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DIFFUSION APPROXIMATION FOR A KNUDSEN GAS IN A THIN DOMAIN WITH REFLEXIVE CHAOTIC LAW

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This paper treats a rarefied Knudsen gas flow between two infinite plates, with boundary reflexion ruled by a reflexive chaotic law called "Arnold's cat map". It is shown that the limiting behaviour, when the distance between the plates goes to 0, is described by an (anisotropic) diffusion equation in the norm topology.

1. Introduction and main results

This paper is concerned with a rarefied Knudsen gas flow model, that is, for a gas with no interparticle collisions between two infinite plates, with boundary reflexion ruled by a 2-torus hyperbolic automorphism. We extend previous work of Bardos et al. [2], deriving an irreversible asymptotic diffusion limit (the heat semigroup) for a simple reversible dynamic described by a continuous unitary group of L^2. We investigate in detail the spectral measure of such a flow. The general framework of this problem is detailed in [1]. The reason for the existence of a diffusion limit is a consequence of the ergodic theory of Anosov's system, using a Markov partition and symbolic dynamics (see Sinai [5]). Our goal is to produce a proof which involves no ergodic theory and which in the present case uses only elementary techniques such as a Fourier series expansion instead of a Markov partition for coding.

Starting with a rescaled kinetic model of the form

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} a(\omega) \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon^2} A f_\varepsilon = 0, \tag{1.1}$$

$$f_{\varepsilon|t=0} = \phi(x), \tag{1.2}$$

where A is an operator acting on the dependence in ω of f_ε, two different asymptotical regimes can be observed as follows.

Case 1. *A is positive, self-adjoint and Fredholm.* In fact A is the orthogonal projection on the functions of mean-value 0. Furthermore, we assume that $a(\omega) \perp \operatorname{Ker} A$. In this

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case, it is shown (see [**1**]) that f_ε converges strongly to $f \equiv f(t,x)$ which is the solution of the diffusion equation

$$\partial_t f = \frac{1}{2} D \Delta_x f\,, \quad f_{t=0} = \phi, \tag{1.3}$$

with the diffusion coefficient D given by

$$D = \Big\langle \psi \mid a(\omega) \Big\rangle\,, \quad A\psi = a(\omega)\,, \quad \psi \in \operatorname{Ker} A^{\perp}\,. \tag{1.4}$$

Here the notation $\langle \cdot \mid \cdot \rangle$ is prescribed by

$$\langle f \mid g \rangle = \int_{X\times Y} f(.,.)\overline{g(.,.)}\, dxdy\,.$$

Case 2. *A is skew-adjoint, with eigenvalue* 0 *included in the continuous spectrum of* A. With ergodicity of the group generated by A, we assume again that $\operatorname{Ker} A$ is reduced on the functions of mean-value 0 (in the variable ω) and that $a(\omega) \perp \operatorname{Ker} A$. In this case ψ does not exist, as A is *not Fredholm*, but we can show that the formula which gives D has mathematical meaning. To do this, we regularise (1.1) by introducing a parameter $\lambda > 0$ *via*

$$\partial_t f_\varepsilon^\lambda + \frac{1}{\varepsilon} a(\omega) \cdot \nabla_x f_\varepsilon^\lambda + \frac{1}{\varepsilon^2}\Big[A f_\varepsilon^\lambda + \lambda\Big(f_\varepsilon^\lambda - \Pi f_\varepsilon^\lambda\Big)\Big] = 0\,, \tag{1.5}$$

where Π denotes the orthogonal projection on $\operatorname{Ker} A$. For $\lambda > 0$ fixed, we make a formal multi-scale expansion in ε and show that f_ε^λ converges for $\varepsilon \to 0$ to the solution of a diffusion equation of type (1.3) and D_λ is the diffusion coefficient, depending on λ. We now investigate the conditions under which D_λ has a finite positive limit. We expand $f_\varepsilon^\lambda(t,x,\omega)$, the solution of (1.5), in powers of ε as

$$f_\varepsilon^\lambda(t,x,\omega) = f_0^\lambda(t,x) + \varepsilon f_1^\lambda(t,x,\omega) + \varepsilon^2 f_2^\lambda(t,x,\omega) + \dots\,. \tag{1.6}$$

The terms of order $1/\varepsilon^2$ vanish, since f_0^λ is independent of ω. The terms of order $1/\varepsilon$ give

$$f_1^\lambda(t,x,\omega) = -\psi^\lambda(\omega) \nabla_x f_0^\lambda(t,x), \tag{1.7}$$

where

$$(\lambda + A)\psi^\lambda = a(\omega). \tag{1.8}$$

Let E be the spectral resolution of the identity associated with A and assume that

$$E(-i\sigma) = E(i\sigma); \qquad E(i\sigma) = \delta_{\sigma=0}\Pi + \rho(\sigma) d\sigma, \tag{1.9}$$

where ρ is a continuous function in a neighbourhood of $\sigma = 0$ with orthogonal projection values. The solution of (1.8) is given by

$$\psi^\lambda(\omega) = \int_{\mathbf{R}} \frac{dE(i\sigma)a(\omega)}{\lambda + i\sigma}. \tag{1.10}$$

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This expression has no limit when $\lambda \to 0$.

