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(D − 2)-EXTREME POINTS AND A HELLY-TYPE THEOREM FOR STARSHAPED SETS

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1. Introduction. We begin with some preliminary definitions. Let S be a subset of $\mathbf{R}^d$. For points x and y in S, we say *x sees y via S* if and only if the corresponding segment $[x, y]$ lies in S. The set S is said to be *starshaped* if and only if there is some point p in S such that, for every x in S, p sees x via S. The collection of all such points p is called the *kernel* of S, denoted ker S. Furthermore, if we define the star of x in S by $S_x = \{y: [x, y] \subseteq S\}$, it is clear that ker $S = \bigcap\{S_x: x \text{ in } S\}$.

Several interesting results indicate a relationship between ker S and the set E of $(d - 2)$-extreme points of S. Recall that for $d \geqq 2$, a point x in S is a $(d - 2)$-extreme point of S if and only if x is not relatively interior to a $(d - 1)$-dimensional simplex which lies in S. Kenelly, Hare et al. [**4**] have proved that if S is a compact starshaped set in $\mathbf{R}^d$, $d \geqq 2$, then ker $S = \bigcap\{S_e: e \text{ in } E\}$. This was strengthened in papers by Stavrakas [**6**] and Goodey [**2**], and their results show that the conclusion follows whenever S is a compact set whose complement $\sim S$ is connected.

Thus it seems natural to expect that the set E might be used in a Helly-type theorem for starshaped sets. A well-known result of Krasnosel'skii [**3**] states that for S compact in $\mathbf{R}^d$, S is starshaped if and only if every $d + 1$ points of S see a common point via S. We show that, with suitable hypothesis, it suffices that every $d + 1$ points of E see a common point via S. In fact, a stronger result is obtained, for an analogue of this statement may be used to determine the dimension of ker S.

Since these results are perhaps most useful when E is finite, it seems appropriate to begin the paper by investigating this situation, and Section 2 shows that for S compact, E countable, and $S \neq E$, then S is planar. The third section studies the relationship between E and ker S to obtain a Helly-type theorem for the dimension of ker S.

The following terminology will be used. Throughout the paper, conv S, aff S, cl S, int S, rel int S, bdry S, rel bdry S, and ker S will denote the convex hull, affine hull, closure, interior, relative interior, boundary, relative boundary, and kernel, respectively, of the set S, while card S will be the cardinality of S. If S is convex, ext S will represent the set of extreme points of S, dim S the dimension of S. Finally, for $x \neq y$, $R(x, y)$ will denote the ray emanating from x through y, and $L(x, y)$ will be the

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line determined by x and y. The reader is referred to [**7**] for a thorough explanation of these concepts.

2. The cardinality of the set E of $(d - 2)$-extreme points. We begin by investigating the case in which E is countable, and we have the following result.

THEOREM 1. *Let S be a compact set in $\mathbf{R}^d$, E the set of $(d - 2)$-extreme points of S. If E is countable and $S \neq E$, then S is planar.*

Proof. Clearly if x is a $(k - 2)$-extreme point of S in the flat aff S, where dim aff $S = k \leqq d$, then x is a $(d - 2)$-extreme point of S in $\mathbf{R}^d$. Hence without loss of generality we assume that aff $S = \mathbf{R}^d$. Also, if dim aff $S \leqq 1$, there is nothing to prove, so let $d \geqq 2$.

We begin by considering the case in which S is convex. Since S is at least 2-dimensional, bdry S is uncountable, and we may select some point s in bdry $S \sim E$. Then s is relatively interior to a $(d - 1)$-simplex F in S, and since $s \in$ bdry S and S is convex, clearly $F \subseteq$ bdry S. Letting H denote the hyperplane aff F, H supports S at s, and we may assume that S lies in the closed halfspace cl H_1 (where H_1, H_2 denote distinct open halfspaces determined by H).

We assert that rel bdry $(H \cap S) \subseteq E$: Select $t \in$ rel bdry $(H \cap S)$. If t were relatively interior to a $(d - 1)$-simplex G in S, then $G \subseteq$ cl H_1. However, since $t \in$ rel bdry $(H \cap S)$, $G \nsubseteq H$. Hence $G \cap H_1 \neq \emptyset$ and since $t \in$ rel int G, this forces $G \cap H_2 \neq \emptyset$. We have a contradiction and t must belong to E, the desired result. Thus

$$\text{rel bdry } (H \cap S) \subseteq E.$$

Now if $d \geqq 3$, then the set $H \cap S$ would be at least 2-dimensional, and its relative boundary would be uncountable. However,

$$\text{rel bdry } (H \cap S) \subseteq E$$

and E is countable so this cannot occur. Hence $d = 2$ and S is planar, finishing the argument for the case in which S is convex.

