphs are *tropically equivalent* if one can be obtained from the other via the following moves:

- adding or removing a 1-valent vertex together with its adjacent edge,
- subdividing a edge,
- contracting exactly one edge adjacent to a 2-valent vertex.

In this section, we show Ceresa–Zharkov triviality is preserved under all these moves and thus is a property of tropical equivalence classes. This fact is analogous to [10, Lemma 4.4] which asserts that tropically equivalent tropical curves have the same Ceresa class.

Lemma 4.5. *Let G be a connected graph and e a separating edge. Then*

1. $\delta_G = \delta_{G/e}$*;*
2. $\mathbf{v}_\tau(G) = \mathbf{v}_\tau(G/e)$*;*
3. $\mathbf{w}_\tau(G) = \mathbf{w}_\tau(G/e)$.

In particular, the graph G is Ceresa–Zharkov trivial if and only if its 2-edge connectivization G^2, which is obtained by contracting all separating edges of G, is Ceresa–Zharkov trivial.

Proof. Statement (1) follows from the fact that $\delta_e = I$ whenever e is a separating edge. Statement (2) follows from the fact that $[T_{\ell_e}, \tau] = 1$ for any hyperelliptic quasi-involution τ. Finally, statement (3) is a consequence of (1) and (2). □

Let G be a graph, and suppose $f \in E(G)$ is subdivided into two edges e_1, e_2, producing the graph G'. Consider the ring map $\phi_f : R[G] \to R[G']$ given by

$$\phi_f(x_e) = \begin{cases} x_e & \text{if } e \neq f \\ x_{e_1} + x_{e_2} & \text{if } e = f. \end{cases}$$

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Lemma 4.6. *With the above notation, we have*

$$(1 \otimes \phi_f) \circ (\delta_G - I) = (\delta_{G'} - I) \circ (1 \otimes \phi_f) \tag{4.5}$$

as morphisms $(L/H)_{R[G]} \to (L/H)_{R[G']}$, *and*

$$(1 \otimes \phi_f)(\mathbf{v}_\tau(G)) = \mathbf{v}_\tau(G').$$

Proof. First, observe that the loops ℓ_f, ℓ_{e_1} and ℓ_{e_2} are isotopic to each other, thus

$$T_{\ell_f} = T_{\ell_{e_1}} = T_{\ell_{e_2}}$$

as elements of $\mathrm{Mod}(\Sigma_g)$, and as a consequence,

$$\delta_{\ell_f} = \delta_{\ell_{e_1}} = \delta_{\ell_{e_2}}.$$

As morphisms $H_{R[G]} \to H_{R[G']}$, we have

$$(1 \otimes \phi_f) \circ (\delta_e - I) = \begin{cases} (\delta_{\ell_e} - I) \circ (1 \otimes \phi_f) & \text{if } e \neq f \\ ((\delta_{\ell_{e_1}} - I) + (\delta_{\ell_{e_2}} - I)) \circ (1 \otimes \phi_f) & \text{if } e = f. \end{cases}$$

Therefore,

$$\begin{aligned} (1 \otimes \phi_f) \circ (\delta_G - I) &= (1 \otimes \phi_f) \circ \sum_{e \in E(G)} (\delta_{\ell_e} - I) \\ &= \left((\delta_{\ell_{e_1}} - I) + (\delta_{\ell_{e_2}} - I) + \sum_{e \in E(G) \setminus f} (\delta_{\ell_e} - I) \right) \circ (1 \otimes \phi_f) \\ &= (\delta_{G'} - I) \circ (1 \otimes \phi_f). \end{aligned}$$

Thus, the identity in Formula 4.5 holds as morphisms $H_{R[G]} \to H_{R[G']}$. One readily extends this as an identity of morphisms $(L/H)_{R[G]} \to (L/H)_{R[G']}$ using Proposition 3.5.

For the statement regarding the Ceresa class, we compute

$$\begin{aligned} (1 \otimes \phi_f)(\mathbf{v}_\tau(G)) &= J([T_{\ell_f}, \tau]) \otimes (x_{e_1} + x_{e_2}) + \sum_{e \in E(G) \setminus f} J([T_{\ell_e}, \tau]) \otimes x_e \\ &= \sum_{e \in E(G')} J([T_{\ell_e}, \tau]) \otimes x_e \\ &= \mathbf{v}_\tau(G'). \end{aligned}$$

□

Proposition 4.7. *If G and G' are tropically equivalent graphs, then G is Ceresa–Zharkov trivial if and only if G' is Ceresa–Zharkov trivial.*

Proof. If G' is obtained from G by adding or contracting an edge adjacent to a 1-valent vertex, then the claim follows from Lemma 4.5. Now, suppose G' is obtained from G by subdividing an edge $f \in E(G)$ into e_1 and e_2. If G' is Ceresa–Zharkov trivial, then G is Ceresa–Zharkov trivial by Proposition 4.4. Conversely, if G is Ceresa–Zharkov trivial, say $\mathbf{w}_\tau(G) = 0$ for some τ, then by Lemma 4.6 we have

$$\mathbf{w}_\tau(G') = (\delta_G - I) \circ (1 \otimes \phi_f)(\mathbf{v}_\tau(G)) = (1 \otimes \phi_f)(\mathbf{w}_\tau(G)) = 0$$

and hence G' is Ceresa–Zharkov trivial. □

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4.4. Relation to the tropical Ceresa class

Let $\Gamma = (G, c)$ be a tropical curve. Recall from Section 2.4 that $\mathbf{v}(\Gamma) \in B(\delta_\Gamma)$ is the image of the Ceresa class $\nu(\Gamma)$ under the map $A(\delta) \to B(\delta)$. Given $c : E(G) \to \mathbb{Z}_{>0}$, define homomorphisms

$$\epsilon_c^1 : F_2(L/H) \otimes R_1 \to B(\delta_\Gamma) \qquad \epsilon_c^2 : F_3(L/H) \otimes R_2 \to C(\delta_\Gamma)$$

by the compositions

$$F_2(L/H) \otimes R_1 \to F_2(L/H) \to \mathrm{gr}_2^F(L/H) \to B(\delta_\Gamma),$$
$$F_3(L/H) \otimes R_2 \to F_3(L/H) \to \mathrm{gr}_3^F(L/H) \to C(\delta_\Gamma),$$

respectively, where the first maps $F_2(L/H)\otimes R_1 \to F_2(L/H)$ and $F_3(L/H)\otimes R_2 \to F_3(L/H)$ are given by the evaluation map $h\otimes f \mapsto h\otimes f(c)$. We say that Γ is *Ceresa–Zharkov trivial* if $\mathbf{w}(\Gamma) = 0$ in $C(\delta_\Gamma)$.

