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Hardy Space Estimate for the Product of Singular Integrals

To the memory of Akihito Uchiyama

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Abstract. H^p estimate for the multilinear operators which are finite sums of pointwise products of singular integrals and fractional integrals is given. An application to Sobolev space and some examples are also given.

1 Introduction

For $0 \leqq \lambda < \infty$, we define $G(\lambda)$ as the set of all those C^∞ functions a on $\mathbb{R}^n \setminus \{0\}$ such that

$$|\partial_\xi^\alpha a(\xi)| \leqq c_\alpha |\xi|^{-\lambda-|\alpha|}$$

for every multi-index α.

Let $\mathcal{S}$ denote the Schwartz class of testing functions. We denote by $\mathcal{S}_0$ the set of all those $f \in \mathcal{S}$ such that $\hat{f}(\xi)$, the Fourier transform, vanishes in a neighbourhood of $\xi = 0$.

If $a \in G(\lambda)$, then we define the linear operator $T \colon \mathcal{S}_0 \to \mathcal{S}_0$ by

$$Tf = (a\hat{f})^\vee \quad (f \in \mathcal{S}_0),$$

where $^\vee$ denotes the inverse Fourier transform. The function a is called the multiplier of T. We denote by $\mathcal{K}(\lambda)$, $0 \leqq \lambda < \infty$, the set of all the operators T corresponding to the multipliers $a \in G(\lambda)$.

Let $H^p = H^p(\mathbb{R}^n)$, $0 < p \leqq 1$, denote the usual real variable Hardy space on $\mathbb{R}^n$. We define $H^p = L^p = L^p(\mathbb{R}^n)$ for $1 < p < \infty$. For H^p, see, *e.g.*, [S, Chap. III].

The following H^p-H^q estimate of the operators of class $\mathcal{K}(\lambda)$ is well known: If $T \in \mathcal{K}(\lambda)$, $0 \leqq \lambda < \infty$, then, for p and q satisfying

$$0 < p \leqq q < \infty \quad \text{and} \quad \frac{1}{p} - \frac{1}{q} = \frac{\lambda}{n}, \tag{1.1}$$

the estimate

$$\|Tf\|_{H^q} \leqq c\|f\|_{H^p} \tag{1.2}$$

holds for all $f \in \mathcal{S}_0$. See [CT, Section 4].

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In this paper, we consider the multilinear operator Λ defined by

$$\Lambda(f_1, \dots, f_k) = \sum_{\sigma \in \mathcal{A}} (T_1^\sigma f_1) \cdots (T_k^\sigma f_k) \tag{1.3}$$

for $f_1, \dots, f_k \in \mathcal{S}_0$, where $\mathcal{A}$ is a finite index set and T_j^σ are linear operators such that

$$T_j^\sigma \in \mathcal{K}(\lambda_j^\sigma), \quad 0 \leqq \lambda_j^\sigma < \infty \quad (\sigma \in \mathcal{A}, j = 1, \dots, k). \tag{1.4}$$

(Each term in the right hand side of (1.3) is the pointwise product of the k functions $T_j^\sigma f_j$.) This Λ is well-defined as a multilinear operator $(\mathcal{S}_0)^k \to \mathcal{S}$.

We consider the case where

$$\sum_{j=1}^{k} \lambda_j^\sigma = \lambda \text{ is independent of } \sigma \in \mathcal{A}. \tag{1.5}$$

Let $p_1, \dots, p_k$ and q be positive real numbers such that

$$\infty > \frac{1}{p_j} > \frac{\lambda_j^\sigma}{n} \quad (\sigma \in \mathcal{A}, j = 1, \dots, k) \tag{1.6}$$

and

$$\sum_{j=1}^{k} \left(\frac{1}{p_j} - \frac{\lambda_j^\sigma}{n} \right) = \frac{1}{q}. \tag{1.7}$$

Then clearly we have the estimate

$$\|\Lambda(f_1, \dots, f_k)\|_q \leqq c \|f_1\|_{H^{p_1}} \cdots \|f_k\|_{H^{p_k}}. \tag{1.8}$$

(We write $\| \cdot \|_r$ to denote the quasinorm in $L^r(\mathbb{R}^n)$; see Section 2.1.) Indeed, if we write $1/q_j^\sigma = 1/p_j - \lambda_j^\sigma/n$, then the estimate (1.2)–(1.1) implies

$$\|T_j^\sigma f_j\|_{H^{q_j^\sigma}} \leqq c \|f_j\|_{H^{p_j}}.$$

Hence, we use Hölder's inequality to obtain

$$\begin{aligned} \|\Lambda(f_1, \dots, f_k)\|_q &\leqq c \sum_{\sigma \in \mathcal{A}} \|(T_1^\sigma f_1) \cdots (T_k^\sigma f_k)\|_q \leqq c \sum_{\sigma \in \mathcal{A}} \prod_{j=1}^{k} \|T_j^\sigma f_j\|_{q_j^\sigma} \\ &\leqq c \sum_{\sigma \in \mathcal{A}} \prod_{j=1}^{k} \|T_j^\sigma f_j\|_{H^{q_j^\sigma}} \leqq c \prod_{j=1}^{k} \|f_j\|_{H^{p_j}}. \end{aligned}$$

(We use the letter c to denote various positive constants which may be different in each occasion.)

The subject of this paper is to show that, under certain assumptions on Λ, the L^q-quasinorm in (1.8) can be replaced by the H^q-quasinorm as

$$\|\Lambda(f_1,\ldots,f_k)\|_{H^q} \leqq c\|f_1\|_{H^{p_1}}\cdots\|f_k\|_{H^{p_k}}. \tag{1.9}$$

Of course only the case $q \leqq 1$ is interesting. (If $1 < q < \infty$, then $H^q = L^q$ and (1.8) and (1.9) are the same.)

If $0 < q \leqq 1$, then $\mathcal{S}$ is not included in H^q. The fact is this: $f \in \mathcal{S} \cap H^q$, $0 < q \leqq 1$, if and only if $f \in \mathcal{S}$ and

$$\int_{\mathbb{R}^n} f(x)x^\alpha\, dx = 0 \quad \text{for } |\alpha| \leqq [n/q - n].$$

Therefore, when $0 < q \leqq 1$, in order that (1.9) holds it is necessary that the moment condition

$$\int_{\mathbb{R}^n} \Lambda(f_1,\ldots,f_k)(x)x^\alpha\, dx = 0 \quad \text{for } |\alpha| \leqq [n/q - n] \tag{1.10}$$

is satisfied for all $f_1,\ldots,f_k \in \mathcal{S}_0$.

The purpose of this paper is to show that (1.10) is also sufficient when $k = 2$ or when $k \geqq 3$ and all the operators T_j^σ are homogeneous operators. The precise statement shall now be given below.

We say that an operator $T \in \mathcal{K}(\lambda)$ is *homogeneous* if its multiplier $a \in G(\lambda)$ is a homogeneous function, *i.e.*, if $a(t\xi) = t^{-\lambda}a(\xi)$ for all $t > 0$ and all $\xi \in \mathbb{R}^n \setminus \{0\}$. (Clearly a homogeneous function in the class $G(\lambda)$ is homogeneous of degree $-\lambda$.)

The main result of this paper reads as follows.

Theorem *Let* Λ *be given by* (1.3) *with* (1.4). *Suppose* λ_j^σ *satisfy* (1.5). *Let* $p_1,\ldots,p_k$ *and* q *satisfy* (1.6) *and* (1.7). *Suppose* $q \leqq 1$ *and the moment condition* (1.10) *is satisfied for all* $f_1,\ldots,f_k \in \mathcal{S}_0$. *Then:*

(a) If $k = 2$, *then the estimate* (1.9) *holds for all* $f_1,\ldots,f_k \in \mathcal{S}_0$;
(b) If $k \geqq 3$ *and if all the operators* T_j^σ *are homogeneous, then the estimate* (1.9) *holds for all* $f_1,\ldots,f_k \in \mathcal{S}_0$.

The homogeneity assumption in (b) can be removed if we assume further moment conditions; see Remark at the end of Section 5. The present author does not know whether the homogeneity assumption in (b) can entirely be removed.

In fact, there already exist several papers dealing with this kind of estimate (as we shall see below). Our result improves the previously known results in the following points. First, our theorem treats the full range $0 \leqq \lambda_j^\sigma < \infty$; the case $\lambda_j^\sigma = 0$ or the case $\lambda = \sum_{j=1}^k \lambda_j^\sigma < n$ are already treated. Second, the assumption of our theorem for the case $k \geqq 3$ is simplified compared with the previous theorems; *cf.* [G]. Thirdly, in the proof of our theorem, we shall give a rather explicit pointwise estimate for the maximal function of $\Lambda(f_1,\ldots,f_k)$, which will be of independent interest.

Several interesting examples together with applications of the estimate of the form (1.9) are given in the paper by Coifman-Lions-Meyer-Semmes [CLMS]. Some examples will also be given in the last section of the present paper.

We shall now review some previous works concerning the same subject.

The simplest case of the estimate (1.9) is for $n = 1$, $k = 2$, and for

$$\Lambda(f_1, f_2) = f_1 \tilde{f}_2 + \tilde{f}_1 f_2 \quad \text{or} \quad f_1 f_2 - \tilde{f}_1 \tilde{f}_2,$$

where $\tilde{}$ denotes the Hilbert transform. In this case, the estimate (1.9) can be immediately derived from Hölder's inequality and the Burkholder-Gundy-Silverstein theorem [BGS] (this theorem gives a characterization of $H^p(\mathbb{R})$ in terms of the classical Hardy class of holomorphic functions of one variable).

The first result for $n \geqq 2$ was given by Coifman-Rochberg-Weiss [CRW, Theorems I and II] for the case $k = 2$, $\lambda_j^\sigma = 0$, and $q = 1$. Chanillo [Ch] treated the case $k = 2$, $0 < \lambda < n$, and $q = 1$. The method used in [CRW] and [Ch] was to use the H^1-BMO duality and thus was restricted to the case $q = 1$. Uchiyama [U] introduced a method which directly estimate certain maximal functions and extended the result of [CRW] to the case $k = 2$, $\lambda_j^\sigma = 0$, and $n/(n+1) < q \leqq 1$. Generalizing Uchiyama's method, Komori [K1] and the present author [M1] treated the case $k = 2$, $0 < \lambda < n$, $n/(n+1) < q \leqq 1$ and the case $k = 2$, $\lambda_j^\sigma = 0$, $0 < q \leqq n/(n+1)$, respectively. These were further generalized by [M2] to the case $k = 2$, $0 \leqq \lambda < n$, and $0 < q \leqq 1$.

In fact, the papers cited above do not treat Λ of the general form (1.3) but treat Λ of a specified form. The methods of [M1] and [M2], however, can be applied to the general Λ with $k = 2$ without essential change.

The case $k \geqq 3$ with $\lambda_j^\sigma = 0$ was considered by Grafakos [G]. The theorems given in [G] contained certain restrictions on the parameters $p_1, \dots, p_k$; Komori [K2] showed that those theorems can be generalized to the entire range $0 < p_j < \infty$.

The contents of the succeeding sections are as follows. In Section 2, we fix several notations and recall some preliminary facts. Sections 3 through 5 are devoted to the proof of Theorem. In Section 6, we give some examples.

2 Preliminaries

2.1 Notations

As well as the notations already introduced in Section 1, the following notations are used throughout this paper.

The letter $\mathbb{N}$ denotes the set of positive integers; $\mathbb{N}$ does not contain 0. For $x \in \mathbb{R}$, $[x]$ denotes the integer which satisfies $[x] \leqq x < [x] + 1$.

In this paper, we consider functions and function spaces defined on $\mathbb{R}^n$; letter n always denotes the dimension of the basic space $\mathbb{R}^n$. If E is a measurable subset of $\mathbb{R}^n$ and $0 < p \leqq \infty$, then $\|\cdot\|_{p,E}$ denotes the quasinorm in $L^p(E)$, *i.e.*, for measurable functions f defined on E, we define

$$\|f\|_{p,E} = \left(\int_E |f(x)|^p \, dx \right)^{1/p}$$

with the usual modification for $p = \infty$. If $E = \mathbb{R}^n$, then $\|f\|_{p,E}$ is simply denoted by $\|f\|_p$. The symbol $B(x, t)$ denotes the open ball in $\mathbb{R}^n$ with respect to the usual Euclidean metric with center $x \in \mathbb{R}^n$ and with radius t, $0 < t < \infty$. The value of the distribution f evaluated at the testing function φ is denoted by $\langle f, \varphi \rangle$.