The remainder of the proof will be concerned with the argument for S not convex. The following lemmas will be useful.

LEMMA 1. *Without loss of generality, we may assume that $\sim S$ is connected.*

Proof of Lemma 1. Let A denote an unbounded component of S. (Since S is compact, standard arguments reveal that A is unique.) Define $T = \sim A$ and define D to be the set of $(d - 2)$-extreme points of T. We will show that T is compact, that $D \subseteq E$, and that it suffices to prove the theorem for the set T. Notice that

$$\sim \text{conv } S \subseteq A = \sim T,$$

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so $T \subseteq \text{conv}\, S$ and T is bounded. Also, since $\mathbf{R}^d$ is locally connected and $\sim S$ is open, the component A is open, and T is closed. Thus T is compact. Also, $A \subseteq \sim S$ so $S \subseteq \sim A = T$.

In order to prove that $D \subseteq E$, first we verify that bdry $T \subseteq$ bdry S. Let $x \in$ bdry T and let N be any neighborhood of x. Then $N \cap A \neq \emptyset$, but $A \subseteq \sim S$, so $N \cap (\sim S) \neq \emptyset$. Now if $N \subseteq \sim S$, then $x \in \sim S$ and some neighborhood M of x would lie in the (open) component of $\sim S$ containing x. But each neighborhood of x contains points of A, so this would imply that $M \subseteq A$. However, then M could contain no point of $\sim A = T$, inpossible since $x \in$ bdry T. Hence $N \nsubseteq \sim S$ and $N \cap \mathrm{S} \neq \emptyset$. We conclude that $x \in$ bdry S and bdry $T \subseteq$ bdry S.

Now it is easy to show that $D \subseteq E$. For y in D, y is not relatively interior to a $(d - 1)$-simplex in T, so $y \in$ bdry $T \subseteq$ bdry S, and $y \in S$. Furthermore, since $S \subseteq T$, y is not relatively interior to a $(d - 1)$-simplex in S, and we conclude that y is a $(d - 2)$-extreme point of S. Thus $D \subseteq E$, the desired result.

In summary, T is a compact set in $\mathbf{R}^d$ whose set D of $(d - 2)$-extreme points lies in E and hence is countable. Since $D \subseteq E \subseteq S \subseteq T$ and $E \neq S$, certainly $D \neq T$ so T satisfies the hypothesis of our theorem. In addition, $\sim T = A$ is connected. If we are able to show that T is planar, then its subset S must also be planar, and the proof of Lemma 1 is complete.

LEMMA 2. *Let S' be any component of S with E' the corresponding set of $(d - 2)$-extreme points of S'. The set S' is compact, E' is countable, and if S' is not a singleton set, then $S' \neq E'$.*

Proof of Lemma 2. Standard arguments reveal that S' is closed and therefore compact. Furthermore, it is easy to show that $E' \subseteq E$ and hence E' is countable: For $y \in S' \sim E$, y is relatively interior to a $(d - 1)$-simplex in S, and this simplex necessarily lies in the component S'. Thus $y \in S' \sim E$ and $E' \subseteq E$.

Finally, we must prove that if S' is not a singleton set, then $S' \neq E'$, and clearly it suffices to show that S' is uncountable. Choose points s, t in S', and let N be a neighborhood of s disjoint from t. The following argument by Robert Sternfeld (private communication) shows that $N \cap S'$ is uncountable: Otherwise, the set of distances

$$P = \{\text{dist}\,(s, u) : u \in N \cap S\}$$

would be countable, and we could choose some positive number $r \notin P$ so that the r-sphere V about s would lie in N. But then $V \cap S'$, $(\sim V) \cap S'$ would give a separation for S', contradicting the fact that S' is connected. We conclude that $S' \cap N$ is uncountable and hence $S' \neq E'$, finishing the proof of Lemma 2.