Proposition 4.8. *Let $\Gamma = (G, c)$ be a tropical curve, and let τ be a hyperelliptic quasi-involution such that $\mathbf{v}_\tau(G)$ lies in $F_2(L/H) \otimes R_1$. Then*

$$\epsilon_c^1(\mathbf{v}_\tau(G)) = \mathbf{v}(\Gamma), \qquad \textit{and} \qquad \epsilon_c^2(\mathbf{w}_\tau(G)) = \mathbf{w}(\Gamma).$$

In particular, if G is Ceresa–Zharkov trivial, then so is Γ.

Proof. This is a direct consequence of formula (2.2) and Proposition 4.2. □

By [10, Proposition 4.6], if a tropical curve Γ is hyperelliptic, then Γ is Ceresa trivial. Assuming Theorem 5.11, we derive a similar statement for tropical curves of hyperelliptic type and Ceresa–Zharkov triviality.

Proposition 4.9. *If a tropical curve Γ is of hyperelliptic type, then Γ is Ceresa–Zharkov trivial.*

Proof. The statement follows from Proposition 4.8 and Theorem 5.11. □

However, the converse to Proposition 4.9 is not true by Remark 5.8.

5. Ceresa–Zharkov trivial and tropical curves of hyperelliptic type

The main goal of this section is to prove Theorem 5.11, that a graph is Ceresa–Zharkov trivial if and only if it is of hyperelliptic type. Being of hyperelliptic type is a minor closed condition on graphs by [9, Proposition 3.8]. Because of this, Theorem 5.11 implies that Ceresa–Zharkov triviality is also a minor closed condition. Nevertheless, to prove this theorem, we need the fact that Ceresa–Zharkov triviality is preserved under edge contraction (Proposition 4.4) and under removal of a loop edge or an edge in a parallel pair (Proposition 5.1). The removal of these two types of edges are easier to handle for the following reason. Using Lemma 5.4, we may choose compatible homology bases and compatible hyperelliptic quasi-involutions of the Riemann surfaces corresponding to the original graph and the removal of such an edge. This allows for a direct comparison between the corresponding Ceresa–Zharkov classes.

Proposition 5.1. *Consider either*

1. *a connected graph G_1 with a loop edge a, or*
2. *a 2-connected graph G_2 with a pair of parallel edges (b, c).*

If G_1, resp. G_2, is Ceresa–Zharkov trivial, then so is $G_1 \setminus a$, resp. $G_2 \setminus b$.

We begin by describing, for any connected graph G, the images $(\delta_G - I)(F_2(L_R/H_R))$ and $(\delta_G - I)^2(F_1(L_R/H_R))$. Order the edges of G by $e_1, \ldots, e_n$ so that $e_{g+1}, \ldots, e_n$ are the edges of a spanning tree. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be the basis of $H_1(\Sigma_g, \mathbb{Z})$ from §3.3, so δ_G has the form in Equation (3.4). Write q_{ij} for the entries of Q_G.

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Lemma 5.2. *We have*

$$(\delta_G - I)\left(\sum_{i;j<k} \alpha_i \wedge \beta_j \wedge \beta_k \otimes b_{ijk}\right) = \sum_{r<s<t} \beta_r \wedge \beta_s \wedge \beta_t \otimes c_{rst},$$

where

$$c_{rst} = \sum_i (b_{irs}q_{ti} - b_{irt}q_{si} + b_{ist}q_{ri}).$$

Proof. By formula (3.6), we have that

$$(\delta_G - I) \sum_{i;j<k} \alpha_i \wedge \beta_j \wedge \beta_k \otimes b_{ijk} = \sum_{i;j<k} Q_G(\alpha_i) \wedge \beta_j \wedge \beta_k \otimes b_{ijk} = \sum_{i;j<k;\ell} \beta_\ell \wedge \beta_j \wedge \beta_k \otimes b_{ijk}q_{\ell i}.$$

Given $r < s < t$ one may extract the coefficient c_{rst} for $\beta_r \wedge \beta_s \wedge \beta_t$. □

Lemma 5.3. *We have*

$$(\delta_G - I)^2 \sum_{i<j;k} \alpha_i \wedge \alpha_j \wedge \beta_k \otimes a_{ijk} = \sum_{r<s<t} \beta_r \wedge \beta_s \wedge \beta_t \otimes c_{rst} \quad \text{where} \quad c_{rst} = 2\sum_{i<j} \begin{vmatrix} q_{ri} & q_{rj} & a_{ijr} \\ q_{si} & q_{sj} & a_{ijs} \\ q_{ti} & q_{tj} & a_{ijt} \end{vmatrix}.$$

Proof. By Formula 3.6, we have that

$$\begin{aligned}(\delta_G - I)^2\left(\sum_{i<j;k} \alpha_i \wedge \alpha_j \wedge \beta_k \otimes a_{ijk}\right) &= \sum_{i<j;k} 2Q_G\,\alpha_i \wedge Q_G\,\alpha_j \wedge \beta_k \otimes a_{ijk} \\ &= \sum_{i<j;k}\sum_{\ell<m} \beta_\ell \wedge \beta_m \wedge \beta_k \otimes 2a_{ijk} \begin{vmatrix} q_{\ell i} & q_{\ell j} \\ q_{mi} & q_{mj} \end{vmatrix}.\end{aligned}$$