We fix a function ϕ on $\mathbb{R}^n$ which has the following properties: ϕ is C^∞, $0 \leqq \phi(x) \leqq 1$ for all $x \in \mathbb{R}^n$, $\phi(x) = 1$ for $|x| \leqq 1$, and $\operatorname{supp} \phi \subset B(0, 2)$.

2.2 Operators of Class $\mathcal{K}(\lambda)$

Let $T \in \mathcal{K}(\lambda)$, $0 \leqq \lambda < \infty$. As was already mentioned in Section 1, the H^p-H^q estimate (1.2)–(1.1) holds for all $f \in \mathcal{S}_0$. Hence, since $\mathcal{S}_0$ is dense in H^p for every $0 < p < \infty$, the operator $T\colon \mathcal{S}_0 \to \mathcal{S}_0$ can be uniquely extended to a bounded operator $H^p \to H^q$ for each (p, q) satisfying (1.1). If the condition (1.1) is satisfied for $(p, q) = (p_1, q_1)$ and for $(p, q) = (p_2, q_2)$, then the extended $T\colon H^{p_1} \to H^{q_1}$ and the extended $T\colon H^{p_2} \to H^{q_2}$ coincide on $H^{p_1} \cap H^{p_2}$. Therefore the extensions of T give rise to a well-defined mapping

$$\bigcup_{0<p<n/\lambda} H^p \longrightarrow \bigcup_{0<q<\infty} H^q.$$

In the sequel we shall use the same symbol T to denote the last mapping.

If $T \in \mathcal{K}(0)$, then Tf for $f \in \mathcal{S}_0$ can be written as

$$(Tf)(x) = \gamma f(x) + \lim_{j\to\infty} \int_{|y|>\epsilon_j} A(y) f(x-y)\, dy, \tag{2.1}$$

where γ is a complex constant, A is a function in $G(n)$ such that

$$\sup_{0<a<b<\infty} \left| \int_{a<|y|<b} A(y)\, dy \right| < \infty,$$

and (ϵ_j) is a sequence such that $\epsilon_j > 0$ and $\lim_{j\to\infty} \epsilon_j = 0$; the converse also holds. The formula (2.1) can also be applied to some extensions of T. For example, it holds for all $f \in \mathcal{S}$ and for all $x \in \mathbb{R}^n$. If $f \in L^p$ with $1 < p < \infty$, then (2.1) holds almost everywhere. For these facts, see, *e.g.*, [S, Chap. VI, Section 4, and Chap. VII, Section 3].

If $T \in \mathcal{K}(\lambda)$ with $0 < \lambda < n$, then there exists an $A \in G(n - \lambda)$ such that

$$(Tf)(x) = \int_{\mathbb{R}^n} A(y) f(x-y)\, dy$$

for $f \in \mathcal{S}_0$; the converse also holds. This formula can be applied also to f which is in L^∞ and has compact support. For these facts, see, *e.g.*, [S, Chap. VI, Section 4, Proposition 1].

2.3 The Vanishing Moment Condition

For nonnegative integers M, we denote by $\mathcal{P}_M$ the set of polynomial functions on $\mathbb{R}^n$ of degree not exceeding M. If M is a negative integer, we define $\mathcal{P}_M = \{0\}$.

Let f be a locally integrable function on $\mathbb{R}^n$ (or let f be a distribution with compact support) and let M be an integer. If $fP \in L^1$ and $\int f(x)P(x)\, dx = 0$ for all $P \in \mathcal{P}_M$ (or if $\langle f, P\rangle = 0$ for all $P \in \mathcal{P}_M$, resp.), then we write $f \perp \mathcal{P}_M$.

If M is a negative integer, then every f satisfies $f \perp \mathcal{P}_M$ since $\mathcal{P}_M = \{0\}$ by our definition.

Now let Λ be the operator defined by (1.3) with (1.4) and let M be an integer.

We say Λ satisfies the *vanishing moment condition up to order M* if $\Lambda(f_1, \ldots, f_k) \perp \mathcal{P}_M$ for all $f_1, \ldots, f_k \in \mathcal{S}_0$. We say Λ satisfies the vanishing moment condition *of all orders* if $\Lambda(f_1, \ldots, f_k) \perp \mathcal{P}_M$ for all $M \in \mathbb{N}$ and for all $f_1, \ldots, f_k \in \mathcal{S}_0$.

If $k = 1$, then Λ is a finite sum of operators of class $\bigcup_{\lambda>0} \mathcal{K}(\lambda)$ and, hence, for every $f \in \mathcal{S}_0$, we have $\Lambda(f) \in \mathcal{S}_0$ and $\Lambda(f) \perp \mathcal{P}_M$ for every M. Thus Λ with $k = 1$ satisfies the vanishing moment condition of all orders. If $M < 0$, then we can say that every Λ satisfies the vanishing moment condition up to order M since $\mathcal{P}_M = \{0\}$ by our definition.

The vanishing moment condition for Λ can be restated as a condition on the multipliers of T_j^σ in the following way.

Let $a_j^\sigma \in G(\lambda_j^\sigma)$ be the multiplier of T_j^σ. The Fourier transform of $\Lambda(f_1, \ldots, f_k)$ can be written as

$$\begin{aligned}
&\big(\Lambda(f_1, \ldots, f_k)\big)^\wedge(\xi) \\
&\quad = \sum_{\sigma \in \mathcal{A}} \int_{(\mathbb{R}^n)^{k-1}} a_1^\sigma(\eta_1) \hat{f}_1(\eta_1) \cdots a_{k-1}^\sigma(\eta_{k-1}) \widehat{f_{k-1}}(\eta_{k-1}) \\
&\qquad\qquad \times a_k^\sigma(\xi - \eta_1 - \cdots - \eta_{k-1}) \hat{f}_k(\xi - \eta_1 - \cdots - \eta_{k-1})\, d\eta_1 \cdots d\eta_{k-1}.
\end{aligned}$$

The condition $\Lambda(f_1, \ldots, f_k) \perp \mathcal{P}_M$, $M \in \mathbb{N} \cup \{0\}$, is equivalent to the condition that the partial derivatives of $\big(\Lambda(f_1, \ldots, f_k)\big)^\wedge(\xi)$ of order $\leqq M$ vanish at $\xi = 0$. From this, it is easy to see that Λ satisfies the vanishing moment condition up to order M, $M \in \mathbb{N} \cup \{0\}$, if and only if the equality

$$\sum_{\sigma \in \mathcal{A}} a_1^\sigma(\eta_1) \cdots a_{k-1}^\sigma(\eta_{k-1}) \partial_{\eta_k}^\beta a_k^\sigma(\eta_k) = 0 \tag{2.2}$$

holds for all multi-indices β with $|\beta| \leqq M$ and for all $\eta_1, \ldots, \eta_k \in \mathbb{R}^n \setminus \{0\}$ satisfying $\eta_1 + \cdots + \eta_k = 0$.

By the symmetry of the situation, the equality (2.2) can be replaced by

$$\sum_{\sigma \in \mathcal{A}} \Big(\prod_{j: j \neq m} a_j^\sigma(\eta_j^\sigma) \Big) \partial_{\eta_m}^\beta a_m^\sigma(\eta_m) = 0$$

with any $m \in \{1, \ldots, k-1\}$.

2.4 Maximal Functions

For measurable functions f on $\mathbb{R}^n$ and for $0 \leqq \lambda < \infty$ and $0 < r < \infty$, the maximal function $f_{\lambda,r}^*$ is defined by

$$f_{\lambda,r}^*(x) = \sup_{\infty > t > 0} t^{\lambda - n/r} \|f\|_{r, B(x,t)} \quad (x \in \mathbb{R}^n).$$

If $0 \leqq \lambda < \infty$, $0 < r < p \leqq q \leqq \infty$, and $1/p - 1/q = \lambda/n$, then

$$\|f_{\lambda,r}^*\|_q \leqq c \|f\|_p \tag{2.3}$$

for all measurable functions f on $\mathbb{R}^n$. See [Ch, Lemma 2].

For $0 \leqq \lambda < \infty$, $m \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{R}^n$, and $0 < t < \infty$, we define the set $\mathcal{T}_m^\lambda(x,t)$ as the set of all those $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset B(x,t)$ and

$$\|\partial^\alpha \varphi\|_\infty \leqq t^{\lambda - n - |\alpha|} \quad \text{for } |\alpha| \leqq m.$$

For $f \in \mathcal{D}'(\mathbb{R}^n)$, $0 \leqq \lambda < \infty$, and $m \in \mathbb{N} \cup \{0\}$, we define the maximal function $M_m^\lambda(f)$ by

$$M_m^\lambda(f)(x) = \sup\Big\{ |\langle f, \varphi \rangle| \mid \varphi \in \bigcup_{0<t<\infty} \mathcal{T}_m^\lambda(x,t) \Big\} \quad (x \in \mathbb{R}^n).$$

If $0 \leqq \lambda < \infty$, $m \in \mathbb{N} \cup \{0\}$, $0 < p < \infty$, $0 < q \leqq \infty$, $1/p - 1/q = \lambda/n$, and $m > n/p - n$, then

$$\|M_m^\lambda(f)\|_q \leqq c\|f\|_{H^p} \tag{2.4}$$

for all $f \in H^p(\mathbb{R}^n)$. If $0 < p < \infty$, $m \in \mathbb{N} \cup \{0\}$, and $m > n/p - n$, then

$$\|M_m^0(f)\|_p \approx \|f\|_{H^p} \tag{2.5}$$

for all $f \in \mathcal{D}'(\mathbb{R}^n)$. These facts can be easily proved by the use of the atomic decomposition for H^p; *cf.* [U, Lemma 7].

For $0 \leqq \lambda < \infty$, $m \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{R}^n$, and $0 < t, \epsilon < \infty$, we define the set $\mathcal{T}_{m,\epsilon}^\lambda(x,t)$ as the set of all those C^∞ functions φ on $\mathbb{R}^n$ such that

$$|\partial^\alpha \varphi(y)| \leqq t^{\lambda - n - |\alpha|} \left(1 + \frac{|x-y|}{t}\right)^{\lambda - n - |\alpha| - \epsilon} \qquad \text{for } |\alpha| \leqq m.$$

Let $0 \leqq \lambda < \infty$, $m \in \mathbb{N} \cup \{0\}$, and $0 < \epsilon < \infty$. Then, for locally integrable functions f on $\mathbb{R}^n$ such that $\int (1+|y|)^{\lambda-n-\epsilon} |f(y)|\, dy < \infty$ and for all $x \in \mathbb{R}^n$, we have

$$\sup\left\{ \left| \int f(y)\varphi(y)\, dy \right| \,\Big|\, \varphi \in \bigcup_{\infty > t > 0} \mathcal{T}_{m,\epsilon}^\lambda(x,t) \right\} \leqq cM_m^\lambda(f)(x). \tag{2.6}$$

Proof of this fact reads as follows.

Proof Take functions $\eta, \delta \in C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\eta \subset \{x \mid |x| < 2\}, \quad \operatorname{supp}\delta \subset \{x \mid 1/2 \leqq |x| < 2\},$$

and

$$\eta(y) + \sum_{j=1}^\infty \delta(2^{-j}y) = 1 \quad \text{for all } y \in \mathbb{R}^n.$$

Let $\varphi \in \mathcal{T}_{m,\epsilon}^\lambda(x,t)$ and set

$$\varphi_0(y) = \varphi(y)\eta\left(\frac{x-y}{t}\right)$$

and

$$\varphi_j(y) = \varphi(y)\delta\left(\frac{x-y}{2^j t}\right) \quad (j \in \mathbb{N}).$$

Then, for $j \in \mathbb{N} \cup \{0\}$, we have $\varphi_j \in c2^{-j\epsilon}\mathcal{T}_m^\lambda(x, 2^{j+1}t)$. (Here, and in the sequel, we use the notation $A\mathcal{T}_m^\lambda(x,t)$, $0 < A < \infty$, to denote the set $\{A\varphi \mid \varphi \in \mathcal{T}_m^\lambda(x,t)\}$; we also use the notation $A\mathcal{T}_{m,\epsilon}^\lambda(x,t)$ in the similar meaning.) Hence

$$\left|\int f(y)\varphi_j(y)\,dy\right| \leqq c2^{-j\epsilon}M_m^\lambda(f)(x).$$

Thus

$$\left|\int f(y)\varphi(y)\,dy\right| = \left|\sum_{j=0}^{\infty}\int f(y)\varphi_j(y)\,dy\right| \leqq \sum_{j=0}^{\infty} c2^{-j\epsilon}M_m^\lambda(f)(x) \leqq c_\epsilon M_m^\lambda(f)(x).$$

This implies the desired estimate.