LEMMA 3. *If some nontrivial component S' of S is planar, then S is planar. Thus without loss of generality we may assume that S is connected.*

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Proof of Lemma 3. By the proof of Lemma 2, if S' is a component of S and S' is not a singleton set, then S' will be uncountable. Assume that S' lies in the plane π, and let B denote the set of relative boundary points of S' (as a subset of π). Now if S is not planar, then aff $S = \mathbf{R}^d$ for some $d \geqq 3$, and it is easy to show that each point in B is a $(d - 2)$-extreme point of S. However, we see that B is uncountable: If S' has no relative interior points in π, then $S' = B$. Otherwise, S' will have a relative interior point p in π, and every ray in π emanating from p will contain a distinct member of B. Hence B will be uncountable, impossible since $B \subseteq E$ and E is countable. We conclude that S must be planar.

To complete the proof of the lemma, note that since a singleton point component of S will be a $(d - 2)$-extreme point of S for $d \geqq 2$, and since $S \neq E$, it follows that S has at least one nontrivial component S'. By Lemma 2, S' satisfies the hypothesis of our theorem. Moreover, by the argument above, if S' is planar, then S is planar also. Therefore, it suffices to prove the theorem for any nontrivial component S' of S, and without loss of generality, we may assume that S is connected. This finishes the proof of Lemma 3.

Now we return to the proof of the theorem. Using our lemmas, we may assume that S is a connected set in $\mathbf{R}^d$ whose complement $\sim S$ is also connected. Furthermore, since we have proved the theorem for the case in which S is convex, we assume that S is not convex. Then there are points z, z' in S such that $[z, z'] \not\subseteq S$. Select x on $(z, z') \sim S$. Also, since S is compact, we may choose a point $x_0 \notin \text{conv } S$ with x_0 not collinear with z and z'.

Using an argument employed in [**2**], since $\sim S$ is open and connected, it is polygonally connected, and there is a path λ in $\sim S$ from x to x_0. Let $v_1 = x, v_2, \ldots, v_n = x_0$ denote consecutive vertices of λ, and assume that no segment of λ is collinear with z. Since $R(z, x) \sim [z, x)$ meets S at z' and $R(z, x_0) \sim [z, x_0)$ clearly cannot meet S, we may select a last vertex of λ, say v_i, for which $R(z, v_i) \sim [z, v_i)$ meets S. Certainly $1 \leqq i < n$, and the ray $R(z, v_{i+1}) \sim [z, v_{i+1})$ contains no point of S. Furthermore, for some convex neighborhood N of v_{i+1}, N in $\sim S$, and for each point w in N, $R(z, w) \sim [z, w)$ contains no point of S: Otherwise, there would be a sequence of rays $R(z, w_n) \sim [z, w_n)$ converging to $R(z, v_{i+1}) \sim [z, v_{i+1})$, each containing a point s_n of S, and a subsequence of $\{s_n\}$ would converge to a point of S on $R(z, v_{i+1}) \sim [z, v_{i+1})$, which is impossible.

Since $\sim S$ is open and $\lambda \subseteq \sim S$, we may choose an open convex cylinder C about $[v_i, v_{i+1}]$ whose closure is disjoint from S. Then $z \notin C$, and we may consider the open convex set

$$U \equiv \cup \{R(z, c) \sim [z, c) \colon c \text{ in } C\}.$$

Recall that $R(z, v_i) \sim [z, v_i)$ intersects S at some point q, and $q \notin \text{cl } C$. Let M be any neighborhood of q contained in U and disjoint from cl C. By the proof of Lemma 2, $M \cap S$ is uncountable and hence contains points not in E. Thus we may select point r in $(M \cap S) \sim E$, and we choose a corresponding point c_0 in C such that $R(z, c_0) \sim [z, c_0)$ contains r.

Since $r \notin E$, r is relatively interior to a $(d - 1)$-simplex P in S. Select a point v_{i+1}' in $C \cap N$ so that $[v_{i+1}', c_0] \subseteq \text{aff } P$ and so that v_{i+1}', c_0, z are not collinear. Let π denote the plane determined by v_{i+1}', c_0, z.

In case aff $P \subseteq \pi$, then the dimension of P is at most 2. However, $\dim P \neq 2$, for otherwise, then aff $P = \pi$, which is impossible by our choice of v_{i+1}'. Thus $\dim P \equiv d - 1 \leqq 1$, and since $d \geqq 2$, this implies $d = 2$ and S is planar, finishing the argument.