Given $r < s < t$, the R-coefficient of $\beta_r \wedge \beta_s \wedge \beta_t$ is

$$c_{rst} = 2\sum_{i<j}\left(a_{ijr}\begin{vmatrix} q_{si} & q_{sj} \\ q_{ti} & q_{tj} \end{vmatrix} - a_{ijs}\begin{vmatrix} q_{ri} & q_{rj} \\ q_{ti} & q_{tj} \end{vmatrix} + a_{ijt}\begin{vmatrix} q_{ri} & q_{rj} \\ q_{si} & q_{sj} \end{vmatrix}\right)$$

and the summand is exactly the 3×3 determinant appearing in the lemma. □

Our next step is to show how the Ceresa classes of G and $G \setminus f$ are related. The two cases listed in Proposition 5.1 are similar, so we handle them in parallel. Let G_1 be a connected graph with a loop edge a and let G_2 be a 2-connected graph with a pair of parallel edges b, c. Let $\Lambda_1 = \{\ell_e : e \in E(G_1)\}$ and $\Lambda_2 = \{\ell_e : e \in E(G_2)\}$ be two arrangements of isotopy classes of simple closed curves whose dual graphs are G_1 and G_2, respectively. Suppose that ℓ_a, ℓ_b, ℓ_c are as in Figure 3, and every other ℓ_e lies in Σ_g^1, the genus g subsurface with a boundary to the left of γ. To emphasize the dependence on the genus g, we write

$$H(\Sigma_g^n) = H_1(\Sigma_g^n, \mathbb{Z}), \quad L(\Sigma_g^n) = \wedge^3 H_1(\Sigma_g^n),$$

where $n = 0$ or 1. The inclusion $\Sigma_g^1 \hookrightarrow \Sigma_{g+1}$ induces an inclusion on homology groups which we use to identify $H(\Sigma_g) \cong H(\Sigma_g^1)$ as a subgroup of $H(\Sigma_{g+1})$. Now, recall that we obtain Σ_g from Σ_g^1 by attaching a disc along γ. Any two extensions of $\ell_c \cap \Sigma_g^1$ to Σ_g are isotopic. Choose such an extension and denote it also by ℓ_c.

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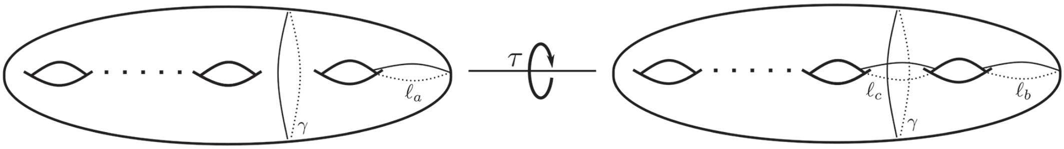

Figure 3. *Arrangements of curves on Σ_{g+1} with dual graphs G_1 and G_2 where (left) a is a loop edge and (right) (b, c) are parallel edges. Here, we identify Σ_g^1 with the subsurface of Σ_{g+1} to the left of γ.*

We may view $\Lambda_1 \setminus \{\ell_a\}$ as an arrangement of curves on Σ_g or Σ_g^1. We may also view $\Lambda_2 \setminus \{\ell_b\}$ as an arrangement of curves on Σ_g, and $\Lambda_2 \setminus \{\ell_b, \ell_c\}$ as an arrangement of curves on Σ_g^1. Because $L(\Sigma_g)/H(\Sigma_g)$ does not naturally embed into $L(\Sigma_{g+1})/H(\Sigma_{g+1})$, we cannot directly compare $\mathbf{v}_\tau(\Lambda_1)$ with $\mathbf{v}_{\tau'}(\Lambda_1 \setminus \{\ell_a\})$, or $\mathbf{v}_\tau(\Lambda_2)$ with $\mathbf{v}_{\tau'}(\Lambda_2 \setminus \{\ell_b\})$. Instead, we compare these with Ceresa classes on Σ_g^1 in the following way.

Lemma 5.4. *There are hyperelliptic quasi-involutions τ on Σ_{g+1}, τ' on Σ_g and τ'' on Σ_g^1, under the natural homomorphisms*

$$L(\Sigma_g^1)_{R[\Lambda_1\setminus\{\ell_a\}]} \to L(\Sigma_g)_{R[\Lambda_1\setminus\{\ell_a\}]}/H(\Sigma_g)_{R[\Lambda_1\setminus\{\ell_a\}]}$$
$$L(\Sigma_g^1)_{R[\Lambda_1\setminus\{\ell_a\}]} \hookrightarrow L(\Sigma_{g+1}^1)_{R[\Lambda_1]} \to L(\Sigma_{g+1})_{R[\Lambda_1]}/H(\Sigma_{g+1})_{R[\Lambda_1]}$$

the class $\boldsymbol{\mu}_{\tau''}(\Lambda_1 \setminus \{\ell_a\})$ maps to $\mathbf{v}_{\tau'}(\Lambda_1 \setminus \{\ell_a\})$ and $\mathbf{v}_\tau(\Lambda_1)$, respectively. Similarly, the natural homomorphisms

$$L(\Sigma_g)_{R[\Lambda_2\setminus\{\ell_b,\ell_c\}]} \to L(\Sigma_g)_{R[\Lambda_2\setminus\{\ell_b\}]}/H(\Sigma_g)_{R[\Lambda_2\setminus\{\ell_b\}]}$$
$$L(\Sigma_g^1)_{R[\Lambda_2\setminus\{\ell_b,\ell_c\}]} \hookrightarrow L(\Sigma_{g+1})_{R[\Lambda_2]} \to L(\Sigma_{g+1})_{R[\Lambda_2]}/H(\Sigma_{g+1})_{R[\Lambda_2]}$$

take $\boldsymbol{\mu}_{\tau''}(\Lambda_2 \setminus \{\ell_b, \ell_c\})$ to $\mathbf{v}_{\tau'}(\Lambda_2 \setminus \{\ell_b\})$ and $\mathbf{v}_\tau(\Lambda_2)$, respectively.