3 Lemmas

Lemma 3.1 *Let $K, m \in \mathbb{N} \cup \{0\}$. Suppose $f \in L^\infty$,* supp *$f \subset B(x_0, t)$, and $f \perp \mathcal{P}_K$. Then:*

(a) f can be written as

$$f = \sum_{i=1}^{\infty} b_i \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \tag{3.1}$$

with b_i $(i = 1, 2, \dots)$ such that $b_i \in L^\infty$, supp *$b_i \subset B(w_i, \rho_i) \subset B(x_0, 2t)$, $b_i \perp \mathcal{P}_K$, and*

$$\left(\sum_{i=1}^{\infty} \|b_i\|_\infty^r \chi_{B(w_i,\rho_i)}(x)\right)^{1/r} \leqq c_r M_m^0(f)(x) \quad (\forall x \in \mathbb{R}^n)$$

for every $0 < r < \infty$.

(b) If p is a real number such that $n/(K+1+n) < p < \infty$, then $f \in H^p$, the series in (3.1) *converges in H^p, and $\|f\|_{H^p} \leqq c\|M_m^0(f)\|_{p,B(x_0,2t)}$.*

For a proof of this lemma, see [M1, Lemmas 2.3 and 2.5].

Let $f \in \mathcal{D}'(\mathbb{R}^n)$, $K \in \mathbb{N} \cup \{0\}$, and $0 < t < \infty$. Let P be the unique polynomial in $\mathcal{P}_K$ such that $\phi(\cdot/t)(f - P) \perp \mathcal{P}_K$. (For the function ϕ, see Section 2.1.) We define

$$g_{K,t}(f) = \phi\left(\frac{\cdot}{t}\right)(f - P),$$

$$\theta_{K,t}(f) = \phi\left(\frac{\cdot}{t}\right)P,$$

$$h_{K,t}(f) = \left(1 - \phi\left(\frac{\cdot}{t}\right)\right)f.$$

Lemma 3.2 *Let f, K, and t be as mentioned above. We simply write $g = g_{K,t}(f)$, $\theta = \theta_{K,t}(f)$, and $h = h_{K,t}(f)$. Let $m \in \mathbb{N} \cup \{0\}$. Then:*

(a) $f = g + \theta + h$;
(b) $\operatorname{supp} g \subset B(0, 2t)$, $g \perp \mathcal{P}_K$, *and*

$$M_m^0(g)(x) \leqq cM_m^0(f)(x) \quad \textit{for } x \in B(0, 4t);$$

(c) $\theta \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} \theta \subset B(0, 2t)$, *and*

$$\|\partial^\alpha \theta\|_\infty \leqq c_\alpha t^{-|\alpha|} \inf_{B(0,4t)} M_m^0(f)$$

for every multi-index α;
(d) $\operatorname{supp} h \subset \{x \in \mathbb{R}^n \mid |x| \geqq t\}$ *and*

$$M_m^0(h)(x) \leqq cM_m^0(f)(x) \quad \textit{for } x \in B(0, 4t).$$

Proof The estimate of $\partial^\alpha \theta$ as given in (c) can be easily proved by the well known techniques used in the atomic decomposition ; *cf.*, *e.g.*, [S, Chap. III, Section 2.1.4]. For the proof of the inequalities in (b) and (d), see [M1, Lemma 2.4]. Other claims are obvious. Details are left to the reader.

Lemma 3.3 *Let* $T \in \mathcal{K}(\lambda)$, $0 \leqq \lambda < \infty$, *and* $K \in \mathbb{N} \cup \{0\}$. *Suppose* $b \in L^\infty$, $\operatorname{supp} b \subset B(w, \rho)$, $b \perp \mathcal{P}_K$, *and* $K + 1 + n > \lambda$. *Then:*

(b) *For* $n/(K + 1 + n - \lambda) < q < \infty$, *we have* $Tb \in H^q$ *and*

$$\|Tb\|_{H^q} \leqq c_q \|b\|_\infty \rho^{\lambda + n/q};$$

(b) *With* $L = K + n - [\lambda]$, *we have*

$$|(Tb)(x)| \leqq c\|b\|_\infty \rho^\lambda \left(\frac{\rho}{|x - w|}\right)^L \quad \textit{for } |x - w| > 2\rho.$$

Proof (a) Let q be in the range as mentioned in the lemma. Define p by $1/p = 1/q + \lambda/n$. Then $\lambda/n < 1/p < (K + 1 + n)/n$ and $\|b\|_{H^p} \leqq c\|b\|_\infty \rho^{n/p}$ (since $(c\|b\|_\infty \rho^{n/p})^{-1} b$ is a p-atom). Hence the desired estimate of $\|Tb\|_{H^q}$ follows from (1.2).

(b) By the translation invariance, we may and shall assume $w = 0$.

We first consider the case $0 \leqq \lambda < n$. In this case, as mentioned in Section 2.2, $(Tb)(x)$ for $|x| > \rho$ can be written as

$$(Tb)(x) = \int_{|y| < \rho} A(x - y) b(y)\, dy$$

with a function $A \in G(n - \lambda)$. Since $b \perp \mathcal{P}_K$, we have

$$(Tb)(x) = \int_{|y| < \rho} \big(A(x - y) - P(y)\big) b(y)\, dy \quad (|x| > \rho)$$

for every $P \in \mathcal{P}_K$. We choose P to be the degree K Taylor polynomial of $A(x - \cdot)$ expanded about 0. Then, for $|x| > 2\rho$, we have

$$\begin{aligned}|(Tb)(x)| &\leqq \int_{|y|<\rho} |A(x-y) - P(y)|\,|b(y)|\,dy \\ &\leqq c\int_{|y|<\rho} |x|^{\lambda-n-K-1}\,|y|^{K+1}\,|b(y)|\,dy \\ &\leqq c\|b\|_\infty \rho^\lambda (|x|^{-1}\rho)^{K+1+n-\lambda} \\ &\leqq c\|b\|_\infty \rho^\lambda (|x|^{-1}\rho)^{L}\end{aligned}$$

(the last inequality holds because $L < K + 1 + n - \lambda$).

Next suppose $n \leqq \lambda < \infty$. (The argument to be given below can actually cover the case $n/2 < \lambda < \infty$.) Let $a \in G(\lambda)$ be the multiplier of T.

Since $b \perp \mathcal{P}_K$ and since b is a compactly supported bounded function, we have $\hat{b}(\xi) = O(|\xi|^{K+1})$ as $\xi \to 0$. Hence $a(\xi)\hat{b}(\xi) = O(|\xi|^{-\lambda+K+1})$ as $\xi \to 0$, which implies that $a(\xi)\hat{b}(\xi)$ is integrable in a neighbourhood of $\xi = 0$. On the other hand, $a(\xi)\hat{b}(\xi)$ is also integrable in $|\xi| > \delta$ for every $\delta > 0$, since $|a(\xi)| \leqq c|\xi|^{-\lambda}$, $\lambda \geqq n$, and since $\hat{b} \in L^2$. Thus $a\hat{b} \in L^1(\mathbb{R}^n)$. It is easy to see that Tb (here T is the extended operator as mentioned in Section 2.2) is given by the absolutely covergent integral

$$(Tb)(x) = \int_{\mathbb{R}^n} a(\xi)\hat{b}(\xi)e^{2\pi i x\xi}\,d\xi \quad (x \in \mathbb{R}^n).$$

For $0 < \epsilon < N < \infty$, we set $\chi_{\epsilon,N}(\xi) = \big(1 - \phi(\epsilon^{-1}\xi)\big)\phi(N^{-1}\xi)$ and

$$f_{\epsilon,N}(x) = \int a(\xi)\hat{b}(\xi)\chi_{\epsilon,N}(\xi)e^{2\pi i x\xi}\,d\xi.$$

Then, for every $x \in \mathbb{R}^n$, $f_{\epsilon,N}(x)$ converges to $(Tb)(x)$ as $\epsilon \downarrow 0$ and $N \to \infty$. If $x \neq 0$, then by integration by parts we have

$$f_{\epsilon,N}(x) = \int e^{2\pi i x\xi}\left(\sum_{j=1}^{n} \frac{-x_j}{2\pi i|x|^2}\partial_{\xi_j}\right)^L [a(\xi)\hat{b}(\xi)\chi_{\epsilon,N}(\xi)]\,d\xi. \tag{3.2}$$

Since $a \in G(\lambda)$, we have

$$|\partial_\xi^\alpha[a(\xi)\chi_{\epsilon,N}(\xi)]| \leqq c_\alpha|\xi|^{-\lambda-|\alpha|} \tag{3.3}$$

with c_α independent of ϵ and N. If $|\alpha| \leqq K + 1$, then

$$\begin{aligned}|\partial_\xi^\alpha\hat{b}(\xi)| &\leqq c|\xi|^{K+1-|\alpha|}\sum_{|\beta|=K+1}\|\partial^\beta\hat{b}\|_\infty \quad (\text{since } b \perp \mathcal{P}_K) \\ &\leqq c|\xi|^{K+1-|\alpha|}\sum_{|\beta|=K+1}\|x^\beta b(x)\|_1 \\ &\leqq c\|b\|_\infty|\xi|^{K+1-|\alpha|}\rho^{K+1+n} \quad (\text{since } \operatorname{supp} b \subset B(0,\rho)).\end{aligned}$$

If $|\alpha| > K + 1$ and $|\xi| \leqq 1/\rho$, then

$$|\partial_\xi^\alpha \hat{b}(\xi)| \leqq \|(-2\pi i x)^\alpha b(x)\|_1 \leqq c_\alpha \|b\|_\infty \rho^{|\alpha|+n}$$
$$\leqq c_\alpha \|b\|_\infty \, |\xi|^{K+1-|\alpha|} \rho^{K+1+n}.$$

Thus the estimate

$$|\partial_\xi^\alpha \hat{b}(\xi)| \leqq c_\alpha \|b\|_\infty |\xi|^{K+1-|\alpha|} \rho^{K+1+n} \quad \text{if } |\xi| \leqq 1/\rho \tag{3.4}$$

holds for every multi-index α. We also have

$$\|\partial^\alpha \hat{b}\|_2 = \|(-2\pi i x)^\alpha b(x)\|_2 \leqq c_\alpha \|b\|_\infty \rho^{|\alpha|+n/2}. \tag{3.5}$$

Now using (3.2), (3.3), (3.4), and (3.5), we obtain

$$\begin{aligned}
|f_{\epsilon,N}(x)| &\leqq c|x|^{-L} \sum_{|\alpha+\beta|=L} \int |\xi|^{-\lambda-|\alpha|} \, |\partial_\xi^\beta \hat{b}(\xi)| \, d\xi \\
&\leqq c|x|^{-L} \sum_{|\alpha+\beta|=L} \int_{|\xi|\leqq 1/\rho} |\xi|^{-\lambda-|\alpha|} \, \|b\|_\infty \, |\xi|^{K+1-|\beta|} \rho^{K+1+n} \, d\xi \\
&\quad + c|x|^{-L} \sum_{|\alpha+\beta|=L} \left(\int_{|\xi|>1/\rho} (|\xi|^{-\lambda-|\alpha|})^2 \, d\xi \right)^{1/2} \|b\|_\infty \rho^{|\beta|+n/2} \\
&\leqq c\|b\|_\infty |x|^{-L} \rho^{\lambda+L}.
\end{aligned}$$

Taking limit as $\epsilon \downarrow 0$ and $N \to \infty$, we obtain the desired estimate. Lemma 3.3 is proved.

Lemma 3.4 *Let $\{b_i\}$ be a sequence in L^∞, let $\{B(w_i, \rho_i)\}$ be a sequence of balls, and let $K \in \mathbb{N} \cup \{0\}$. Suppose* $\operatorname{supp} b_i \subset B(w_i, \rho_i)$, $\|b_i\|_\infty = a_i < \infty$, *and $b_i \perp \mathcal{P}_K$. Let $T \in \mathcal{K}(\lambda)$, $0 \leqq \lambda < \infty$, and $K' \in \mathbb{R}$, and suppose $K + 1 + n > \lambda$ and $K' < K - [\lambda]$. Finally let $1 \leqq r < \infty$ and $0 \leqq s < \infty$. Then*

$$\left\| \sum_i \rho_i^s \left(1 + \frac{|\cdot - w_i|}{\rho_i} \right)^{K'} |Tb_i| \right\|_r \leqq c \left\| \sum_i a_i \chi_{B(w_i,\rho_i)} \right\|_p$$

with $1/p = 1/r + s/n + \lambda/n$.