Therefore, we need only consider the case in which aff $P \nsubseteq \pi$. That is, we will assume that $d \geqq 3$ to reach a contradiction. Let L be a line in π through z and disjoint from cl C. Select a point p in $(P \sim \pi) \cap U$ so that the corresponding plane aff $(L \cup \{p\})$ intersects $N \cap C$. (Certainly this is possible for p sufficiently close to r.) For p_1, p_2 distinct points on $[p, r]$, clearly the planes aff $(L \cup \{p_1\})$, aff $(L \cup \{p_2\})$ intersect only in L.

We will show that for p' on $[p, r]$, the plane $\pi' \equiv \text{aff } (L \cup \{p'\})$ contains a point of $E \sim L$: By our choice of p,

$$N \cap C \cap \text{aff } (L \cup \{p\}) \neq \emptyset,$$

and since $N \cap C$ is convex, it is easy to see that there is a point v_{i+1}'' in $N \cap C \cap \text{aff } (L \cup \{p'\})$. Recall v_{i+1}'' in N implies that $R(z, v_{i+1}'') \sim [z, v_{i+1}'')$ does not intersect S. However, $p' \in U$ so for some c' in C, $R(z, c') \sim [z, c')$ intersects S at p'. Also,

$$[v_{i+1}'', c'] \subseteq C$$

and therefore $[v_{i+1}'', c']$ is disjoint from L. Since S is compact, there is a last point y on $[c', v_{i+1}'']$ such that $R(z, y) \sim [z, y)$ meets S, and $c' \leqq y < v_{i+1}''$. Let u denote the last point of S on the ray; that is, the point of S on $R(z, y) \sim [z, y)$ whose distance to y is maximal.

We assert that $u \in E \sim L$: If u were relatively interior to a $(d - 1)$-simplex in S, then that simplex would meet the plane π' in at least a segment, so u would be relatively interior to a segment (a, b) in $\pi' \cap S$. But by our choice of u as the last point of S on our ray, a and b could both lie on the ray, so a and b would lie on opposite sides of the corresponding line $L(z, y)$ in π'. However, this contradicts our choice of y. Our assumption is false and $u \in E$. Furthermore, $u \notin L$, for otherwise $y \in L$, which is impossible since $y \in C$ and $L \cap C = \emptyset$.

We conclude that for each point p_α on $[p, r]$, we may associate a $(d - 2)$-extreme point u_α in $\pi_\alpha \sim L$, where $\pi_\alpha = \text{aff } (L \cup \{p_\alpha\})$. For distinct points on $[p, r]$, their associated planes meet only in L, and hence the points u_α are necessarily distinct. Thus E must be uncountable, violating our hypothesis. Our assumption that $d \geqq 3$ must be false, so $d = 2$ and S is planar, finishing the proof of the theorem.

COROLLARY. *Let S be a nonempty compact set in $\mathbf{R}^d$, $d \geqq 2$, S' a component of S with corresponding set of $(d-2)$-extreme points E'. Then $E' \neq \emptyset$, and if S' is nontrivial*, card $E' \geqq 2$.

Proof. It is easy to show that every extreme point of the compact set conv S' is in E': Let $x \in$ ext (conv S'). Then x is not relatively interior to a segment whose endpoints are in conv S', so x is certainly not relatively interior to a $(d-1)$-simplex in S'. Furthermore, $x \in S'$, for otherwise, by Carathéodory's theorem in $\mathbf{R}^d$, x would be relatively interior to a k-simplex with vertices in S' for some $1 \leqq k \leqq d$, clearly impossible. Hence $x \in E'$. Since

$$\text{conv } S' = \text{conv (ext conv } S') \neq \emptyset,$$

$E' \neq \emptyset$.

Now if S is not planar and S' is nontrivial, then by Theorem 1 and arguments in Lemmas 2 and 3, E' will be uncountable (regardless of dim aff S'). In case S is planar and aff S' is a line, then S' must be a segment, and card $E' = 2$. For S planar and aff S' also planar, then conv S' has at least 3 extreme points, and card $E' \geqq 3$. Of course, whenever S' is a singleton set, $E' = S' \neq \emptyset$ for every $d \geqq 2$.

To conclude this section, we show that the full hypothesis of Theorem 1 is required. It is easy to see that S must be closed: In particular, any open set in $\mathbf{R}^d$ has no $(d-2)$-extreme points. The following examples reveal that S must be bounded with $S \neq E$.