Furthermore, we may choose τ, τ' and τ'' so that all classes defined above live in the F_2 part of the relevant filtration.

Proof. Consider the surfaces homeomorphic to Σ_{g+1} in Figure 3. These depict Σ_{g+1} embedded in $\mathbb{R}^3$ and bounding a handlebody V, the ‘inside’ of the surface. Assume that the curves in Λ_1 and Λ_2 are *meridians*, that is, they bound properly embedded discs in V. We use these assumptions in the next paragraph to show that the Ceresa classes all belong to F_2.

Next, let τ be the hyperelliptic involution on Σ_{g+1} given by rotation by 180° about the axis illustrated in Figure 3; in particular, τ takes ℓ_f to ℓ_f (reversing its orientation) for $f = a, b, c$. The map τ can be isotoped in a regular neighborhood of γ so that it fixes γ pointwise. Consequently, the mapping class $\tau \in \mathrm{Mod}(\Sigma_{g+1})$ restricts to a hyperelliptic quasi-involution τ'' on Σ_g^1; denote by τ' the image of τ'' under the natural map $\mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_g)$. Because all the curves in Λ_1 and Λ_2 are meridians and the hyperelliptic quasi-involutions τ, τ' and τ'' extend to homeomorphisms on the handlebodies, each $J([T_\ell, \tau''])$, $J([T_\ell, \tau'])$ and $J([T_\ell, \tau])$ lie in F_2 by [10, Theorem 6.6], and therefore $\boldsymbol{\mu}_{\tau''}$, $\mathbf{v}_{\tau'}$ and $\mathbf{v}_{\tau'}$ lie in $F_2 \otimes R_1$.

Since $[T_{\ell_f}, \tau] = T_{\ell_f} T_{\tau(\ell_f)}^{-1}$ and $\tau(\ell_f)$ is isotopic to ℓ_f for $f = a, b, c$, we have that $[T_{\ell_f}, \tau] = 1$ for $f = a, b, c$. Similarly, we have $[T_{\ell_c}, \tau'] = 1$. Therefore,

$$\boldsymbol{\mu}_{\tau''}(\Lambda_1 \setminus \{\ell_a\}) = \sum_{e\in E(G_1)\setminus\{a\}} J([T_{\ell_e}, \tau]) \otimes x_e \quad \text{and} \quad \boldsymbol{\mu}_{\tau''}(\Lambda_2 \setminus \{\ell_b, \ell_c\}) = \sum_{e\in E(G_1)\setminus\{b,c\}} J([T_{\ell_e}, \tau]) \otimes x_e.$$

The lemma follows from these computations. □

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Proof of Proposition 5.1. Suppose that the graph G and edge $f \in E(G)$ are either

1. $G = G_1$ and $f = a$ is a loop edge, or
2. $G = G_2$, a 2-connected graph, and $f = b$ is parallel to another edge c.

Let $g+1$ be the genus of G. Order the edges of G by $e_0, \ldots, e_g, e_{g+1}, \ldots, e_n$ such that the last $n-(g+1)$ edges form a spanning tree of G and $e_0 = f$. In the second case, assume that $e_1 = c$ and orient $[\ell_b]$ and $[\ell_c]$ in the same direction. This is possible since G_2 is 2-connected, so b, c are not a separating pair. Denote by $\alpha_0, \ldots, \alpha_g, \beta_0, \ldots, \beta_g$ the basis described in Section 3.3. This identifies $Q_{G\setminus f}$ with the lower-right $g \times g$ submatrix of Q_G. Let q_{ij} denote the coordinates of Q_G. When $G = G_1$, we have

$$q_{00} = x_0, \quad \text{and} \quad q_{0j} = q_{j0} = 0 \quad \text{for} \quad j \geq 1$$

and when $G = G_2$, we have

$$q_{00} = x_0 + q_{01}, \quad q_{11} = x_1 + q_{01} \quad \text{and} \quad q_{0j} = q_{1j} \quad \text{for} \quad j \geq 2.$$

By Lemma 5.4, there are hyperelliptic quasi-involutions τ on Σ_{g+1} and τ' on Σ_g such that

$$\mathbf{v}_\tau(G) = \sum_{\substack{1 \leq i \leq g \\ 1 \leq j < k \leq g}} \alpha_i \wedge \beta_j \wedge \beta_k \otimes b_{ijk} \quad \text{in} \quad L(\Sigma_{g+1})_{R[G]}/H(\Sigma_{g+1})_{R[G]}$$

$$\mathbf{v}_{\tau'}(G \setminus f) = \sum_{\substack{1 \leq i \leq g \\ 1 \leq j < k \leq g}} \alpha_i \wedge \beta_j \wedge \beta_k \otimes b_{ijk} \quad \text{in} \quad L(\Sigma_g)_{R[G\setminus f]}/H(\Sigma_g)_{R[G\setminus f]},$$

where b_{ijk}'s are linear forms in $R[G \setminus f] \subset R[G]$. Implicit in this description is that $b_{0jk} = 0$ and $b_{i0k} = 0$. Because G is Ceresa–Zharkov trivial, by Proposition 4.2, there is a $\mathbf{u} \in F_1(L/H)$, say

$$\mathbf{u} = \sum_{\substack{0 \leq i < j \leq g \\ 0 \leq k \leq g}} a_{ijk} \cdot \alpha_i \wedge \alpha_j \wedge \beta_k$$