Proof We write $B_i = B(w_i, \rho_i)$, $2B_i = B(w_i, 2\rho_i)$, and

$$F_i = \rho_i^s \left(1 + \frac{|\cdot - w_i|}{\rho_i} \right)^{K'} |Tb_i|.$$

We decompose F_i as

$$F_i = F_i \chi_{2B_i} + F_i \chi_{(2B_i)^c} = f_i + g_i.$$

The function f_i is supported on $2B_i$ and, by Lemma 3.3 (a), we have

$$\|f_i\|_q \leqq c_q a_i \rho_i^{s+\lambda} |2B_i|^{1/q} \quad \text{for } 0 < \frac{1}{q} < \frac{K+1+n-\lambda}{n}.$$

From these facts, it follows that

$$\Big\|\sum_i f_i\Big\|_r \leqq c\Big\|\sum_i a_i \rho_i^{s+\lambda}\chi_{2B_i}\Big\|_r \leqq c\Big\|\sum_i a_i \chi_{B_i}\Big\|_p.$$

(For the former inequality, see [StT, Chap. VIII, Lemma 5]; for the latter, see [M4, Lemma 3.2, (2)].)

For g_i, we have

$$|g_i(x)| \leqq c a_i \rho_i^{s+\lambda} \left(\frac{|x-w_i|}{\rho_i}\right)^{K'-L} \chi_{(2B_i)^c}(x)$$

with $L = K + n - [\lambda]$ (by Lemma 3.3 (b)). Thus

$$\Big\|\sum_i g_i\Big\|_r \leqq c \left\|\sum_i a_i \rho_i^{s+\lambda}\left(1+\frac{|x-w_i|}{\rho_i}\right)^{K'-L}\right\|_r \leqq c\Big\|\sum_i a_i \chi_{B_i}\Big\|_p.$$

(For the last inequality, see [M4, Lemma 3.2, (2)].) Lemma 3.4 is proved.

Lemma 3.5 *Let $a \in G(\lambda)$, $0 \leqq \lambda < \infty$, $0 < t < \infty$, and let*

$$A_t = \Big(\big(1-\phi(t\xi)\big)a(\xi)\Big)^{\vee} \in \mathcal{S}'.$$

Then A_t restricted to $\mathbb{R}^n \setminus \{0\}$ is a C^∞ function, and for every multi-index α and for $L \in \mathbb{N} \cup \{0\}$ satisfying $L > -\lambda + n + |\alpha|$, we have

$$|\partial^\alpha A_t(x)| \leqq c_{\alpha,L} t^{\lambda-n-|\alpha|+L}|x|^{-L}.$$

Proof Let α and L be as mentioned in the lemma. For multi-indices β with $|\beta| = L$, we have

$$|\partial_\xi^\beta\big[\xi^\alpha\big(1-\phi(t\xi)\big)a(\xi)\big]| \leqq c_{\alpha,\beta}|\xi|^{-\lambda+|\alpha|-L}\chi_{B(0,1/t)^c}(\xi),$$

where $c_{\alpha,\beta}$ does not depend on t. The right hand side of the above inequality is integrable on $\mathbb{R}^n$. Hence, taking the inverse Fourier transform, we see that $x^\beta \partial^\alpha A_t$ is a continuous function and that

$$|x^\beta \partial_x^\alpha A_t(x)| \leqq c_{\alpha,\beta}\int_{|\xi|>1/t} |\xi|^{-\lambda+|\alpha|-L}\, d\xi = c_{\alpha,\beta} t^{\lambda-|\alpha|+L-n}.$$

Since this estimate holds for all β with $|\beta| = L$, the conclusion of the lemma follows.

Lemma 3.6 *Let $T \in \mathcal{K}(\lambda)$, $0 \leqq \lambda < \infty$, $f \in \mathcal{S}_0$, $K \in \mathbb{N} \cup \{0\}$, and $0 < t < \infty$. Suppose $K + 1 + n > \lambda$. Let $\theta = \theta_{K,10t}(f)$ and $h = h_{K,10t}(f)$. Then $T(\theta + h)$ is a C^∞ function and, for* *every multi-index α, for every $m \in \mathbb{N}\cup\{0\}$, and for every p satisfying $n/(K+1+n) < p < \infty$, we have*

$$\sup_{|x|<t} |\partial_x^\alpha T(\theta + h)(x)| \leqq c_{\alpha,p,m} t^{-|\alpha|} \Big(\big(M_m^0(f)\big)^*_{\lambda,p}(0) + M_m^\lambda(f)(0) + M_m^0(Tf)(0) \Big).$$

Proof Let $a \in G(\lambda)$ be the multiplier of T. We set $g = g_{K,10t}(f)$. By the same reason as in the proof of Lemma 3.3 (b), the formula

$$T\varphi = (a\hat{\varphi})^\vee \quad \text{with} \quad a\hat{\varphi} \in L^1$$

holds for $\varphi = f, g$, and $\theta + h$ (notice that $\theta + h \in \mathcal{S}$ and $\theta + h = f - g \perp \mathcal{P}_K$). We decompose $T(\theta + h)$ as

$$\begin{aligned} T(\theta + h) &= \big(\phi(t\xi)a(\xi)\hat{f}(\xi)\big)^\vee - \big(\phi(t\xi)a(\xi)\hat{g}(\xi)\big)^\vee \\ &\quad + \Big(\big(1 - \phi(t\xi)\big)a(\xi)\hat{\theta}(\xi)\Big)^\vee + \Big(\big(1 - \phi(t\xi)\big)a(\xi)\hat{h}(\xi)\Big)^\vee \\ &= \mathrm{I} - \mathrm{II} + \mathrm{III} + \mathrm{IV}\,. \end{aligned}$$

The functions I and II are C^∞ since these are inverse Fourier transforms of compactly supported L^1 functions. The functions III and IV are also C^∞ since these belong to $\mathcal{S}$. Therefore $T(\theta + h)$ is C^∞.

In the rest of the proof, we shall estimate the derivatives of I, II, III, and IV separately.

Estimate of $\partial^\alpha\,\mathrm{I}(x)$ We can write

$$\mathrm{I} = \big(\phi(t\xi)(Tf)^\wedge(\xi)\big)^\vee = \frac{1}{t^n}\check{\phi}\left(\frac{\cdot}{t}\right) * (Tf)$$

and thus

$$\partial^\alpha\,\mathrm{I}(x) = \left\langle Tf, \frac{1}{t^{n+|\alpha|}}(\check{\phi})^{(\alpha)}\left(\frac{x - \cdot}{t}\right)\right\rangle.$$

If $|x| < t$, then, as is easily seen,

$$\frac{1}{t^{n+|\alpha|}}(\check{\phi})^{(\alpha)}\left(\frac{x - \cdot}{t}\right) \in c_{\alpha,\epsilon} t^{-|\alpha|}\mathcal{T}^0_{m,\epsilon}(0, t)$$

for every $\epsilon > 0$, where $c_{\alpha,\epsilon}$ can be taken independent of x so long as $|x| < t$. Hence, by (2.6),

$$|\partial^\alpha\,\mathrm{I}(x)| \leqq c_\alpha t^{-|\alpha|} M_m^0(Tf)(0) \quad \text{for } |x| < t.$$

Estimate of $\partial^\alpha\,\mathrm{II}(x)$ First assume $(K+1+n)/n > 1/p > \max\{1, \lambda/n\}$. By Lemma 3.1 (b) and Lemma 3.2 (b), we see that $g \in H^p$ and

$$\|g\|_{H^p} \leqq c_p \|M_m^0(g)\|_{p,B(0,40t)} \leqq c_p \|M_m^0(f)\|_{p,B(0,40t)}.$$

Hence

$$|\hat{g}(\xi)| \leqq c_p \|g\|_{H^p} |\xi|^{n/p-n} \leqq c_p \|M_m^0(f)\|_{p,B(0,40t)} |\xi|^{n/p-n}.$$

Thus

$$\begin{aligned}|\partial^\alpha \mathrm{II}(x)| &= \left|\int (2\pi i\xi)^\alpha \phi(t\xi) a(\xi) \hat{g}(\xi) e^{2\pi i x\xi}\, d\xi\right| \\ &\leqq \int_{|\xi|<2/t} |(2\pi i\xi)^\alpha a(\xi)\hat{g}(\xi)|\, d\xi \\ &\leqq c_{p,\alpha} \int_{|\xi|<2/t} |\xi|^{|\alpha|-\lambda+n/p-n} \|M_m^0(f)\|_{p,B(0,40t)}\, d\xi,\end{aligned}$$

which implies

$$|\partial^\alpha \mathrm{II}(x)| \leqq c_{\alpha,p} \|M_m^0(f)\|_{p,B(0,40t)} t^{\lambda-|\alpha|} |B(0,40t)|^{-1/p} \quad (\forall x \in \mathbb{R}^n). \tag{3.6}$$

The estimate (3.6) holds also for p with $\max\{1, \lambda/n\} \geqq 1/p \geqq 0$ because, except for the constant factor, the right hand side is a nondecreasing function in p (by Hölder's inequality).

The estimate (3.6) clearly implies

$$|\partial^\alpha \mathrm{II}(x)| \leqq c_{\alpha,p} t^{-|\alpha|} \left(M_m^0(f)\right)^*_{\lambda,p}(0) \quad (\forall x \in \mathbb{R}^n).$$

Estimate of $\partial^\alpha \mathrm{III}(x)$ By Lemma 3.2 (c), we have

$$|(2\pi i\xi)^\beta \hat{\theta}(\xi)| \leqq \|\partial^\beta \theta\|_1 \leqq c_\beta t^{n-|\beta|} \inf_{B(0,40t)} M_m^0(f).$$

Hence

$$|\hat{\theta}(\xi)| \leqq c_L t^{n-L} |\xi|^{-L} \inf_{B(0,40t)} M_m^0(f)$$

for every $L \in \mathbb{N}$. Thus

$$\begin{aligned}|\partial^\alpha \mathrm{III}(x)| &= \left|\left(\left(1-\phi(t\xi)\right) a(\xi)\hat{\theta}(\xi)(2\pi i\xi)^\alpha\right)^\vee (x)\right| \\ &\leqq \int_{|\xi|>1/t} |a(\xi)\hat{\theta}(\xi)(2\pi i\xi)^\alpha|\, d\xi \\ &\leqq c_{\alpha,L} t^{n-L} \int_{|\xi|>1/t} |\xi|^{-\lambda+|\alpha|-L}\, d\xi \inf_{B(0,40t)} M_m^0(f).\end{aligned}$$

Taking L sufficiently large, we obtain

$$|\partial^\alpha \mathrm{III}(x)| \leqq c_\alpha t^{\lambda-|\alpha|} \inf_{B(0,40t)} M_m^0(f) \quad (\forall x \in \mathbb{R}^n),$$

which *a fortiori* implies

$$|\partial^\alpha \mathrm{III}(x)| \leqq c_{\alpha,p} t^{-|\alpha|} \left(M_m^0(f)\right)^*_{\lambda,p}(0) \quad (\forall x \in \mathbb{R}^n).$$

Estimate of $\partial^\alpha \mathrm{IV}(x)$ Using A_t of Lemma 3.5, we can write $\partial^\alpha \mathrm{IV}(x)$ for $|x| < t$ as

$$\partial^\alpha \mathrm{IV}(x) = \int (A_t)^{(\alpha)}(x-y) \left(1 - \phi\left(\frac{y}{10t}\right)\right) f(y)\, dy.$$

If $|x| < t$, then using Lemma 3.5 we see that

$$(A_t)^{(\alpha)}(x - \cdot) \left(1 - \phi\left(\frac{\cdot}{10t}\right)\right) \in c_{\alpha,\epsilon} t^{-|\alpha|} \mathcal{T}^\lambda_{m,\epsilon}(0,t)$$

for every $\epsilon > 0$. Hence by (2.6) we have

$$|\partial^\alpha \mathrm{IV}(x)| \leqq c_\alpha t^{-|\alpha|} M_m^\lambda(f)(0) \quad \text{for } |x| < t.$$

Combining the estimates of the derivatives of I $\sim$ IV, we obtain the estimate as stated in Lemma 3.6. Lemma 3.6 is proved.

Lemma 3.7 *Let* Λ *be defined by* (1.3) *with* (1.4). *Let* $N \in \mathbb{N} \cup \{0\}$ *and* $0 < t < \infty$. *Suppose* Λ *satisfies the vanishing moment condition up to order* $N-1$. *Let* $f_j \in L^\infty$ $(j = 1, \dots, k)$ *and suppose* supp $f_j \subset B(0,t)$ *and* $f_j \perp \mathcal{P}_K$ *with*

$$K = N + n + 2 + \max\{[\lambda_j^\sigma] \mid \sigma \in \mathcal{A},\ j = 1, \dots, k\}.$$

Let $\varphi \in \mathcal{T}^0_N(0,t)$. *Let* r_j^σ *and* s_j^σ $(\sigma \in \mathcal{A},\ j = 1, \dots, k)$ *be real numbers such that* $1 \leqq r_j^\sigma < \infty$, $0 \leqq s_j^\sigma < \infty$, $\sum_{j=1}^k 1/r_j^\sigma \leqq 1$, *and* $\sum_{j=1}^k s_j^\sigma \leqq N$. *Finally let* $m \in \mathbb{N} \cup \{0\}$. *Then*

$$\left| \int \varphi \Lambda(f_1, \dots, f_k) \right| \leqq c \sum_{\sigma \in \mathcal{A}} \prod_{j=1}^k t^{-n/r_j^\sigma - s_j^\sigma} \|M_m^0(f_j)\|_{v_j^\sigma, B(0,2t)},$$

where $1/v_j^\sigma = 1/r_j^\sigma + s_j^\sigma/n + \lambda_j^\sigma/n$.