Example 1. To see that S must be bounded, let D be the d-dimensional unit disk in $\mathbf{R}^d$, $d \geqq 3$, and let $S = \text{cl } (\mathbf{R}^d \sim D)$. Then S has no $(d-2)$-extreme points yet S is certainly nonplanar.

Example 2. To see that we must require $S \neq E$, for $d \geqq 3$ let T denote any sequence in $\mathbf{R}^d$ converging to the origin Φ, with aff $T = \mathbf{R}^d$. Then the set $S \equiv T \cup \{\Phi\}$ is a countable, compact set, every point of S is a $(d-2)$-extreme point, and S is nonplanar.

3. A Helly-type theorem for dim ker S. In this section we obtain a Helly-type theorem which uses the set E of $(d-2)$-extreme points of S to determine dim ker S. First we develop an analogue of some results in [**6**] and [**2**], then use a technique given in [**7**] to prove our main results.

In [**6**], Stavrakas introduced the following definition: A set S in $\mathbf{R}^d$ is said to have the *half-ray property* if and only if for every point x in $\sim S$, there exists a ray emanating from x and disjoint from S. Furthermore, he used this property to characterize compact sets S for which

$$\ker S = \bigcap \{S_e : e \text{ in } E\}.$$

Goodey [**2**] obtained a parallel theorem, replacing the half-ray property with the weaker requirement that $\sim S$ be connected, and the following lemma is an analogue of his result for convex hulls of the sets S_e.

LEMMA 4. *Let S be a compact set in $\mathbf{R}^d$, E the set of $(d-2)$-extreme points of S, and assume that $\sim S$ is connected. If*

$$\bigcap \{\text{int conv } S_e : e \text{ in } E\} \neq \emptyset,$$

then S has the half-ray property.

Proof. Select a point $z \in \bigcap \{\text{int conv } S_e : e \text{ in } E\}$. We use an argument similar to one in [**2**] to show that for x in $\sim S$, the ray $R(z, x) \sim [z, x)$ is disjoint from S. Choose $x_0 \notin \text{conv } S$. Then $R(z, x_0) \sim [z, x_0)$ cannot intersect S. As in the proof of Theorem 1, since $\sim S$ is open and connected, we may choose a polygonal path λ in $\sim S$ from x to x_0, with no segment of λ collinear with z. We let $v_1 = x, v_2, \ldots, v_n = x_0$ be consecutive vertices of λ.

Now if $R(z, x) \sim [z, x)$ does not intersect S, the argument is finished. Hence we assume that the ray meets S, to reach a contradiction. Choose a last vertex v_i of λ such that $R(z, v_i) \sim [z, v_i)$ meets S. Let A be the translate of $L(v_i, v_{i+1})$ through z, and let C be an open convex cylinder about $[v_i, v_{i+1}]$ whose closure is disjoint from $S \cup A$. Using an argument from Theorem 1, let N be the closure of a spherical neighborhood of v_{i+1} contained in C such that for w in N, $R(z, w) \sim [z, w)$ does not intersect S.

Consider the family of translates of N centered on $[v_i, v_{i+1}]$, and for $0 \leqq \lambda \leqq 1$, let N_λ denote that translate of N whose center is $\lambda(v_i) + (1-\lambda)v_{i+1}$, so that $N_0 = N$. Certainly each $N_\lambda \subseteq C$. Since S is compact, there is a smallest α, $0 < \alpha \leqq 1$, such that bdry N_α contains a point y with $R(z, y) \sim [z, y)$ intersecting S. Let u be the point of $(R(z, y) \sim [z, y)) \cap S$ whose distance to z is maximal.

We will show that $u \in E$. Recall that the line A through z is disjoint from C. Since $y \in C$, $y \notin A$ and $u \notin A$. Thus $\pi \equiv \text{aff }(A \cup \{u\})$ is a plane. Moreover, $N_\lambda \cap \pi$ is a translate of $N \cap \pi$ for every $0 \leqq \lambda \leqq 1$. Letting v_{i+1}' denote the center of $N_0 \cap \pi$, since $v_{i+1}' \in N$, $v_{i+1}' \notin L(z, u)$. Furthermore, if L_1 denotes the open halfplane of π determined by $L(z, u)$ and containing v_{i+1}', notice that $(N_\alpha \cap \pi) \subseteq \text{cl } L_1$.