with $a_{ijk} \in \mathbb{Z}$ such that $\mathbf{w}_\tau(G) = (\delta_G - I)^2(\mathbf{u})$. We will show that $\mathbf{w}_{\tau'}(G \setminus f)$ lies in the image $(\delta_{G\setminus f} - I)^2(F_1(L(\Sigma_g)/H(\Sigma_g)))$, whence $G \setminus f$ is Ceresa–Zharkov trivial by Proposition 4.2. Because $b_{0jk} = 0$, we have

$$c_{rst} := \sum_{i=0}^{g} (b_{irs}q_{ti} - b_{irt}q_{si} + b_{ist}q_{ri}) = \sum_{i=1}^{g} (b_{irs}q_{ti} - b_{irt}q_{si} + b_{ist}q_{ri}) \tag{5.1}$$

for $0 \leq r < s < t \leq g$. Therefore, by Lemma 5.2, we have

$$\mathbf{w}_\tau(G) = (\delta_G - I)(\mathbf{v}_\tau(G)) = \sum_{0 \leq r < s < t \leq g} \beta_r \wedge \beta_s \wedge \beta_t \otimes c_{rst} \quad \text{and}$$

$$\mathbf{w}_{\tau'}(G \setminus f) = (\delta_{G\setminus f} - I)(\mathbf{v}_{\tau'}(G \setminus f)) = \sum_{1 \leq r < s < t \leq g} \beta_r \wedge \beta_s \wedge \beta_t \otimes c_{rst}.$$

By the equality $\mathbf{w}_\tau(G) = (\delta_G - I)^2(\mathbf{u})$ and Lemma 5.3, we have

$$c_{rst} = 2 \sum_{0 \leq i < j \leq g} \begin{vmatrix} q_{ri} & q_{rj} & a_{ijr} \\ q_{si} & q_{sj} & a_{ijs} \\ q_{ti} & q_{tj} & a_{ijt} \end{vmatrix}. \tag{5.2}$$

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Set

$$\mathbf{u}' = \sum_{\substack{1\le i<j\le g\\ 1\le k\le g}} a_{ijk}\cdot \alpha_i\wedge\alpha_j\wedge\beta_k.$$

Then

$$(\delta_{G\setminus f}-I)^2(\mathbf{u}') = \sum_{1\le r<s<t\le g} \beta_r\wedge\beta_s\wedge\beta_t\otimes c'_{rst} \quad \text{where} \quad c'_{rst} = 2\sum_{1\le i<j\le g}\begin{vmatrix} q_{ri} & q_{rj} & a_{ijr}\\ q_{si} & q_{sj} & a_{ijs}\\ q_{ti} & q_{tj} & a_{ijt}\end{vmatrix}.$$

Next, let's compute the difference

$$\mathbf{w}_{\tau'}(G\setminus f) - (\delta_{G\setminus f}-I)^2(\mathbf{u}') = \sum_{1\le r<s<t\le g}\beta_r\wedge\beta_s\wedge\beta_t\otimes(c_{rst}-c'_{rst}),$$

where

$$c_{rst}-c'_{rst} = 2\sum_{j=1}^{g}\begin{vmatrix} q_{r0} & q_{rj} & a_{0jr}\\ q_{s0} & q_{sj} & a_{0js}\\ q_{t0} & q_{tj} & a_{0jt}\end{vmatrix} \quad \text{for} \quad 1\le r<s<t\le g.$$

If we are in the first case, that is, $G = G_1$ and $f = a$ is a loop edge, then $q_{j0} = 0$ for $j\ge 1$, so the above formula implies $c_{rst}-c'_{rst} = 0$, whence $\mathbf{w}_{\tau'}(G\setminus f) = (\delta_{G\setminus f}-I)^2(\mathbf{u}')$.

For the rest of the proof, suppose we are in the second case, that is, $G = G_2$ and $f = b = e_0$ is parallel to the edge $c = e_1$. Set

$$\mathbf{u}'' = \sum_{\substack{2\le j\le g\\ 1\le k\le g}} a_{0jk}\cdot\alpha_1\wedge\alpha_j\wedge\beta_k.$$

We have

$$(\delta_{G\setminus f}-I)^2(\mathbf{u}'') = \sum_{1\le r<s<t\le g}\beta_r\wedge\beta_s\wedge\beta_t\otimes c''_{rst} \quad\text{where}\quad c''_{rst} = 2\sum_{j=2}^{g}\begin{vmatrix} q_{r1} & q_{rj} & a_{0jr}\\ q_{s1} & q_{sj} & a_{0js}\\ q_{t1} & q_{tj} & a_{0jt}\end{vmatrix}.$$

When $r\ge 2$, we have that $q_{r0} = q_{r1}$, and therefore $c_{rst}-c'_{rst} = c''_{rst}$. Now, consider $r = 1$. Since x_0 only appears in q_{00}, where $q_{00} = x_0 + q_{01}$, the coefficient of x_0 in c_{rst} must equal 0 for all $0\le r<s<t\le g$ by formula (5.1). By extracting the coefficient of x_0 in the expression of c_{0st} from formula (5.2), we see that

$$\sum_{j=1}^{g}\begin{vmatrix} q_{sj} & a_{0js}\\ q_{tj} & a_{0jt}\end{vmatrix} = 0. \tag{5.3}$$

Therefore,

$$c''_{1st} - (c_{1st}-c'_{1st}) = 2(q_{11}-q_{10})\sum_{j=1}^{g}\begin{vmatrix} q_{sj} & a_{0js}\\ q_{tj} & a_{0jt}\end{vmatrix} = 0,$$

where the last equality follows from formula (5.3). So $c_{rst} = c'_{rst} + c''_{rst}$ and therefore

$$\begin{aligned}\mathbf{w}_{\tau'}(G\setminus f) &= \sum_{1\le r<s<t\le g}\beta_r\wedge\beta_s\wedge\beta_t\otimes c_{rst}\\ &= \sum_{1\le r<s<t\le g}\beta_r\wedge\beta_s\wedge\beta_t\otimes(c'_{rst}+c''_{rst}) = (\delta_{G\setminus f}-I)^2(\mathbf{u}'+\mathbf{u}'').\end{aligned}$$

□

Next, we show that Ceresa–Zharkov triviality may be detected at the level of 2-connected components.