Proof By Lemma 3.1, we can decompose f_j as $f_j = \sum_{i=1}^\infty b_{ji}$ with the series converging in H^p for $0 < 1/p < (K+1+n)/n$ and with b_{ji} such that

$$\|b_{ji}\|_\infty = a_{ji} < \infty, \quad \operatorname{supp} b_{ji} \subset B_{ji} = B(w_{ji}, \rho_{ji}) \subset B(0,2t), \quad b_{ji} \perp \mathcal{P}_K,$$

and

$$\sum_{i=1}^\infty a_{ji} \chi_{B_{ji}}(x) \leqq c M_m^0(f_j)(x) \quad (\forall x \in \mathbb{R}^n). \tag{3.7}$$

By the boundedness (1.2), we have $T_j^\sigma f_j = \sum_{i=1}^\infty T_j^\sigma b_{ji}$ with the series converging in H^q for $0 < 1/q < (K+1+n-\lambda_j^\sigma)/n$; in particular it converges in L^q for every sufficiently large $q < \infty$. From this it follows that

$$\Lambda(f_1, \dots, f_k) = \lim_{L\to\infty} \sum_{i_1=1}^{L} \cdots \sum_{i_k=1}^{L} \Lambda(b_{1i_1}, \dots, b_{ki_k})$$

with the convergence holding with respect to L^q-norm for every sufficiently large $q < \infty$. Hence

$$\int \varphi\Lambda(f_1, \dots, f_k) = \lim_{L\to\infty} \sum_{i_1=1}^{L} \cdots \sum_{i_k=1}^{L} \int \varphi\Lambda(b_{1i_1}, \dots, b_{ki_k}).$$

Therefore we have

$$\left|\int \varphi\Lambda(f_1, \dots, f_k)\right| \leqq \sum_{i_1,\dots,i_k} \left|\int \varphi\Lambda(b_{1i_1}, \dots, b_{ki_k})\right|. \tag{3.8}$$

For the moment we shall estimate each term on the right hand side of (3.8), which we shall simply write as

$$\left|\int \varphi\Lambda(b_1, \dots, b_k)\right|$$

with $\|b_j\|_\infty = a_j < \infty$, supp $b_j \subset B(w_j, \rho_j) \subset B(0, 2t)$, and $b_j \perp \mathcal{P}_K$.

We assume $\rho_1 = \min\{\rho_1, \dots, \rho_k\}$.

We first observe that

$$\int P\Lambda(b_1, \dots, b_k) = 0 \quad \text{for all } P \in \mathcal{P}_{N-1}. \tag{3.9}$$

This can be deduced from the vanishing moment condition on Λ by a limiting argument; here we omit the limiting argument but prove that $P\Lambda(b_1, \dots, b_k)$ is integrable for all $P \in \mathcal{P}_{N-1}$. Indeed, using Lemma 3.3, we see that $\Lambda(b_1, \dots, b_k) \in L^1_{\mathrm{loc}}$ and $\Lambda(b_1, \dots, b_k)(x) = O(|x|^{-M})$ as $|x| \to \infty$ with

$$-M = \max_{\sigma\in\mathcal{A}} \Big\{ \sum_{j=1}^{k} ([\lambda_j^\sigma] - K - n) \Big\} < -(N-1) - n,$$

from which the integrability of $P\Lambda(b_1, \dots, b_k)$ for $P \in \mathcal{P}_{N-1}$ immediately follows.

We take P as follows: If $N > 0$, then let P be the degree $N-1$ Taylor polynomial of φ expanded about w_1; if $N = 0$, then let $P = 0$. Then, using (3.9), we have

$$\begin{aligned}\left|\int \varphi\Lambda(b_1, \dots, b_k)\right| &= \left|\int (\varphi - P)\Lambda(b_1, \dots, b_k)\right| \\ &\leqq \sum_{\sigma\in\mathcal{A}} \int |\varphi - P|\, |T_1^\sigma b_1| \cdots |T_k^\sigma b_k| \\ &\leqq c \sum_{\sigma\in\mathcal{A}} \int t^{-n-N} |x - w_1|^N\, |T_1^\sigma b_1| \cdots |T_k^\sigma b_k|\, dx\end{aligned}$$

($|T_j^\sigma b_j|$ stands for $|(T_j^\sigma b_j)(x)|$). Since $B(w_1, \rho_1) \subset B(0, 2t)$, we have

$$1 + \frac{|x|}{t} \leqq c\left(1 + \frac{|x - w_1|}{\rho_1}\right). \tag{3.10}$$

Also, since $2t \geqq \rho_1 = \min\{\rho_j\}$ and since $s_j^\sigma \geqq 0$ and $s_1^\sigma + \cdots + s_k^\sigma \leqq N$, we have

$$t^{-N}\rho_1^N \leqq ct^{-s_1^\sigma - s_2^\sigma - \cdots - s_k^\sigma}\rho_1^{s_1^\sigma}\rho_2^{s_2^\sigma}\cdots\rho_k^{s_k^\sigma}. \tag{3.11}$$

Using (3.10) and (3.11), we have

$$\begin{aligned} &t^{-n-N}|x - w_1|^N \\ &\quad\leqq ct^{-n-N}\rho_1^N\left(1 + \frac{|x - w_1|}{\rho_1}\right)^N \\ &\quad\leqq ct^{-n-s_1^\sigma - s_2^\sigma - \cdots - s_k^\sigma}\left(1 + \frac{|x|}{t}\right)^{-n-1}\rho_1^{s_1^\sigma}\left(1 + \frac{|x - w_1|}{\rho_1}\right)^{N+n+1}\rho_2^{s_2^\sigma}\cdots\rho_k^{s_k^\sigma}. \end{aligned}$$

Putting the above inequalities together, we obtain the following estimate: If $\rho_1 = \min\{\rho_j\}$, then

$$\begin{aligned} &\left|\int \varphi\Lambda(b_1, \ldots, b_k)\right| \\ &\quad\leqq c\sum_{\sigma\in\mathcal{A}} t^{-n-s_1^\sigma - s_2^\sigma - \cdots - s_k^\sigma}\int\left(1 + \frac{|x|}{t}\right)^{-n-1}\rho_1^{s_1^\sigma}\left(1 + \frac{|x - w_1|}{\rho_1}\right)^{N+n+1}|T_1^\sigma b_1| \\ &\qquad\times \rho_2^{s_2^\sigma}|T_2^\sigma b_2|\cdots\rho_k^{s_k^\sigma}|T_k^\sigma b_k|\,dx. \end{aligned}$$

We now apply the above estimate to those terms in (3.8) for which $\rho_{1i_1} = \min\{\rho_{1i_1}, \ldots, \rho_{ki_k}\}$ and take the sum of those terms to obtain

$$\begin{aligned} \sum_{\rho_{1i_1}=\min}\left|\int \varphi\Lambda(b_{1i_1}, \ldots, b_{ki_k})\right| &\leqq c\sum_{\sigma\in\mathcal{A}} t^{-n-s_1^\sigma - s_2^\sigma - \cdots - s_k^\sigma}\int\left(1 + \frac{|x|}{t}\right)^{-n-1} \\ &\qquad\times\left(\sum_{i=1}^\infty \rho_{1i}^{s_1^\sigma}\left(1 + \frac{|x - w_{1i}|}{\rho_{1i}}\right)^{N+n+1}|T_1^\sigma b_{1i}|\right) \\ &\qquad\times\left(\sum_{i=1}^\infty \rho_{2i}^{s_2^\sigma}|T_2^\sigma b_{2i}|\right)\cdots\left(\sum_{i=1}^\infty \rho_{ki}^{s_k^\sigma}|T_k^\sigma b_{ki}|\right)dx \\ &= (*). \end{aligned}$$

Define r' by $1/r' = 1 - \sum_{j=1}^{k} 1/r_j^\sigma$. Then, using Hölder's inequality, Lemma 3.4, and (3.7), we obtain

$$
\begin{aligned}
(*) \leqq c \sum_{\sigma \in \mathcal{A}} t^{-n-s_1^\sigma - s_2^\sigma - \cdots - s_k^\sigma} \left\| \left(1 + \frac{|x|}{t}\right)^{-n-1} \right\|_{r'} \\
\times \left\| \sum_{i=1}^{\infty} \rho_{1i}^{s_1^\sigma} \left(1 + \frac{|x - w_{1i}|}{\rho_{1i}}\right)^{N+n+1} |T_1^\sigma b_{1i}| \right\|_{r_1^\sigma} \\
\times \left\| \sum_{i=1}^{\infty} \rho_{2i}^{s_2^\sigma} |T_2^\sigma b_{2i}| \right\|_{r_2^\sigma} \cdots \left\| \sum_{i=1}^{\infty} \rho_{ki}^{s_k^\sigma} |T_k^\sigma b_{ki}| \right\|_{r_k^\sigma} \\
\leqq c \sum_{\sigma \in \mathcal{A}} t^{-n-s_1^\sigma - s_2^\sigma - \cdots - s_k^\sigma + n/r'} \\
\times \left\| \sum_{i=1}^{\infty} a_{1i} \chi_{B_{1i}} \right\|_{\nu_1^\sigma} \cdots \left\| \sum_{i=1}^{\infty} a_{ki} \chi_{B_{ki}} \right\|_{\nu_k^\sigma} \\
\leqq c \sum_{\sigma \in \mathcal{A}} \prod_{j=1}^{k} t^{-s_j^\sigma - n/r_j^\sigma} \|M_m^0(f_j)\|_{\nu_j^\sigma, B(0,2t)}.
\end{aligned}
$$

The same estimate holds also for the sum of the terms of (3.8) with $\rho_{mi_m} = \min\{\rho_{1i_1}, \ldots, \rho_{ki_k}\}$ for every $m \in \{2, \ldots, k\}$. Lemma 3.7 is proved.

4 Proof of Theorem, Part (a)

We shall prove the part (a) of Theorem.

We use the following notation: For $T \in \mathcal{K}(\lambda)$, $f \in \mathcal{S}_0$, $0 < p < \infty$, and $m \in \mathbb{N} \cup \{0\}$, we write

$$G_m(f, T, p) = \left(M_m^0(f)\right)_{\lambda,p}^* + M_m^\lambda(f) + M_m^0(Tf).$$

We also write

$$N = [n/q - n] + 1$$

(q is the number as mentioned in Theorem).

In order to prove the part (a), we shall prove that there exist ν_j^σ ($\sigma \in \mathcal{A}$, $j = 1, 2$) such that

$$0 < \nu_j^\sigma < p_j \tag{4.1}$$

and that the pointwise estimate

$$M_N^0\big(\Lambda(f_1, f_2)\big)(x) \leqq c_m \sum_{\sigma \in \mathcal{A}} \prod_{j=1}^{2} G_m(f_j, T_j^\sigma, \nu_j^\sigma)(x) \quad (\forall x \in \mathbb{R}^n) \tag{4.2}$$

holds for all $f_1, f_2 \in \mathcal{S}_0$ and for every $m \in \mathbb{N} \cup \{0\}$. In the sequel we shall write

$$F_j^\sigma = G_m(f_j, T_j^\sigma, \nu_j^\sigma).$$

Once the estimate (4.2) is obtained, the desired H^q estimate can be derived in the following way. We choose $m \in \mathbb{N} \cup \{0\}$ so large that $m > \max\{n/p_j - n \mid j = 1, 2\}$. Then (2.3), (2.4), (2.5), and (1.2) give

$$\|F_j^\sigma\|_{q_j^\sigma} \leqq c\|f_j\|_{H^{p_j}} \quad \text{with} \quad \frac{1}{q_j^\sigma} = \frac{1}{p_j} - \frac{\lambda_j^\sigma}{n}.$$

Hence (4.2) and Hölder's inequality give

$$\|\Lambda(f_1, f_2)\|_{H^q} \leqq c\|M_N^0\big(\Lambda(f_1, f_2)\big)\|_q \leqq c \sum_{\sigma \in \mathcal{A}} \|F_1^\sigma\|_{q_1^\sigma} \, \|F_2^\sigma\|_{q_2^\sigma} \leqq c\|f_1\|_{H^{p_1}} \, \|f_2\|_{H^{p_2}},$$

which is the desired estimate.