Now if u were not in E, then using arguments in the proof of Theorem 1, u would be relatively interior to a segment (s, t) in $S \cap \pi$, with s and t on opposite sides of $L(z, u)$, say with s in L_1. However, then for some $0 < \beta < \alpha$, and for some b in bdry $(N_\beta \cap \pi)$, $R(z, b) \sim [z, b)$ would meet (s, u), which is impossible by our choice of α. Thus $u \in E$, the desired result.

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Now since u is in E, $z \in \text{int conv } S_u$. To finish the argument, we will show that this cannot occur. Let H be a hyperplane supporting the convex cone $K \equiv \cup \{R(z, v): v \text{ in } N_\alpha\}$ at point u, with K in the closed halfspace cl H_1 determined by H. Clearly $v_{i+1} \in H_1$ by our choice of α. Furthermore, u can see no point p in $S \cap H_1$, for otherwise the halfplane of aff $(L(z, u) \cup \{p\})$ determined by $L(z, u)$ and containing p would meet int N_α, and for some $0 < \beta < \alpha$ and some b in bdry N_β, $R(z, b) \sim [z, b)$ would meet (p, u), impossible by our choice of α. We conclude that $S_u \cap H_1 = \emptyset$ and $S_u \subseteq \text{cl } H_2$. However, $z \in H$ so this implies $z \notin \text{int conv } S_u$. We have a contradiction, our assumption must be false, and the ray $R(z, x) \sim [z, x)$ necessarily is disjoint from S. Therefore S has the half-ray property, and the proof of Lemma 4 is complete.

It is interesting to notice that the hypothesis

$$\cap\{\text{int conv } S_e: e \text{ in } E\} \neq \emptyset$$

in Lemma 4 may be replaced with the requirements that

$$\cap\{\text{conv } S_e: e \text{ in } E\} \neq \emptyset \text{ and } S \subset \mathbf{R}^2,$$

and we have the following corollary.

COROLLARY. *Let S be a compact set in $\mathbf{R}^2$, E the set of $(d - 2)$-extreme points of S, and assume that $\sim S$ is connected. If*

$$\cap\{\text{conv } S_e: e \text{ in } E\} \neq \emptyset,$$

then S has the half-ray property.

Proof. The argument involves only slight modifications in the proof of Lemma 4. Select $z \in \cap \{\text{conv } S_e: e \text{ in } E\}$ and proceed as in Lemma 4 to obtain $y \in C$, $u \in E$, and hyperplane H, with $S_u \subseteq \text{cl } H_2$. Notice that for S planar, $H = L(z, u)$. Furthermore, since $y \notin S$ and $y \in (u, z)$, u sees no point of S on $R(u, z) \sim [u, y)$. Hence

$$S_u \subseteq H_2 \cup R(y, u), z \notin \text{conv } S_u,$$

and we have the required contradiction.

To obtain an analogue of the Krasnosel'skii theorem, we use the approach given in [**7**, Lemma 6.2 and Theorem 6.17], suitably adapted for the set E of $(d - 2)$-extreme points of S.

LEMMA 5. *Let S be a nonempty compact set in $\mathbf{R}^d$, $d \geqq 2$, having the half-ray property. If $y \in S$ and $[x, y] \subseteq S$, then there exist a $(d - 2)$-extreme point e of S and a hyperplane H through e separating S_e from x.*

Proof. Select a point p in $(x, y) \sim S$. Since S has the half-ray property, there exists a ray l emanating from p and disjoint from S, and since S is compact, l may be chosen so that it is not collinear with x and y. Further-

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more, there is a convex neighborhood of l disjoint from S, and we may select a closed, spherical neighborhood V of p, $x \notin V$, and a point w collinear with l, $w \notin V \cup l$, so that the cone

$$C = \cup \{R(w, v): v \in V\}$$

is a closed neighborhood of l disjoint from S. Notice that $R(w, p)$ is the axis of C.

Let π denote the plane aff $(l \cup \{y\})$. Rotate the cone C in the following manner: Let μ represent the measure of the smaller angle in π determined by rays $R_\mu \equiv R(w, p)$ and $R_0 \equiv R(w, y)$. Then for $0 < \lambda < \mu$, there is a corresponding ray R_λ emanating from w and between R_0 and R_μ such that the angle determined by R_0 and R_λ has measure λ. Moreover, for $0 \leqq \lambda \leqq \mu$, there is a cone C_λ having axis R_λ and congruent to C. Choosing the largest α such that bdry C_α contains a point of S, clearly $0 < \alpha < \mu$. Finally, select a point e in $S \cap$ bdry C_α whose distance to w is maximal. If G denotes the hyperplane which contains the ray R_α and whose normal vector lies in π, notice that x and y lie in opposite open halfspaces G_1 and G_2, respectively, determined by G, and e lies in cl G_2.

