Canad. Math. Bull. Vol. **56** (2), 2013 pp. 395–399
http://dx.doi.org/10.4153/CMB-2011-172-3

Coessential Abelianization Morphisms in the Category of Groups

D. Oancea

Abstract. An epimorphism $\phi\colon G \to H$ of groups, where G has rank n, is called *coessential* if every (ordered) generating n-tuple of H can be lifted along ϕ to a generating n-tuple for G. We discuss this property in the context of the category of groups, and establish a criterion for such a group G to have the property that its abelianization epimorphism $G \to G/[G,G]$, where $[G,G]$ is the commutator subgroup, is coessential. We give an example of a family of 2-generator groups whose abelianization epimorphism is not coessential. This family also provides counterexamples to the generalized Andrews–Curtis conjecture.

1 Introduction

For a group G of rank n, that is, generated by n elements but no fewer, and for any $k \geq n$, we consider here whether any generating k-tuple of the abelianization, $G_{\mathrm{ab}} := G/[G,G]$, of G can be lifted naturally to a generating k-tuple of G. Clearly this can always be done if every generating k-tuple of G_{ab} can be transformed into every other by means of Nielsen transformations (since then any given generating k-tuple of G_{ab} is Nielsen equivalent to the image under the abelianization epimorphism $\varphi\colon G \to G_{\mathrm{ab}}$ of any generating k-tuple of G). Now by a theorem of [3] extended to the case of finitely generated abelian groups [6, 7], this is the case if $k > n$ or $k = n$ and either $\operatorname{rank}(G_{\mathrm{ab}}) < n$ or G_{ab} is free abelian. Thus the remaining nontrivial case is when $k = n$, $\operatorname{rank}(G_{\mathrm{ab}}) = \operatorname{rank}(G) = n$, and G_{ab} is not free abelian.

This question arose in the course of our quantitative investigation [2, 6] of the Andrews–Curtis problem for an arbitrary group G of rank n, whether every normal generating n-tuple $(h_1, h_2, \dots, h_n)$, that is, with normal closure all of G, can be transformed by means of Andrews–Curtis transformations to a generating n-tuple. Since an easy necessary condition for this to be true is that there exist a generating n-tuple of G which has the same image as $(h_1, \dots, h_n)$ in G_{ab}, this naturally prompts our question whether any generating n-tuple of G_{ab} can be lifted to a generating n-tuple of G. We find here a sufficient condition for the answer to be affirmative, and we also exhibit a family of 2-generator groups for which the answer is negative. Some of these groups are also counterexamples to the generalized Andrews–Curtis conjecture, that is, they have normal generating pairs not transformable to generating pairs by means of Andrews–Curtis moves.

A somewhat related question asks when the rank of G_{ab} is equal to the rank of G. We refer the reader to [5] for a result in this general direction.

Received by the editors October 26, 2010; revised May 12, 2011.
Published electronically September 15, 2011.
AMS subject classification: **20F05**, 20F99, 20J15.
Keywords: coessential epimorphism, Nielsen transformations, Andrew-Curtis transformations.

2 Definitions and Statement of the Theorem

In the case where $[G, G]$ is finite, one obtains an affirmative answer to our question immediately from the following more general result of Gaschütz [4] (sometimes called in one form or another the Gaschütz lemma): if $\varphi\colon G \to H$ is any epimorphism of groups with finite kernel and $k \geq \operatorname{rank}(G)$, then, corresponding to any generating k-tuple $(h_1, \dots, h_k)$ for H, there is a generating k-tuple $(g_1, \dots, g_k)$ for G such that $\varphi(g_i) = h_i,\ i = 1, \dots, k$.

We first reformulate our question more generally in terms of category theory and diagrams. A generating k-tuple $(g_1, \dots, g_k)$ of a group determines in the usual way an epimorphism π from the free group F_k on k free generators $x_1, \dots, x_k$ to $\langle g_1, \dots, g_k\rangle$, namely that given by $x_i \mapsto g_i,\ i = 1, \dots, k$. Hence the question is equivalent to the following one: can every diagram

$$\begin{array}{ccc} & & F_k \\ & & \big\downarrow{\scriptstyle \pi} \\ G & \overset{\epsilon}{\twoheadrightarrow} & G_{ab} \end{array}$$

be completed to a commutative diagram

$$\begin{array}{ccc} & & F_k \\ & {\scriptstyle \varphi}\swarrow & \big\downarrow{\scriptstyle \pi} \\ G & \overset{\epsilon}{\twoheadrightarrow} & G_{ab} \end{array}$$

for all $k \geq n$, where $n = \operatorname{rank}(G)$?