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Proposition 5.5. *A connected graph G is Ceresa–Zharkov trivial if and only if its 2-connected components are Ceresa–Zharkov trivial.*

Proof. By induction, it suffices to consider the case where G has two 2-connected components G_1 and G_2. Choose a separating curve γ and an arrangement of simple closed curves $\Lambda = \{\ell_e : e \in E(G)\}$ such that

- the dual graph of Λ is G;
- cutting along γ separates Σ_g into two subsurfaces $S_1 \cong \Sigma^1_{g_1}$ and $S_2 \cong \Sigma^1_{g_2}$;
- ℓ_e lies in S_i whenever $e \in E(G_i)$.

Denote by $H^{(i)} = H_1(\Sigma_{g_i}, \mathbb{Z})$ and $L^{(i)} = \wedge^3 H^{(i)}$. The inclusion $S_i \subset \Sigma_g$ allows us to view $F_q L^{(i)} \otimes R$ as an R-submodule of $F_q L_R$ for $q = 0, \ldots, 3$. Choose hyperelliptic quasi-involutions τ'_1 of S_1 and τ'_2 of S_2 such that $\boldsymbol{\mu}_{\tau'_i}(\Lambda'_i)$ lies in $F_2 L^{(i)} \otimes R_1$; this is possible by Proposition 4.1. Let τ be the hyperelliptic quasi-involution of Σ_g obtained that restricts to τ'_i on S_i. Denote by $\tau_i \in \mathrm{Mod}(\Sigma_{g_i})$ the image of τ' under the natural map $\mathrm{Mod}(\Sigma^1_{g_i}) \to \mathrm{Mod}(\Sigma_{g_i})$; thus τ_i is a hyperelliptic quasi-involution of Σ_{g_i}. Then $\mathbf{v}_{\tau_1}(G_1), \mathbf{v}_{\tau_2}(G_2), \mathbf{v}_\tau(G)$ are in $F_2 L \otimes R_1$, and therefore $\mathbf{w}_{\tau_1}(G_1), \mathbf{w}_{\tau_2}(G_2), \mathbf{w}_\tau(G)$ are in $F_3 L \otimes R_2$.

Consider the following commutative diagram:

$$\begin{array}{ccccc} (F_2L^{(1)} \oplus F_2L^{(2)})_R & \hookrightarrow & F_2 L_R & \longrightarrow & F_2(L/H)_R \\ \downarrow {\scriptstyle (\delta_{G_1}-I)\oplus(\delta_{G_2}-I)} & & \downarrow {\scriptstyle \delta_G - I} & & \downarrow {\scriptstyle \delta_G - I} \\ (F_3L^{(1)} \oplus F_3L^{(2)})_R & \hookrightarrow & F_3 L_R & \xrightarrow{\sim} & F_3(L/H)_R. \end{array} \tag{5.4}$$

The right arrow on the bottom row is an isomorphism by formula (2.1). Consider $\boldsymbol{\mu} := \boldsymbol{\mu}_{\tau'_1}(\Lambda'_1) + \boldsymbol{\mu}_{\tau'_2}(\Lambda'_2)$ in $(F_2L^{(1)} \oplus F_2L^{(2)})_R$. The composition of the top two arrows of the diagram in Equation (5.4) maps $\boldsymbol{\mu}$ to $\mathbf{v}_\tau(G)$, which maps to $\mathbf{w}_\tau(G)$ by the right vertical arrow. The composition along the bottom takes $\boldsymbol{\mu}$ to $\mathbf{w}_{\tau_1}(G_1) + \mathbf{w}_{\tau_2}(G_2)$. Thus,

$$\mathbf{w}_{\tau_1}(G_1) + \mathbf{w}_{\tau_2}(G_2) = \mathbf{w}_\tau(G). \tag{5.5}$$

Suppose G_1 and G_2 are Ceresa–Zharkov trivial. By Proposition 4.2, there are elements $\mathbf{u}_i \in F_1(L^{(i)}/H^{(i)})$ such that $(\delta_{G_i} - I)^2(\mathbf{u}_i) = \mathbf{w}_{\tau_i}(G_i)$. The restriction of $(\delta_G - I)^2$ to $F_1 L^{(1)} \oplus F_1 L^{(2)}$ is $(\delta_{G_1} - I)^2 \oplus (\delta_{G_2} - I)^2$; this follows from formula (3.6) and the fact that $Q_G = Q_{G_1} \oplus Q_{G_2}$. So $(\delta_G - I)^2(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{w}_\tau(G)$, and therefore G is Ceresa–Zharkov trivial.

Conversely, suppose G is Ceresa–Zharkov trivial. Let T be a spanning tree of G_1. Then G/T is Ceresa–Zharkov trivial by Proposition 4.4. Observe that G/T is obtained from G_2 by attaching g_1 loop edges to a single vertex of G_2. So G_2 is Ceresa–Zharkov trivial by Proposition 5.1. Swapping the roles of G_1 and G_2, we conclude that G_1 is also Ceresa–Zharkov trivial. □

Recall from [9, §3] that a graph G is *strongly of hyperelliptic type* if there is a choice of edge lengths of G so that the resulting tropical curve is hyperelliptic. Such graphs are Ceresa–Zharkov trivial by the following proposition.

Proposition 5.6. *Let G be a graph that has a separating pair of edges (f, f'), and let $\Lambda = \{\ell_e : e \in E(G)\}$ be a collection of pairwise disjoint simple closed curves on Σ_g with dual graph G. If τ is a hyperelliptic quasi-involution such that $\tau(\ell_f) = \ell_{f'}$, then*

$$(\delta_G - I)(J([T_{\ell_f}, \tau]) \otimes 1) = 0.$$

In particular, if G is strongly of hyperelliptic type, then G is Ceresa–Zharkov trivial.