Let H be the hyperplane supporting the cone C_α at e, with C_α in the closed halfspace cl H_1 determined by H. We assert that e and H satisfy the lemma. To see that e is a $(d - 2)$-extreme point of S, let L represent the line in π which contains w and is parallel to $L(x, y)$. Thus for $0 \leqq \lambda \leqq \alpha$, $L \cap C_\lambda = \{w\}$. Also, let π' denote the plane determined by L and e. By previous arguments, if e were not a $(d - 2)$-extreme point of S, e would be relatively interior to some segment in $\pi' \cap S$, with endpoints of this segment on opposite sides of $L(w, e)$. However, then for some $\beta > \alpha$, (bdry C_β) $\cap \pi'$ would contain points of S, violating our choice of α.

It remains to show that H separates S_e from x. Recall that $C_\alpha \subseteq$ cl H_1. Now e can see no point p in $C_\alpha \cap H_1$, for otherwise the halfplane determined by w, e, p and containing p would meet the interior of C_α, and for some $\beta > \alpha$, bdry $C_\beta \cap S \neq \emptyset$, and this is impossible. Thus $S_e \not\subseteq$ cl H_2. Finally, we see that $x \in H_1$: If e is in π, this is obvious. Otherwise, examine the 3-dimensional flat

$$\text{aff } (\pi \cup \{e\}) \equiv B.$$

Then $e \in (\text{cl } G_2) \cap B$, so clearly $x \in H_1 \cap B \subseteq H_1$, and Lemma 5 is proved.

Theorem 2 provides the desired analogue of Krasnosel'skii's theorem for the set of $(d - 2)$-extreme points of S. To simplify the statement of the theorem, we interpret a 0-dimensional ϵ-neighborhood to be a singleton point.

THEOREM 2. *Let S be a nonempty compact set in $\mathbf{R}^d$, $d \geqq 2$, having the half-ray property, and assume that for some $\epsilon > 0$, every $f(d, k)$ or fewer*

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$(d-2)$-extreme points of S see via S a common k-dimensional ϵ-neighborhood, where $f(d, 0) = f(\mathrm{d}, k) = d + 1$ *and* $f(d, k) = 2d$ *for* $1 \leqq k \leqq d - 1$. *Then S is starshaped and* $\dim \ker S \geqq k$.

Proof. The first part of our proof is an adaptation of the argument in [**7**, Theorem 6.17]. Let E denote the set of $(d-2)$-extreme points of S, and for z in E, let D_z represent the intersection of the closed halfspaces which contain S_z and whose boundaries contain z. (If no such halfspace exists, then $D_z = \mathbf{R}^d$.) We assert that $\ker S = \cap \{D_z: z \text{ in } E\}$: Clearly

$$\ker S \subseteq \cap \{S_z: z \text{ in } E\} \subseteq \cap \{D_z: z \text{ in } E\},$$

so we need only establish the reverse containment. Let x be a point in $\mathbf{R}^d \sim \ker S$. Then there is some y in S such that $[x, y] \nsubseteq S$, and by Lemma 5 there exist some $(d-2)$-extreme point e of S and corresponding hyperplane H through e separating S_e from x. Hence

$$x \notin D_e, \cap \{D_z: z \text{ in } E\} \subseteq \ker S,$$

and the sets are equal.

To complete the proof, define

$$\mathscr{D} \equiv \{D_z \cap \operatorname{conv} S: z \text{ in } E\}.$$

Then $\mathscr{D}$ is a uniformly bounded collection of compact convex sets in $\mathbf{R}^d$, and clearly

$$\cap \{D: D \text{ in } \mathscr{D}\} \equiv \cap \mathscr{D} = \ker S.$$

For $1 \leqq k \leqq d$, every $f(d, k)$ members of $\mathscr{D}$ contain a common k-dimensional ϵ-neighborhood, so by [**1**, Lemma], $\dim \cap \mathscr{D} \geqq k$. For $k = 0$, by a direct application of Helly's theorem in $\mathbf{R}^d$, $\cap \mathscr{D} \neq \emptyset$, and $\dim \cap \mathscr{D} \geqq 0$. Hence $\dim \ker S \geqq k$ for $0 \leqq k \leqq d$, and the theorem is proved.