Recall that in a category **C**, an object P is called *projective* if for any epimorphism $\epsilon\colon A \to B$ in **C** the map $\operatorname{Hom}(P, \epsilon)\colon\ \operatorname{Hom}(P, A) \to \operatorname{Hom}(P, B)$ (defined by $\varphi \mapsto \epsilon\varphi$) is an epimorphism in **Set**. In **Grp** the free groups are projective, so for arbitrary groups G, H, and epimorphism $\epsilon\colon G \to H$, and for any morphism $\pi\colon F_k \to H$, there exists a morphism $\varphi\colon F_k \to G$ yielding a commutative diagram

$$\begin{array}{ccc} & & F_k \\ & {\scriptstyle \varphi}\swarrow & \big\downarrow{\scriptstyle \pi} \\ G & \overset{\epsilon}{\twoheadrightarrow} & H \end{array}$$

If ϵ has the form $\epsilon\colon G \to G/N$ for N a *finite* normal subgroup of G with $\operatorname{rank}(G) \leq k$, then by the result of Gaschütz quoted above there exists an *epi*morphism φ whenever π is epi. (However, this is not true in general. An easy example is $\epsilon\colon C_\infty \to C_5$, where $C_\infty (= F_1 = G)$ is the infinite cyclic group generated by x, $C_5 = \langle x \mid x^5\rangle$, the cyclic group of order 5, and $\pi\colon F_1 \to C_5$ is defined by $x \mapsto \epsilon(x^2)$. The generator $\epsilon(x^2)$ of C_5 cannot be pulled back along ϵ to a generator of $G = C_\infty$.)

Definition 2.1 (See [1]) Let **C** be a category. An epimorphism $\epsilon\colon A \to B$ in **C** is called *coessential* if the map $\mathrm{Epi}(P,\epsilon)\colon \mathrm{Epi}(P,A) \to \mathrm{Epi}(P,B)$ is epi in **Set** for any projective object P in **C**.

From our initial discussion we know that the abelianization morphism is coessential for finitely generated groups with finite commutator subgroup (and so, in particular, for finite groups), and for any finitely generated group G with G_{ab} torsion-free or satisfying $\mathrm{rank}(G_{\mathrm{ab}}) < \mathrm{rank}(G)$.

Below we establish the coessentiality of the abelianization morphism for finitely generated groups G in the case that G_{ab} has torsion, $\mathrm{rank}(G_{\mathrm{ab}}) = \mathrm{rank}(G)$, and G satisfies an additional condition, which we now describe. Thus, suppose $\mathrm{rank}(G_{\mathrm{ab}}) = \mathrm{rank}(G) = n$, and that G_{ab} has torsion. As before, denote by $\varphi\colon G \to G_{\mathrm{ab}}$ the natural epimorphism, and write $G_{\mathrm{ab}} = Z_1 \times \cdots \times Z_{l-1} \times Z_l \times \cdots \times Z_n$, where for $1 \le i \le l$ the Z_i are finite cyclic groups of orders m_i, with m_j dividing m_{j+1} for $j = 1,\dots,l-1$, and for $l < i \le n$, Z_i is infinite cyclic.

Let $\mathbf{g} = (g_1,\dots,g_n)$ be a generating n-tuple for G. Then since $\bar{\mathbf{g}} = \varphi(\mathbf{g})$ generates G_{ab}, we have by a theorem of [3] as generalized in [6,7] that some Nielsen transform of $\bar{\mathbf{g}}$ has the form $(z_1,\dots,z_n)$, where z_i generates Z_i, $1 \le i \le n$. Hence some Nielsen transform of $\mathbf{g}$ maps to $(z_1,\dots,z_n)$, and we may assume without loss of generality that in fact $\bar{\mathbf{g}} = (z_1,\dots,z_n)$. This assumed, we call the element g_1, a *distinguished generator* of G. Our main result is as follows.