Since $1/p_j - \lambda_j^\sigma/n > 0$ and

$$\frac{1}{p_1} - \frac{\lambda_1^\sigma}{n} + \frac{1}{p_2} - \frac{\lambda_2^\sigma}{n} = \frac{1}{q} < 1 + \frac{N}{n},$$

we can take r_j^σ and s_j^σ ($\sigma \in \mathcal{A}$, $j = 1, 2$) such that

$$0 < \frac{1}{r_j^\sigma} < 1, \quad \frac{1}{r_1^\sigma} + \frac{1}{r_2^\sigma} \leqq 1, \quad 0 \leqq s_j^\sigma < \infty, \quad s_1^\sigma + s_2^\sigma \leqq N,$$

$$\text{and} \quad \frac{1}{r_j^\sigma} + \frac{s_j^\sigma}{n} > \frac{1}{p_j} - \frac{\lambda_j^\sigma}{n}.$$

As in Lemma 3.7, we define ν_j^σ by $1/\nu_j^\sigma = 1/r_j^\sigma + s_j^\sigma/n + \lambda_j^\sigma/n$. Then (4.1) is satisfied. We shall prove the estimate (4.2) with these ν_j^σ.

By translation it is sufficient to prove (4.2) for $x = 0$.

Let $\varphi \in \mathcal{T}_N^0(0, t)$. We shall estimate $\int \varphi\Lambda(f_1, f_2)$. As in Lemma 3.7, we set

$$K = N + n + 2 + \max\{[\lambda_j^\sigma] \mid \sigma \in \mathcal{A}, j = 1, 2\}.$$

We decompose $f_j \in \mathcal{S}_0$ ($j = 1, 2$) as

$$f_j = g_j + u_j \tag{4.3}$$

with

$$g_j = g_{K,10t}(f_j), \quad u_j = \theta_{K,10t}(f_j) + h_{K,10t}(f_j). \tag{4.4}$$

We have

$$\int \varphi\Lambda(f_1, f_2) = \int \varphi\Lambda(g_1, g_2) + \int \varphi\Lambda(g_1, u_2) + \int \varphi\Lambda(u_1, g_2) + \int \varphi\Lambda(u_1, u_2)$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\,.$$

Estimate of I By Lemma 3.7 and Lemma 3.2 (b), we have

$$\begin{aligned} |\,\mathrm{I}\,| &\leqq c \sum_{\sigma} \prod_{j=1}^{2} t^{-n/r_j^\sigma - s_j^\sigma} \|M_m^0(g_j)\|_{v_j^\sigma, B(0,40t)} \\ &\leqq c \sum_{\sigma} \prod_{j=1}^{2} t^{-n/r_j^\sigma - s_j^\sigma} \|M_m^0(f_j)\|_{v_j^\sigma, B(0,40t)} \\ &\leqq c \sum_{\sigma} \prod_{j=1}^{2} \big(M_m^0(f_j)\big)^*_{\lambda_j^\sigma, v_j^\sigma}(0). \end{aligned}$$

Estimate of II By Lemma 3.6, we have

$$\varphi T_2^\sigma u_2 \in c F_2^\sigma(0) \mathcal{T}_N^0(0, t)$$

for each $\sigma \in \mathcal{A}$. Hence, by Lemma 3.7 with $k = 1$ and by Lemma 3.2 (b), we have

$$\begin{aligned} \left| \int \varphi(T_1^\sigma g_1)(T_2^\sigma u_2) \right| &\leqq c F_2^\sigma(0) t^{-n/r_1^\sigma - s_1^\sigma} \|M_m^0(g_1)\|_{v_1^\sigma, B(0,40t)} \\ &\leqq c F_2^\sigma(0) t^{-n/r_1^\sigma - s_1^\sigma} \|M_m^0(f_1)\|_{v_1^\sigma, B(0,40t)} \\ &\leqq c F_2^\sigma(0) \big(M_m^0(f_1)\big)^*_{\lambda_1^\sigma, v_1^\sigma}(0) \end{aligned}$$

for each $\sigma \in \mathcal{A}$. Therefore

$$|\,\mathrm{II}\,| \leqq c \sum_{\sigma \in \mathcal{A}} \big(M_m^0(f_1)\big)^*_{\lambda_1^\sigma, v_1^\sigma}(0) F_2^\sigma(0).$$

Estimate of III In the same way as in the estimate of II, we obtain

$$|\,\mathrm{III}\,| \leqq c \sum_{\sigma \in \mathcal{A}} F_1^\sigma(0) \big(M_m^0(f_2)\big)^*_{\lambda_2^\sigma, v_2^\sigma}(0).$$

Estimate of IV By Lemma 3.6, we have, for each $\sigma \in \mathcal{A}$,

$$\varphi(T_1^\sigma u_1)(T_2^\sigma u_2) \in c F_1^\sigma(0) F_2^\sigma(0) \mathcal{T}_N^0(0, t)$$

and hence

$$\left| \int \varphi (T_1^\sigma u_1)(T_2^\sigma u_2) \right| \leqq c F_1^\sigma(0) F_2^\sigma(0).$$

Thus

$$|\,\mathrm{IV}\,| \leqq c \sum_{\sigma\in\mathcal{A}} F_1^\sigma(0) F_2^\sigma(0).$$

Combining the estimates of I $\sim$ IV, we obtain

$$\left| \int \varphi \Lambda(f_1, f_2) \right| \leqq c \sum_{\sigma\in\mathcal{A}} F_1^\sigma(0) F_2^\sigma(0).$$

Since this holds for all $\varphi \in \mathcal{T}_N^0(0,t)$, $0 < t < \infty$, we have (4.2) for $x = 0$. The part (a) of Theorem is proved.

5 Proof of Theorem, Part (b)

Throughout this section we write $X = \{1, \dots, k\}$.

In order to prove the part (b) of Theorem, we first rewrite the operator Λ. Let Λ be given by (1.3) with (1.4). For each $j \in X$, take a maximal linearly independent subset of $\{T_j^\sigma \mid \sigma \in \mathcal{A}\}$ and denote it by $\{S_j^i \mid i = 1, \dots, L_j\}$. (Here the linear independence refers to that in the linear space of all the linear operators $\mathcal{S}_0 \to \mathcal{S}_0$; for operators in $\bigcup_{\lambda>0} \mathcal{K}(\lambda)$, this linear independence is the same as that of the corresponding multipliers in the linear space $C^\infty(\mathbb{R}^n \setminus \{0\})$.) Then, for each j, T_j^σ can be written as a linear combination of $\{S_j^i \mid i = 1, \dots, L_j\}$ and thus Λ can be written in the form as

$$\Lambda(f_1, \dots, f_k) = \sum_{\tau\in\mathcal{B}} b_\tau (S_1^{\tau(1)} f_1) \cdots (S_k^{\tau(k)} f_k), \tag{5.1}$$

where

$$\mathcal{B} = \{\tau \colon X \to \mathbb{N} \mid \tau(j) \leqq L_j \text{ for all } j \in X\} \tag{5.2}$$

and b_τ are complex numbers. Let μ_j^i be the number such that

$$S_j^i \in \mathcal{K}(\mu_j^i). \tag{5.3}$$

We set

$$\bar{\mu}_j = \max\{\mu_j^i \mid i = 1, \dots, L_j\}. \tag{5.4}$$

Let J be a subset of X and let $J = \{j_1, \dots, j_m\}$ with $j_1 < \cdots < j_m$. Then we use the following notations: $|J| = m$, $J^c = X \setminus J$,

$$\mathcal{B}_J = \{\rho \colon J \to \mathbb{N} \mid \rho(j) \leqq L_j \text{ for all } j \in J\},$$

and, for $f_1, \dots, f_k \in \mathcal{S}_0$,

$$\mathbf{f}_J = (f_{j_1}, \dots, f_{j_m}).$$

Now let Λ be written as in (5.1)–(5.2). Suppose J is a subset of X with $1 \leqq |J| \leqq k-1$. We can write Λ as

$$\Lambda(\mathbf{f}_X) = \sum_{\rho \in \mathcal{B}_J} \Big[\prod_{j \in J} (S_j^{\rho(j)} f_j)\Big] \Lambda_\rho^J(\mathbf{f}_{J^c})$$

with

$$\Lambda_\rho^J(\mathbf{f}_{J^c}) = \sum_{\tau \in \mathcal{B}, \tau|J=\rho} b_\tau \prod_{j \in J^c} (S_j^{\tau(j)} f_j).$$

We call Λ_ρ^J the (J, ρ)-*cofactor*.

Lemma 5.1 *Let Λ be given as in* (5.1) *with* (5.2), (5.3), *and* (5.4). *Suppose all S_j^i are homogeneous operators. Also suppose Λ satisfies the vanishing moment condition up to order K, $K \in \mathbb{N} \cup \{0\}$. Then the cofactor Λ_ρ^J satisfies the vanishing moment condition up to order $K - m_\rho$ with*

$$m_\rho = \Big[\sum_{j \in J} (\bar{\mu}_j - \mu_j^{\rho(j)})\Big].$$

Proof We shall give the proof for the case $k = 4$ and $J = \{1, 2\}$. The argument can be applied to the general case without essential change.

Let a_j^i denote the multiplier of S_j^i. We write

$$\begin{aligned} A(\xi_1, \xi_2, \xi_3, \xi_4) &= \sum_{\tau \in \mathcal{B}} b_\tau a_1^{\tau(1)}(\xi_1) a_2^{\tau(2)}(\xi_2) a_3^{\tau(3)}(\xi_3) a_4^{\tau(4)}(\xi_4) \\ &= \sum_{\rho \in \mathcal{B}_J} a_1^{\rho(1)}(\xi_1) a_2^{\rho(2)}(\xi_2) A_\rho(\xi_3, \xi_4), \end{aligned}$$

where

$$A_\rho(\xi_3, \xi_4) = \sum_{\tau \in \mathcal{B}, \tau|J=\rho} b_\tau a_3^{\tau(3)}(\xi_3) a_4^{\tau(4)}(\xi_4).$$

We shall simply write

$$A_\rho{}^{(\alpha)}(\xi_3, \xi_4) = \partial_{\xi_4}^\alpha A_\rho(\xi_3, \xi_4).$$

As we saw in Section 2.2, the vanishing moment condition of Λ which is assumed in the lemma is equivalent to this condition (M): If $\xi_1, \dots, \xi_4 \in \mathbb{R}^n \setminus \{0\}$ and $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ and if $|\alpha| \leqq K$, then

$$\partial_{\xi_4}^\alpha A(\xi_1, \xi_2, \xi_3, \xi_4) = \sum_{\rho \in \mathcal{B}_J} a_1^{\rho(1)}(\xi_1) a_2^{\rho(2)}(\xi_2) A_\rho^{(\alpha)}(\xi_3, \xi_4) = 0.$$

Also the vanishing moment condition of Λ_ρ^J which we are going to prove is equivalent to this condition (M*): If $\xi_3, \xi_4 \in \mathbb{R}^n \setminus \{0\}$ and $\xi_3 + \xi_4 = 0$ and if $|\alpha| \leqq K - m_\rho$, then $A_\rho{}^{(\alpha)}(\xi_3, \xi_4) = 0$, where $m_\rho = [\bar{\mu}_1 + \bar{\mu}_2 - \mu_1^{\rho(1)} - \mu_1^{\rho(1)}]$. We write

$$\bar{\mu} = \bar{\mu}_1 + \bar{\mu}_2 \quad \text{and} \quad \mu_\rho = \mu_1^{\rho(1)} + \mu_2^{\rho(2)} \quad (\rho \in \mathcal{B}_J).$$

Thus $m_\rho = [\bar{\mu} - \mu_\rho]$.

The set $E = \{\mu_\rho \mid \rho \in \mathcal{B}_J\}$ is a finite set of real numbers in which the maximum element is $\bar{\mu}$. We shall prove (M*) by an induction on this set E.

First we shall prove (M*) for those $\rho \in \mathcal{B}_J$ with μ_ρ maximum, *i.e.*, with $\mu_\rho = \bar{\mu}$. For such ρ, we have $m_\rho = 0$.

Let $\xi_3, \xi_4 \in \mathbb{R}^n \setminus \{0\}$ and $\xi_3 + \xi_4 = 0$ and let α be a multi-index with $|\alpha| \leqq K$.