In case E is finite, we use a theorem of Meir Katchalski to obtain the following corollary. Notice that the half-ray property may be replaced with the weaker requirement that $\sim S$ be connected.

COROLLARY 1. *Let S be a nonempty compact set in $\mathbf{R}^d$, $d \geqq 2$, with $\sim S$ connected and with the corresponding set E of $(d-2)$-extreme points of S finite. Assume that every $g(2, k)$ or fewer points in E see via S a common k-dimensional neighborhood, where $g(2, 0) = g(2, 2) = 3$ and $g(2, 1) = 4$. Then* $\dim \ker S \geqq k$.

Proof. It is easy to show that $S \neq E$, and hence S must be planar by Theorem 1. Thus by previous arguments, we may assume that $d = 2$. Define the family

$$\mathscr{C} \equiv \{\operatorname{conv} S_z: z \text{ in } E\} \neq \emptyset.$$

Then $\mathscr{C}$ is a finite family of convex sets in the plane, every $g(2, k)$ or fewer members of $\mathscr{C}$ have at least a k-dimensional intersection, so by [**3**], $\dim \cap \mathscr{C} \geqq k$. Since

$$\cap\{\text{conv } S_z : z \text{ in } E\} \neq \emptyset,$$

by the corollary to Lemma 4, S has the half-ray property. To complete the argument, define the family

$$\mathscr{D} \equiv \{D_z : z \text{ in } E\}$$

as in the proof of Theorem 2 above and use Lemma 5 to show that $\cap \mathscr{D} = \ker S$. Since $\text{conv } S_z \subseteq D_z$ for z in E,

$$\cap \{\text{conv } S_z : z \text{ in } E\} \subseteq \cap \{D_z : z \text{ in } E\} = \ker S.$$

Clearly $\ker S \subseteq \cap \{\text{conv } S_z : z \text{ in } E\}$, so the sets are equal, and

$$\dim \ker S = \dim \cap \mathscr{C} \geqq k.$$

COROLLARY 2. *Let S be a nonempty compact set in $\mathbf{R}^d$, $d \geqq 2$, with $\sim S$ connected. Assume that for some $\epsilon > 0$, every $d + 1$ or fewer $(d - 2)$-extreme points of S see via S a common d-dimensional ϵ-neighborhood. Then* $\dim \ker S = d$.

Proof. Again let E denote the set of $(d - 2)$-extreme points of S. Apply [**1**, Lemma] to the family $\mathscr{C} = \{\text{conv } S_z : z \text{ in } E\}$ to conclude that $\dim \cap \mathscr{C} = d$. Hence

$$\cap \{\text{int conv } S_z : z \text{ in } E\} \neq \emptyset,$$

and by Lemma 4, S has the half-ray property. Finally, using the argument in Corollary 1 above, $\cap \mathscr{C} = \ker S$ and $\dim \ker S = d$.

In conclusion, we remark that if S is the boundary of a planar triangle (an example given in [**6**]), then $\cap \{\text{conv } S_z : z$ a $(d - 2)$-extreme point of $S\}$ is the 2-dimensional set conv S, although the compact set S is not starshaped. Thus the hypothesis that $\sim S$ be connected (or that S have the half-ray property) is needed in Theorem 2 and its corollaries.

REFERENCES

1. M. Breen, *K-dimensional intersections of convex sets and convex kernels*, submitted.

2. P. R. Goodey, *A note on starshaped sets*, Pacific J. Math. *61* (1975), 151–152.

3. M. Katchalski, *The dimension of intersections of convex sets*, Israel J. Math. *10* (1971), 465–470.

4. J. W. Kenelly, W. R. Hare et. al. *Convex components, extreme points, and the convex kernel*, Proc. Amer. Math. Soc. *21* (1969), 83–87.

5. M. A. Krasnosel'skii, *Sur un critère pour qu'un domain soit étoilé*, Math. Sb. (61) *19* (1946), 309–310.

6. N. Stavrakas, *A note on starshaped sets, (k)-extreme points and the half ray property*, Pacific J. Math. *53* (1974), 627–628.

7. F. A. Valentine, *Convex sets* (McGraw Hill, 1964).

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