Theorem 2.2 *The abelianization morphism $G \xrightarrow{\varphi} G_{\mathrm{ab}}$ of a finitely generated group G is coessential if G satisfies any one of the following:*

(i) G_{ab} *is torsion-free;*
(ii) $\mathrm{rank}(G_{\mathrm{ab}}) < \mathrm{rank}(G)$*;*
(iii) $\mathrm{rank}(G_{\mathrm{ab}}) = \mathrm{rank}(G)$ *and G has a distinguished generator of finite order.*

3 Proof of the Theorem, and Corollaries

If either of the first two conditions is satisfied, then φ is coessential by the above discussion. Thus we may suppose that G_{ab} has torsion, that $\mathrm{rank}(G_{\mathrm{ab}}) = \mathrm{rank}(G) = n$, and that we have a generating n-tuple $(g_1,\dots,g_n)$ for G with image $\bar{\mathbf{g}} = (z_1,\dots,z_n)$ under φ, where z_i generates Z_i, $i = 1,\dots n$ in the above notation, and where, furthermore, g_1 has finite order.

Now let $\bar{\mathbf{y}}$ be an arbitrary generating n-tuple for G_{ab}. Then by a theorem of [3], generalized to the case of finitely generated abelian groups in [6,7], this n-tuple can be Nielsen-transformed to a generating n-tuple of the form $(z_1^t, z_2,\dots,z_n)$ for some $t \in (\mathbb{Z}/m_1\mathbb{Z})^*$ where $m_1 = |Z_1|$, that is, for some integer t, $0 < t < m_1$ relatively prime to m_1. For simplicity, we write $m_1 = m$. Since g_1 has finite order, we must have $|g_1| = sm$ for some $s \ge 1$. For every integer r, every n-tuple of the form $(g_1^{t+rm}, g_2,\dots,g_n)$ is sent under φ to $(z_1^t, z_2,\dots,z_n)$. We have the following lemma.

Lemma 3.1 *There exists an integer r such that $(t + rm, sm) = 1$, so that $(g_1^{t+rm}, g_2,\dots,g_n)$ is a generating n-tuple for G.*

https://doi.org/10.4153/CMB-2011-172-3 Published online by Cambridge University Press

Proof of Theorem 2.2 Assuming Lemma 3.1 for the moment, we then have that our initial arbitrary generating n-tuple $\bar{\mathbf{y}}$ of G_{ab} can be Nielsen transformed to the n-tuple $(z_1^t, z_2, \dots, z_n)$ of G_{ab}, which is the image under φ of the generating n-tuple $(g_1^{t+rm}, g_2, \dots, g_n)$ of G. Applying to this n-tuple the inverse of the Nielsen transformation used to get from $\bar{\mathbf{y}}$ to $(z_1^t, z_2, \dots, z_n)$, we obtain a generating n-tuple $\mathbf{y}$ of G, which maps to $\bar{\mathbf{y}}$ under φ. ■

Thus any generating n-tuple of G_{ab} can be lifted to a generating n-tuple of G, which is equivalent to saying that the abelianization morphism of G is coessential.

Proof of Lemma 3.1 We wish to show that there exists an integer r such that $(t + rm, sm) = 1$. Note first that if an integer d divides $(t + rm, sm)$, then we must have $(d, m) = 1$, since otherwise $(t, m) \neq 1$. Hence any such d must divide s and be relatively prime to m. Let P denote the set of prime factors of s that do not divide t.

If P is empty, choose $r = 1$. By the above, if p is a common prime factor of $t + rm = t + m$ and sm, p must divide s and therefore also t (since P is empty). However, then p must divide m, giving a contradiction. Thus $(t + m, sm) = 1$.

If P is not empty, choose $r = \Pi_{p_i \in P} p_i$. Suppose again that p is a prime dividing $(t + rm, sm)$. Recall from earlier that p must divide s but not m. If p divides t, then $p \notin P$, so that p does not divide r. Hence p does not divide rm, contradicting the assumption that p divides $t + rm$. If on the other hand p does not divide t, then p divides rm (since it divides r), so it does not divide $t + rm$, and we again have a contradiction. ■

Corollary 3.2 *The abelianisation morphism of a finitely generated group is coessential if $[G, G]$ is periodic, that is, if every element of $[G, G]$ has finite order.*