Take arbitrary $\xi_1, \xi_2 \in \mathbb{R}^n \setminus \{0\}$. Then for all sufficiently small $\epsilon > 0$, the condition (M) implies

$$\sum_{\rho \in \mathcal{B}_J} a_1^{\rho(1)}(\epsilon\xi_1) a_2^{\rho(2)}(\epsilon\xi_2) A_\rho^{(\alpha)}(\xi_3, \xi_4 - \epsilon\xi_1 - \epsilon\xi_2) = 0.$$

By the homogeneity of a_j^i, we have

$$\sum_{\rho \in \mathcal{B}_J} \epsilon^{-\mu_\rho} a_1^{\rho(1)}(\xi_1) a_2^{\rho(2)}(\xi_2) A_\rho^{(\alpha)}(\xi_3, \xi_4 - \epsilon\xi_1 - \epsilon\xi_2) = 0. \tag{5.5}$$

We multiply (5.5) by $\epsilon^{\bar{\mu}}$ and take the limit as $\epsilon \to 0$. Then, since $\bar{\mu} = \max\{\mu_\rho\}$, we get

$$\sum_{\rho \in \mathcal{B}_J, \mu_\rho = \bar{\mu}} a_1^{\rho(1)}(\xi_1) a_2^{\rho(2)}(\xi_2) A_\rho^{(\alpha)}(\xi_3, \xi_4) = 0.$$

Since the last equality holds for all $\xi_1, \xi_2 \in \mathbb{R}^n \setminus \{0\}$ and since the functions $a_1^i(\xi_1) a_2^j(\xi_2)$ are linearly independent, we have $A_\rho^{(\alpha)}(\xi_3, \xi_4) = 0$ for each $\rho \in \mathcal{B}_J$ with $\mu_\rho = \bar{\mu}$. This proves (M*) for $\rho \in \mathcal{B}_J$ with μ_ρ maximum.

Next, we assume (M*) holds for all those $\rho \in \mathcal{B}_J$ with $\mu_\rho > \nu$ and shall prove (M*) for $\rho \in \mathcal{B}_J$ with $\mu_\rho = \nu$. Here ν is an element of the set E.

Fix ξ_3 and ξ_4 such that $\xi_3, \xi_4 \in \mathbb{R}^n \setminus \{0\}$ and $\xi_3 + \xi_4 = 0$. Also fix a multi-index α such that $|\alpha| \leqq K - [\bar{\mu} - \nu]$. What we have to show is $A_\rho^{(\alpha)}(\xi_3, \xi_4) = 0$ for each $\rho \in \mathcal{B}_J$ with $\mu_\rho = \nu$.

As above, the equality (5.5) holds for all $\xi_1, \xi_2 \in \mathbb{R}^n \setminus \{0\}$ and for all sufficiently small $\epsilon > 0$. For the moment, we shall simply write $\epsilon^{-\mu_\rho}\tilde{A}_{\epsilon,\rho}$ to denote each term on the left hand side of (5.5), *i.e.*,

$$\epsilon^{-\mu_\rho}\tilde{A}_{\epsilon,\rho} = \epsilon^{-\mu_\rho} a_1^{\rho(1)}(\xi_1) a_2^{\rho(2)}(\xi_2) A_\rho^{(\alpha)}(\xi_3, \xi_4 - \epsilon\xi_1 - \epsilon\xi_2).$$

We multiply (5.5) by ϵ^ν and take the limit as $\epsilon \to 0$.

For ρ with $\mu_\rho = \nu$, we have, clearly,

$$\epsilon^{\nu-\mu_\rho}\tilde{A}_{\epsilon,\rho} \longrightarrow a_1^{\rho(1)}(\xi_1) a_2^{\rho(2)}(\xi_2) A_\rho^{(\alpha)}(\xi_3, \xi_4).$$

We shall show that $\epsilon^{\nu-\mu_\rho}\tilde{A}_{\epsilon,\rho} \to 0$ for $\rho \in \mathcal{B}_J$ with $\mu_\rho \neq \nu$. Since this is clear when $\mu_\rho < \nu$, it is sufficient to consider the case $\mu_\rho > \nu$.

Suppose $\mu_\rho > \nu$. The induction hypothesis implies that ${A_\rho}^{(\beta)}(\xi_3, \xi_4) = 0$ for $|\beta| \leqq K - m_\rho$. Hence, by Taylor's formula, we have

$$A_\rho^{(\alpha)}(\xi_3, \xi_4 - \epsilon\xi_1 - \epsilon\xi_2) = O(\epsilon^{K - m_\rho + 1 - |\alpha|}).$$

(This estimate holds even if $|\alpha| > K - m_\rho$, since in this case the estimate is weaker than the obvious estimate $O(1)$.) Thus

$$\epsilon^{\nu-\mu_\rho}\tilde{A}_{\epsilon,\rho} = O(\epsilon^{K+1+\nu-\mu_\rho-m_\rho-|\alpha|}).$$

This implies $\epsilon^{\nu-\mu_\rho}\tilde{A}_{\epsilon,\rho} \to 0$ since

$$|\alpha| \leqq K - [\bar{\mu} - \nu] < K + 1 - \bar{\mu} + \nu \leqq K + 1 + \nu - \mu_\rho - m_\rho.$$

Thus

$$\epsilon^{\nu} \times (\text{the left hand side of (5.5)}) \longrightarrow \sum_{\rho \in \mathcal{B}_J, \mu_\rho = \nu} a_1^{\rho(1)}(\xi_1) a_2^{\rho(2)}(\xi_2) A_\rho^{(\alpha)}(\xi_3, \xi_4).$$

Therefore we obtain

$$\sum_{\rho \in \mathcal{B}_J, \mu_\rho = \nu} a_1^{\rho(1)}(\xi_1) a_2^{\rho(2)}(\xi_2) A_\rho^{(\alpha)}(\xi_3, \xi_4) = 0.$$

By the linear independence of the functions $a_1^i(\xi_1)a_2^j(\xi_2)$, we have $A_\rho^{(\alpha)}(\xi_3, \xi_4) = 0$ for each $\rho \in \mathcal{B}_J$ with $\mu_\rho = \nu$, as desired. Lemma 5.1 is proved.

Proof of the Part (b) of Theorem The main idea is the same as in the proof of the part (a).

We write Λ as in (5.1) with (5.2), (5.3), and (5.4). We also write

$$\mathcal{B}^\times = \{\tau \in \mathcal{B} \mid b_\tau \neq 0\}.$$

The conditions (1.5), (1.6), and (1.7) can now be written as follows:

$$\sum_{j=1}^{k} \mu_j^{\tau(j)} = \lambda \quad \text{for every } \tau \in \mathcal{B}^\times,$$

$$\infty > \frac{1}{p_j} > \frac{\bar{\mu}_j}{n} \quad \text{for every } j \in X,$$

$$\sum_{j=1}^{k} \left(\frac{1}{p_j} - \frac{\mu_j^{\tau(j)}}{n} \right) = \frac{1}{q} \quad \text{for every } \tau \in \mathcal{B}^\times.$$

As in Section 4, we write $N = [n/q - n] + 1$.

We shall prove that there exist real numbers ν_j^τ ($\tau \in \mathcal{B}^\times$, $j \in X$) satisfying

$$0 < \nu_j^\tau < p_j \quad (\tau \in \mathcal{B}^\times, j \in X) \tag{5.6}$$

with which the pointwise estimate

$$M_N^0\big(\Lambda(f_1, \ldots, f_k)\big)(x) \leqq c_m \sum_{\tau \in \mathcal{B}^\times} |b_\tau| \prod_{j=1}^{k} G_m(f_j, S_j^{\tau(j)}, \nu_j^\tau)(x) \tag{5.7}$$

holds for every $m \in \mathbb{N} \cup \{0\}$ and for all $f_1, \dots, f_k \in \mathcal{S}_0$. (As for the notation $G_m(\cdot,\cdot,\cdot)$, see Section 4.) By the same reason as in the proof of the part (a), this pointwise estimate implies the desired H^q estimate.

By translation invariance, it is sufficient to show (5.7) for $x = 0$. We set

$$K = N + n + 2 + \max\{[\bar{\mu}_j] \mid j \in X\}.$$

We decompose f_j ($j \in X$) as in (4.3)–(4.4). Let $\varphi \in \mathcal{T}_N^0(0, t)$. The integral

$$\int \varphi \Lambda(f_1, \dots, f_k),$$

which we shall estimate, can be written as the sum of 2^k terms each of which is the form

$$\int \varphi \Lambda(\tilde{f}_1, \dots, \tilde{f}_k) \quad \text{with } \tilde{f}_j = g_j \text{ or } u_j.$$

We shall estimate each term separately.

We shall prove that for each one of the above 2^k terms we can take v_j^τ ($\tau \in \mathcal{B}^\times$, $j \in X$) satisfying (5.6) and

$$\left| \int \varphi \Lambda(\tilde{f}_1, \dots, \tilde{f}_k) \right| \leqq \text{ (the right hand side of (5.7) with } x = 0).$$

Our (v_j^τ) may be different for each term; *i.e.*, our (v_j^τ) may depend on the set $\{j \in X \mid \tilde{f}_j = g_j\}$. This, however, is sufficient for our purpose. Indeed, the maximal function

$$\left(M_m^0(f_j)\right)^*_{\mu_j^{\tau(j)}, v_j^\tau}$$

can only be bigger, except for a constant factor, when one replaces v_j^τ by a bigger number (by Hölder's inequality). Hence we have only to fix v_j^τ, for each (τ, j), to be the maximum one of the possibly 2^k different v_j^τ's.

First we shall estimate the term with $\tilde{f}_j = g_j$ for all $j \in X$. We can choose r_j^τ and s_j^τ ($\tau \in \mathcal{B}^\times$, $j \in X$) such that

$$0 < \frac{1}{r_j^\tau} < 1, \quad \sum_{j=1}^k \frac{1}{r_j^\tau} \leqq 1, \quad 0 \leqq s_j^\tau < \infty, \quad \sum_{j=1}^k s_j^\tau \leqq N,$$
$$\text{and} \quad \frac{1}{r_j^\tau} + \frac{s_j^\tau}{n} > \frac{1}{p_j} - \frac{\mu_j^{\tau(j)}}{n}.$$

We define v_j^τ by $1/v_j^\tau = 1/r_j^\tau + s_j^\tau/n + \mu_j^{\tau(j)}/n$ ($\tau \in \mathcal{B}^\times$, $j \in X$).

Then (5.6) holds and, by Lemma 3.7 and Lemma 3.2 (b), we have

$$\left|\int \varphi\Lambda(g_1,\dots,g_k)\right| \leqq c \sum_{\tau\in\mathcal{B}^\times} |b_\tau| \prod_{j=1}^{k} t^{-n/r_j^\tau - s_j^\tau} \|M_m^0(f_j)\|_{\nu_j^\tau, B(0,40t)}$$

$$\leqq c \sum_{\tau\in\mathcal{B}^\times} |b_\tau| \prod_{j=1}^{k} \big(M_m^0(f_j)\big)^*_{\mu_j^{\tau(j)},\nu_j^\tau}(0).$$

Next we estimate the term with $\tilde{f}_j = u_j$ for all $j \in X$. We take ν_j^τ such that $n/(K+1+n) < \nu_j^\tau < p_j$. Then, by Lemma 3.6, we have

$$\varphi\Lambda(u_1,\dots,u_k) \in c \sum_{\tau\in\mathcal{B}^\times} |b_\tau| \Big[\prod_{j=1}^{k} G_m(f_j, S_j^{\tau(j)}, \nu_j^\tau)(0)\Big] \mathcal{T}_N^0(0,t)$$

and hence

$$\left|\int \varphi\Lambda(u_1,\dots,u_k)\right| \leqq c \sum_{\tau\in\mathcal{B}^\times} |b_\tau| \prod_{j=1}^{k} G_m(f_j, S_j^{\tau(j)}, \nu_j^\tau)(0).$$

Finally we estimate the terms with $\tilde{f}_j = u_j$ for some $j \in X$ and $\tilde{f}_j = g_j$ for some other $j \in X$. As a typical example of such terms, we shall treat the case where $k = 4$ and $(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4) = (u_1, u_2, g_3, g_4)$. (General case can be treated in a similar way.) We write $J = \{1, 2\}$.

We can write

$$\varphi\Lambda(u_1,u_2,g_3,g_4) = \sum_{\rho\in\mathcal{B}_J} \varphi\Big[\prod_{j=1}^{2} (S_j^{\rho(j)} u_j)\Big] \Lambda_\rho^J(g_3,g_4).$$

Fix a $\rho \in \mathcal{B}_J$.

By Lemma 3.6, we have

$$\varphi \prod_{j=1}^{2} (S_j^{\rho(j)} u_j) \in c\Big[\prod_{j=1}^{2} G_m(f_j, S_j^{\rho(j)}, p_j^\rho)(0)\Big] \mathcal{T}_N^0(0,t), \tag{5.8}$$

where we take p_j^ρ such that $n/(K+1+n) < p_j^\rho < p_j$.