Proof. The removal of $\{f, f'\}$ from G separates this graph into two graphs G_1 and G_2 of gerena g_1 and g_2, respectively. Order the edges of G by $e_1, \ldots, e_g, e_{g+1}, \ldots, e_n$ so that

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- $e_1, \ldots, e_{g_1} \in E(G_1)$, $e_{g_1+1}, \ldots, e_{g-1} \in E(G_2)$, and $e_g = f$;
- $e_{g+1}, \ldots, e_n$ are the edges of a spanning tree of G.

Necessarily, f' must be among the edges of the spanning tree. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be the basis from §3.3. With respect to this basis, we have

$$Q_G = \begin{bmatrix} Q_{G_1} & 0 & * \\ 0 & Q_{G_2} & * \\ * & * & x_f \end{bmatrix}.$$

Cutting along ℓ_f and $\ell_{f'}$ separates the surface Σ_g into two subsurfaces S_1 and S_2. The homology of S_i splits as a direct sum $V \oplus W_i$, where $V = \mathbb{Z} \cdot [\ell_f]$ and W_i is identified with a symplectic subspace of H under the map $H_1(S_i, \mathbb{Z}) \to H$. The intersection 2-form ω_i of W_i is the restriction of ω to W_i, so

$$\omega_1 = \sum_{j=1}^{g_1} \alpha_j \wedge \beta_j \quad \text{and} \quad \omega_2 = \sum_{j=g_1+1}^{g-1} \alpha_j \wedge \beta_j.$$

Orient $[\ell_f]$ such that S_1 lies to its right. Because $\tau(\ell_f) = \ell_{f'}$, we have that $\tau T_{\ell_f} \tau^{-1} = T_{\ell_{f'}}$, and hence

$$J([T_{\ell_f}, \tau]) = J(T_{\ell_f} T_{\ell_{f'}}^{-1}) = \omega_1 \wedge [\ell_f] = -\omega_2 \wedge [\ell_{f'}].$$

The expression $(\delta_G - I)(J(T_{\ell_f} T_{\ell_{f'}}^{-1}))$ is equal to

$$(\delta_G - I)(\omega_1 \wedge [\ell_f]) = \sum_{j=1}^{g_1} (Q_G\, \alpha_j) \wedge \beta_j \wedge \beta_g = \sum_{j=1}^{g_1} (Q_{G_1} \alpha_j) \wedge \beta_j \wedge \beta_g = (\delta_{G_1} - I)(\omega_1 \wedge [\ell_f]).$$

We use formula (3.6) in the first equality and $Q_G(\alpha_j) - Q_{G_1}(\alpha_j) \in \mathbb{Z} \cdot \beta_g$ for $j \leq g_1$ in the second. By Proposition 3.3 applied to V and $W = W_1$, we have

$$(\delta_{G_1} - I)(\omega_1 \wedge [\ell_f]) = \omega_1 \wedge (\delta_{G_1} - I)([\ell_f]) + \eta \wedge \delta_{G_1}([\ell_f])$$

for some $\eta \in V \wedge W_1$. Since the curves in Λ are disjoint, we have that $\delta_{G_1}([\ell_f]) = [\ell_f]$. This implies that the first summand above is 0, and the second summand is in $V \wedge W_1 \wedge V$, which must also be 0 since $\dim V = 1$. So we have $(\delta_G - I)(J([T_{\ell_f}, \tau])) = 0$, as required.

Now, suppose G is strongly of hyperelliptic type. By Proposition 4.7, we may assume that G is stable. Let Γ be a hyperelliptic tropical curve with underlying graph G, and σ be a hyperelliptic involution of Γ. There is a hyperelliptic quasi-involution τ of Σ_g such that $\tau(\ell_e) = \ell_{\sigma(e)}$ by [10, Lemma 4.5]. By [9, Proposition 2.5], for any edge $e \in E(G)$, we have that $\sigma(e) = e$ or $\sigma(e) = f$, where (e, f) is a separating pair. In the first case, $[T_{\ell_e}, \tau] = 1$. In the second case, $(\delta_G - I)(J([T_{\ell_e}, \tau])) = 0$ by the first part of this proposition. We conclude that $\mathbf{w}_\tau(G) = 0$, that is, G is Ceresa–Zharkov trivial. □

To prove Ceresa–Zharkov trivial implies hyperelliptic type, we use in an essential way the main theorem of [9], which states that a graph is of hyperelliptic type if and only if it has no K_4 or L_3 minor. We prove directly that these graphs are not Ceresa–Zharkov trivial, using Proposition 4.8.

Proposition 5.7. *The graph $G = K_4$ is not Ceresa–Zharkov trivial.*

Proof. By [10, Example 7.2], a Ceresa cocycle $\mathbf{v}_\tau(G)$ for the graph K_4 is given by

$$\mathbf{v}_\tau(G) = \alpha_1 \wedge \beta_1 \wedge \beta_2 \otimes x_2 + (-\alpha_2 \wedge \beta_1 \wedge \beta_2 - \alpha_2 \wedge \beta_2 \wedge \beta_3 + \alpha_2 \wedge \beta_1 \wedge \beta_3) \otimes x_5, \tag{5.6}$$

which lies in $F_2(L/H) \otimes R_1$. The matrix Q_G is recorded in formula (3.5). We get

$$\mathbf{w}_\tau(G) = -2 \cdot \beta_1 \wedge \beta_2 \wedge \beta_3 \otimes x_2 x_5,$$

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which lies in $F_3(L/H) \otimes R_2$. By Proposition 4.8, it is sufficient to show that the tropical curve $\Gamma = (K_4, c)$ is not Ceresa–Zharkov trivial for some edge-length function $c : E(K_4) \to \mathbb{Z}_{>0}$. Let c be the function that assigns the length 1 to each edge. Then $\mathbf{w}(\Gamma) = -2\beta_1 \wedge \beta_2 \wedge \beta_3$, whereas $(\delta_\Gamma - I)^2(F_1(L/H))$ is spanned by $4\beta_1 \wedge \beta_2 \wedge \beta_3$; see [10, Remark 7.3]. So $\mathbf{w}(\Gamma) \neq 0$, as required. □

Remark 5.8. Define $c : E(K_4) \to \mathbb{Z}_{>0}$ by $c(e_1) = 2$ and $c(e_i) = 1$ for $i = 2, \ldots, 6$, and let $\Gamma = (K_4, c)$. Then $(\delta_\Gamma - I)^2(F_1(L/H)) = F_3L$, and hence $\mathbf{w}(\Gamma) = 0$. Therefore, Γ is Ceresa–Zharkov trivial but clearly not of hyperelliptic type.