Corollary 3.3 *Let G be a finitely generated group satisfying any one of the conditions of the theorem, and let $k \geq \operatorname{rank}(G)$. Then for any annihilating k-tuple $(u_1, \dots, u_k)$ of G (that is, with normal closure all of G) there exists a generating k-tuple $(g_1, \dots, g_k)$ and elements $c_i \in [G, G]$, $1 \leq i \leq k$, such that $u_i = g_i c_i$.*

Proof For every $g \in G$, write as before $\bar{g} := \varphi(g)$. Since $(u_1, \dots, u_k)$ is annihilating for G, its image $(\bar{u}_1, \dots, \bar{u}_k)$ must be generating for G_{ab}. By the theorem, there is a generating k-tuple $(g_1, \dots, g_k)$ such that $\bar{g}_i = \bar{u}_i$, $i = 1, \dots, k$, which simply means that for $i = 1, \dots, k$, $u_i = g_i c_i$ for some elements $c_i \in [G, G]$. ■

4 Groups with Non-Coessential Abelianization Morphism

According to Pride [8], the group with presentation $G = \langle a, t | (at^{-1}a^3t)^n \rangle$, $n > 1$ has the property that every pair of generators is Nielsen equivalent to every other, that is, every generating pair can be transformed into every other such pair by means of a Nielsen transformation. Thus every generating pair can be obtained from the pair (a, t) by means of a Nielsen transformation. However, here we have $G_{\mathrm{ab}} \cong \mathbf{Z}_{4n} \times \mathbb{Z}$, where $\mathbf{Z}_{4n}$ denotes the cyclic group of order $4n$, and by the generalization of the theorem of [3] given in [6, 7], this abelian group has more than one Nielsen equivalence class, so that it must be the case that some pair of generators of G_{ab} cannot be lifted to a generating pair for G.

In fact the group G is, at least for certain n, a counterexample to the generalized Andrews–Curtis conjecture in the sense that it has a normal generating pair not transformable to a generating pair by means of Andrews–Curtis moves. For if n is not divisible by 3, the pair (a^3, t) is annihilating for G_n since it follows from $a^3 = 1$ and $(at^{-1}a^3t)^n = 1$ that $a^n = 1$, and then from $a^3 = 1$ and $a^n = 1$ that $a = 1$. If the pair (a^3, t) could be AC-transformed to (a, t), it would follow that the image pair $(\bar{a}^3, \bar{t})$ could be Nielsen transformed into $(\bar{a}, \bar{t})$ in the abelianization $G/[G, G] \cong \mathbf{Z}_{4n} \times \mathbb{Z}$. However, by [7] these two pairs lie in different Nielsen classes in this abelian group.

Acknowledgement This work formed part of the author's Ph.D. dissertation (York University, 2010). The author thanks his supervisor R. G. Burns for help with the presentation of this paper.

References

[1] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories; The Joy of Cats.* Repr. Theory Appl. Categ. **17**(2006), 1–507.

[2] R. G. Burns and D. Oancea, *Recalcitrance in groups II.* J. Group Theory, Published online 12/05/2011. http://dx.doi.org/10.1515/JGT.2011.098

[3] P. Diaconis and R. Graham, *The graph of generating sets of an abelian group.* Colloq. Math. **80**(1999), no. 1, 31–38.

[4] W. Gaschütz, *Zu einem von B. H. und H. Neumann gestellten Problem.* Math. Nachr. **14**(1955), 249–252. http://dx.doi.org/10.1002/mana.19550140406

[5] F. Dalla Volta and A. Lucchini., *Finite groups that need more generators than any proper quotient.* J. Austral. Math. Soc. Ser. A **64**(1998), no. 1, 82–91. http://dx.doi.org/10.1017/S1446788700001312

[6] D. Oancea. *The Andrews-Curtis Conjecture and the Recalcitrance of Groups.* Ph.D. Dissertation, York University, 2010.

[7] ———, *A note on Nielsen equivalence in finitely generated abelian groups.* Bull. Australian Math. Soc. **84**(2011), 127–136. http://dx.doi.org/10.1017/S0004972711002279

[8] S. J. Pride, *On the Nielsen equivalence of pairs of generators in certain HNN groups.* In: Proceedings of the Second International Conference on the Theory of Groups. Lecture Notes in Math. 372. Springer, Berlin, 1974, pp. 580-588.

1549 Victoria St. E., Whitby, ON, L1N 9E3
e-mail: daniel.oancea@opg.com