By Lemma 5.1, the operator Λ_ρ^J satisfies the vanishing moment condition up to order $N - 1 - m_\rho$ where $m_\rho = [\bar{\mu}_1 + \bar{\mu}_2 - \mu_1^{\rho(1)} - \mu_1^{\rho(1)}]$. Hence, by Lemma 3.7 and Lemma 3.2 (b), the estimate

$$\left|\int \psi\Lambda_\rho^J(g_3,g_4)\right| \leqq c \sum_{\tau\in\mathcal{B}^\times, \tau|J=\rho} |b_\tau| \prod_{j=3}^{4} t^{-n/r_j^\tau - s_j^\tau} \|M_m^0(f_j)\|_{\nu_j^\tau, B(0,40t)} \tag{5.9}$$

holds for all $\psi \in \mathcal{T}_N^0(0,t)$ and for $r_j^\tau, s_j^\tau, v_j^\tau$ ($\tau \in \mathcal{B}^\times$, $\tau|J = \rho$, $j = 3, 4$) satisfying

$$0 < \frac{1}{r_j^\tau} < 1, \quad \sum_{j=3}^{4} \frac{1}{r_j^\tau} \leqq 1, \tag{5.10}$$

$$0 \leqq s_j^\tau < \infty, \quad \sum_{j=3}^{4} s_j^\tau \leqq \max\{N - m_\rho, 0\}, \tag{5.11}$$

and $1/v_j^\tau = 1/r_j^\tau + s_j^\tau/n + \mu_j^{\tau(j)}/n$.

If $\tau \in \mathcal{B}^\times$ and $\tau|J = \rho$, then

$$\sum_{j=1}^{2}\left(\frac{1}{p_j} - \frac{\mu_j^{\rho(j)}}{n}\right) + \sum_{j=3}^{4}\left(\frac{1}{p_j} - \frac{\mu_j^{\tau(j)}}{n}\right) = \frac{1}{q} < 1 + \frac{N}{n}.$$

We have

$$\begin{aligned}\sum_{j=1}^{2}\left(\frac{1}{p_j} - \frac{\mu_j^{\rho(j)}}{n}\right) &= \sum_{j=1}^{2}\left(\frac{1}{p_j} - \frac{\bar{\mu}_j}{n}\right) + \sum_{j=1}^{2}\frac{\bar{\mu}_j - \mu_j^{\rho(j)}}{n} \\ &\geqq \sum_{j=1}^{2}\left(\frac{1}{p_j} - \frac{\bar{\mu}_j}{n}\right) + \frac{m_\rho}{n} > \frac{m_\rho}{n}.\end{aligned}$$

Hence, for $\tau \in \mathcal{B}^\times$ with $\tau|J = \rho$, we have

$$\sum_{j=3}^{4}\left(\frac{1}{p_j} - \frac{\mu_j^{\tau(j)}}{n}\right) < 1 + \frac{N - m_\rho}{n}.$$

Therefore we can choose r_j^τ and s_j^τ ($\tau \in \mathcal{B}^\times$, $\tau|J = \rho$, $j = 3, 4$) which satisfy (5.10) and (5.11) and also satisfy $1/r_j^\tau + s_j^\tau/n > 1/p_j - \mu_j^{\tau(j)}/n$. Choosing r_j^τ and s_j^τ in this way, we have $0 < v_j^\tau < p_j$ ($\tau \in \mathcal{B}^\times$, $\tau|J = \rho$, $j = 3, 4$).

Now, using (5.8) and (5.9), we obtain

$$\begin{aligned}&\left|\int \varphi\Big[\prod_{j=1}^{2}(S_j^{\rho(j)}u_j)\Big]\Lambda_\rho^J(g_3, g_4)\right| \\ &\quad\leqq c\Big[\prod_{j=1}^{2} G_m(f_j, S_j^{\rho(j)}, p_j^\rho)(0)\Big] \sum_{\tau\in\mathcal{B}^\times,\tau|J=\rho} |b_\tau| \prod_{j=3}^{4} t^{\mu_j^{\tau(j)} - n/v_j^\tau} \|M_m^0(f_j)\|_{v_j^\tau, B(0,40t)} \\ &\quad\leqq c\Big[\prod_{j=1}^{2} G_m(f_j, S_j^{\rho(j)}, p_j^\rho)(0)\Big] \sum_{\tau\in\mathcal{B}^\times,\tau|J=\rho} |b_\tau| \prod_{j=3}^{4} \big(M_m^0(f_j)\big)^*_{\mu_j^{\tau(j)}, v_j^\tau}(0) \\ &\quad\leqq \text{ (the right hand side of (5.7) with } x = 0).\end{aligned}$$

The part (b) of Theorem is proved.

Remark In the arguments of this section, the homogeneity of T_j^σ was used only in the proof of Lemma 5.1. Therefore the part (b) of Theorem still holds if we replace the assumption "all the operators T_j^σ are homogeneous" by the following assumption: Λ is written as in (5.1) with (5.2), (5.3), and (5.4), in which each cofactor Λ_ρ^J with $1 \leqq |J| \leqq k-1$ satisfies the vanishing moment condition up to order

$$\left[\frac{n}{q} - n\right] - \Big[\sum_{j\in J} (\bar{\mu}_j - \mu_j^{\rho(j)})\Big].$$

6 Examples

In this section we give some examples of the operators Λ of Theorem with $\lambda = \sum_{j=1}^k \lambda_j^\sigma > 0$. Examples with $\lambda = 0$ can be found in [CRW, Theorem II], [M1], [CLMS], [CG], and [G].

Example 1 **(Product of functions in Sobolev spaces)** For $0 < p < \infty$ and $m \in \mathbb{N}$, let $W^{p,m}$ be the set of all those $f \in \mathcal{D}'(\mathbb{R}^n)$ for which $\partial^\alpha f \in H^p$ for all multi-indices α with $|\alpha| = m$. We shall consider the case $m/n < 1/p < 1 + m/n$. In this case, if $f \in W^{p,m}$, then there exists a unique polynomial $P_f \in \mathcal{P}_{m-1}$ such that $f - P_f \in L^q$ with $1/q = 1/p - m/n$ and

$$\|f - P_f\|_q \leqq c \sum_{|\alpha|=m} \|\partial^\alpha f\|_{H^p} \tag{6.1}$$

(see, *e.g.*, [M4, Theorem 4.3 and Section 1, Remark (1°)]). We define $W_0^{p,m}$ as the set of all $f \in W^{p,m}$ with $P_f = 0$.

Suppose $f \in W_0^{p_1,1}$ and $g \in W_0^{p_2,1}$ with $1/n < 1/p_i < 1 + 1/n$ $(i = 1, 2)$. We set $1/q_i = 1/p_i - 1/n$ $(i = 1, 2)$ and $1/r = 1/p_1 + 1/p_2 - 1/n$. Then formal application of the Leibniz rule and Hölder's inequality, together with (6.1), gives

$$\begin{aligned}\|\partial_j(fg)\|_r = \|(\partial_j f)g + f\partial_j g\|_r &\leqq c(\|\partial_j f\|_{p_1} \|g\|_{q_2} + \|f\|_{q_1} \|\partial_j g\|_{p_2}) \\ &\leqq c\Big(\sum_{j=1}^n \|\partial_j f\|_{H^{p_1}}\Big)\Big(\sum_{j=1}^n \|\partial_j g\|_{H^{p_2}}\Big).\end{aligned}$$

Thus we may well expect that $\partial_j(fg) \in L^r$. If $r \geqq 1$, this is indeed true (rigorous proof is easy). But, in the case $r = 1$, we have in fact a stronger conclusion that $\partial_j(fg) \in H^1$.

More generally, the following is true: If $m \in \mathbb{N}$, $m/n < 1/p_j < 1 + m/n$ $(j = 1, 2)$, and $1/r = 1/p_1 + 1/p_2 - m/n < 1 + m/n$, and if $f_1 \in W_0^{p_1,m}$ and $f_2 \in W_0^{p_2,m}$, then $f_1 f_2 \in L^1_{\mathrm{loc}}$ and $f_1 f_2 \in W_0^{r,m}$ and

$$\sum_{|\alpha|=m} \|\partial^\alpha (f_1 f_2)\|_{H^r} \leqq c\Big(\sum_{|\alpha|=m} \|\partial^\alpha f_1\|_{H^{p_1}}\Big)\Big(\sum_{|\alpha|=m} \|\partial^\alpha f_2\|_{H^{p_2}}\Big). \tag{6.2}$$

For this fact, see [F], [M3, Theorems 1.1 and 4.1], or [SiT].

Now, apart from limiting arguments, the above fact is a consequence of our theorem. To see this, assume $f_1, f_2 \in \mathcal{S}_0$ and set

$$F_j = |\nabla|^m f_j = \left(|2\pi\xi|^m \hat{f}_j(\xi)\right)^\vee \quad (j = 1, 2).$$

Then

$$f_j = |\nabla|^{-m} F_j = \left(|2\pi\xi|^{-m} \hat{F}_j(\xi)\right)^\vee$$

and

$$\|F_j\|_{H^{p_j}} = \| \, |\nabla|^m f_j\|_{H^{p_j}} \approx \sum_{|\alpha|=m} \|\partial^\alpha f_j\|_{H^{p_j}}.$$

For $|\alpha| = m$, the derivative $\partial^\alpha(f_1 f_2)$ can be written as

$$\begin{aligned}\partial^\alpha(f_1 f_2) &= \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} (\partial^\beta f_1)(\partial^{\alpha-\beta} f_2)\\ &= \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} (\partial^\beta |\nabla|^{-m} F_1)(\partial^{\alpha-\beta} |\nabla|^{-m} F_2),\end{aligned}$$

which is the form $\Lambda(F_1, F_2)$ with a bilinear operator Λ of (1.3)–(1.4). This Λ satisfies the condition (1.5) with $\lambda = m$ and also satisfies the moment condition

$$\Lambda(F_1, F_2) = \partial^\alpha(f_1 f_2) \perp \mathcal{P}_{m-1} \quad (|\alpha| = m).$$

Hence our theorem gives the estimate (6.2).

In the following examples, we give general methods to define the operator Λ satisfying the vanishing moment condition.

Example 2 Let $a_j \in G(\lambda_j)$, $0 \leqq \lambda_j < \infty$ $(j = 1, \dots, N)$. Define the bilinear operator Λ by

$$\left(\Lambda(f, g)\right)^\wedge(\xi) = \int_{\mathbb{R}^n} \hat{f}(\xi - \eta)\hat{g}(\eta) \prod_{j=1}^{N} \left(a_j(\xi - \eta) - a_j(-\eta)\right) d\eta$$

for $f, g \in \mathcal{S}_0$. It is easy to see that Λ is of the form (1.3)–(1.4) and the assumption (1.5) is satisfied with $\lambda = \sum_{j=1}^N \lambda_j$. It is also easy to see that Λ satisfies the vanishing moment condition up to order $N - 1$ (observe that the integrand in the above integral is $O(|\xi|^N)$ as $\xi \to 0$).

This operator was treated in [M1] and [M2] under the restriction $0 \leqq \lambda < n$.

Example 3 Suppose Λ is defined by (1.3). If there exists a closed half space

$$E = \{\xi \in \mathbb{R}^n \mid u\xi \geqq 0\}, \quad u \in \mathbb{R}^n \setminus \{0\},$$

(where $u\xi$ denotes the usual inner product of two vectors in $\mathbb{R}^n$) such that all the supports of the multipliers of T_j^σ are included in E, then Λ satisfies the vanishing moment condition of all orders. This can be easily seen by checking the condition (2.2).

Example 4 For integers i, set

$$D_i = \{\xi \in \mathbb{R}^n \mid 2^{i-1} \leqq |\xi| \leqq 2^{i+1}\}.$$

Take $A, B \in \mathbb{N}$ such that $A > 10B$ and $B > 10$. Set

$$E_m = \bigcup_{i \equiv mB \ (\mathrm{mod}\ A)} D_i \quad (m = 1, \dots, 10).$$

If Λ is defined by (1.3) with $k \leqq 10$ and if the support of the multiplier of T_j^σ is included in E_j for every $\sigma \in \mathcal{A}$ and for $j = 1, \dots, k$, then Λ satisfies the vanishing moment condition of all orders. This is also easily checked by means of the condition (2.2).

To the above Λ, the part (b) of Theorem in its original form can not be applied except for the trivial case $\Lambda = 0$, since homogeneous operator $T_j^\sigma \neq 0$ does not satisfy the above support condition. But, the modified (b) as given in Remark at the end of Section 5 can be applied.

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