Proposition 5.9. *The graph $G = L_3$ is not Ceresa–Zharkov trivial.*

Proof. From [10, Example 7.6] by setting $c_7 = c_8 = c_9 = 0$, we get

$$Q_G = \begin{pmatrix} x_1 + x_6 & 0 & x_6 & x_6 \\ 0 & x_2 + x_5 & x_5 & x_5 \\ x_6 & x_5 & x_3 + x_5 + x_6 & x_5 + x_6 \\ x_6 & x_5 & x_5 + x_6 & x_4 + x_5 + x_6 \end{pmatrix}.$$

$$\begin{aligned} \mathbf{v}_\tau(G) =& (\alpha_2 \wedge \beta_2 \wedge \beta_3 + \alpha_2 \wedge \beta_2 \wedge \beta_4 - \alpha_2 \wedge \beta_1 \wedge \beta_2) \otimes x_6 \\ & -(\alpha_1 \wedge \beta_1 \wedge \beta_2 + \alpha_1 \wedge \beta_1 \wedge \beta_3 + \alpha_1 \wedge \beta_1 \wedge \beta_4) \otimes x_5. \end{aligned}$$

So we have

$$\mathbf{w}_\tau(G) = -2x_5x_6(\beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_1 \wedge \beta_2 \wedge \beta_4).$$

By Proposition 4.8, it is sufficient to show that the tropical curve $\Gamma = (L_3, c)$ is not Ceresa–Zharkov trivial for some edge-length function $c : E(L_3) \to \mathbb{Z}_{>0}$. Let c be the function that assigns the length 1 to each edge. Using OSCAR [11, 15] (which runs in `julia` [3]), we show that $(\delta_\Gamma - I)^2(F_1(L/H))$ is spanned by

$$\begin{aligned} &2\beta_1 \wedge \beta_2 \wedge \beta_3 + 2\beta_1 \wedge \beta_2 \wedge \beta_4 + 2\beta_2 \wedge \beta_3 \wedge \beta_4, && 4\beta_1 \wedge \beta_2 \wedge \beta_4, \\ &2\beta_1 \wedge \beta_3 \wedge \beta_4 + 2\beta_2 \wedge \beta_3 \wedge \beta_4, && 4\beta_2 \wedge \beta_3 \wedge \beta_4. \end{aligned}$$

The code may be found in the github repository

https://github.com/dcorey2814/ceresaZharkovClass

One readily verifies that $\mathbf{w}_\tau(\Gamma) = -2\beta_1 \wedge \beta_2 \wedge \beta_3 - 2\beta_1 \wedge \beta_2 \wedge \beta_4$ and $\mathbf{w}_\tau(\Gamma)$ is not in the $\mathbb{Z}$-span of the vectors listed above. So L_3 is not Ceresa–Zharkov trivial. □

Lemma 5.10. *Suppose G is a graph with a minor G' obtained by adding (parallel) edges to the complete graph K_m. Then there are edges $S_1 \subset E(G)$ and $S_2 \subset E(G/S_1)$ such that $(G/S_1) \setminus S_2$ is G', and every edge in S_2 is either a loop or parallel to another edge in G/S_1.*

Proof. The order in which edges are contracted or removed does not matter when forming a graph minor, so suppose $G' = (G/S_1) \setminus S_2$. As contracting an edge drops the number of vertices by one and removing an edge preserves the number of vertices, we have that $V(G/S_1) = V(G')$. As there is an edge between any two vertices in G', every edge in S_1 must be either a loop or parallel to some other edge. □

Theorem 5.11. *A connected graph G is Ceresa–Zharkov trivial if and only if it is of hyperelliptic type.*

Proof. Suppose G is not of hyperelliptic type. By [9, Theorem 1.1], G has a K_4 or L_3 minor. By Lemma 5.10, there are subsets $S_1 \subset E(G)$ and $S_2 \subset E(G/S_1)$ such that $(G/S_1) \setminus S_2$ is K_4 or L_3, and every edge of S_2 is a loop or parallel to another edge in G/S_1. The graph $(G/S_1) \setminus S_2$ is not Ceresa–Zharkov trivial by Propositions 5.7 and 5.9. So (G/S_1) is not Ceresa–Zharkov trivial by Propositions 5.1 and 5.5, and therefore G is not Ceresa–Zharkov trivial by Proposition 4.4.

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For the converse, first suppose that G is of hyperelliptic type and 2-connected. By [9, Theorem 4.5], there is a $\tilde{G}$ that is strongly of hyperelliptic type such that $G = \tilde{G}/S$ for some subset $S \subset E(G)$. The graph $\tilde{G}$ is Ceresa–Zharkov trivial by Proposition 5.6, and therefore so is G by Proposition 4.4.

In general, if G is of hyperelliptic type, then its 2-connected components are of hyperelliptic type, and hence Ceresa–Zharkov trivial. The graph G is Ceresa–Zharkov trivial by Proposition 5.5. □

Corollary 5.12. *The property of being Ceresa–Zharkov trivial is a minor closed condition on graphs.*

Proof. This follows from Theorem 5.11 and the fact that being of hyperelliptic type is a minor closed condition on graphs [9, Proposition 3.8]. □

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