Formally, if we project (1.5) on the kernel of A, we get

$$\partial_t f_0^\lambda - \nabla_x \cdot \left[\left\langle a(\omega)\psi^\lambda \mid 1\right\rangle \nabla_x f_0\right] = 0, \tag{1.11}$$

which is a diffusion equation with diffusion matrix

$$D^\lambda = \int_{\mathbf{R}} \frac{\left\langle dE(i\sigma)a(\omega) \,\middle|\, a(\omega)\right\rangle}{\lambda + i\sigma} = \int_{\mathbf{R}} \frac{\lambda\left\langle dE(i\sigma)a(\omega) \mid a(\omega)\right\rangle}{\lambda^2 + \sigma^2}, \tag{1.12}$$

as may be seen by changing σ into $-\sigma$ and averaging the two integrals. Since $\lambda/(\lambda^2 + \sigma^2)$ converges to $\pi\delta_{\sigma=0}$ when $\lambda \to 0$ and ρ is continuous near $\sigma = 0$, we have

$$D^\lambda \to \pi\left\langle \rho(0)a(\omega) \mid a(\omega)\right\rangle. \tag{1.13}$$

The existence of a solution to (1.8) is more or less equivalent to the convergence of the integral (1.10). Therefore, to find an approximation when (1.8) has no solution (while 0 is in the spectra because $A1 = 0$), it is natural to consider the integral (1.12) which may converge even if (1.10) does not. In fact, with sufficiently strong regularity on the spectral measure $S(\sigma) = d\big(E(i\sigma)a(\omega) \otimes a(\omega)\big)$ of the operator A, namely that $S(\sigma)$ is continuous near zero, it can be proved that f_ε converges to the solution of

$$\partial_t f - \nabla(S(0)\nabla_x f) = 0,$$

weakly if $S(0) \neq 0$ and strongly if $S(0) = 0$. The integral (1.12) is an abstract version of the so-called Einstein–Kubo formula, which appears in many deterministic and stochastic diffusion approximations.

As an example showing the importance of the preceeding discussion, a diffusion approximation of a Knudsen gas flow model will be constructed. The mathematical model is as follows.

A family of particles evolves as a Knudsen gas between two horizontal plates. The vertical components of the particle velocities are all assumed to have modulus $c > 0$. The horizontal components of their velocities $ca(\omega)$ are parametrised by $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$. Whenever a particle hits the top or bottom plate, its vertical velocity is reversed while the horizontal velocity is modified by the right action of a hyperbolic automorphism of $\mathbb{T}^2$ (see Figure 1).

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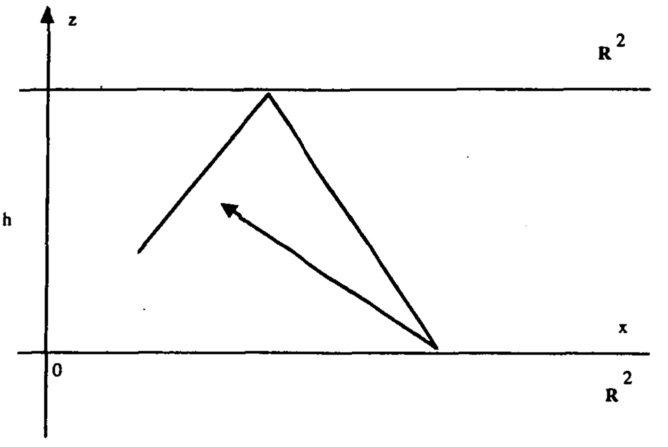

Figure 1.

The position of a particle is denoted by $(x, z) \in \mathbb{R}^d \times (0, h)$ and the vertical component of its velocity by $\pm c$. The horizontal component of this velocity is given by $ca(\omega)$, $\omega \in \mathbb{T}^2$, where $a : \mathbb{T}^2 \to \mathbb{R}^d$ denotes a smooth zero-mean vector field.

The nonnegative function $f^+(t, x, z, \omega)$ (respectively, $f^-(t, x, z, \omega)$) represents the density number of the particles which at time t occupy position (x, z) and move with wave vector ω (with horizontal velocity $\big(ca(\omega), +c\big)$ (respectively, $\big(ca(\omega), -c\big)$)). The densities $f^\pm$ satisfy the Liouville (Knudsen gas) system of equations

$$\partial_t f^\pm + a(\omega) \cdot \nabla_x f^\pm \pm c\partial_z f^\pm = 0\,, \quad x \in \mathbb{R}^d\,,\ 0 < z < h\,,\ \omega \in \mathbb{T}^2, \tag{1.14}$$

with boundary conditions

$$f^+(t, x, 0, \omega) = f^-(t, x, 0, T\omega) \tag{1.15}$$

$$f^-(t, x, h, \omega) = f^+(t, x, h, T\omega)\,, \tag{1.16}$$

on the plates, exhibiting a change of wave number on each line. Their values at $t = 0$ are given by

$$f^\pm(0, x, z, \omega) = \phi(x)\,, \tag{1.17}$$

which is compatible with an approximation by a horizontal diffusion as $h \to 0$ and which precludes the appearance of an initial layer in the limiting process. This asymptotic limit, leading to a horizontal diffusion, is obtained by letting h tend to zero and observing the system at large positive times. If a small parameter $\varepsilon > 0$ is introduced, with t replaced by t/ε, the problem of interest becomes

$$\partial_t f_\varepsilon^\pm + \frac{1}{\varepsilon} a(\omega) \cdot \nabla_x f_\varepsilon^\pm \pm \frac{1}{\varepsilon^2} c\partial_z f_\varepsilon^\pm = 0\,. \tag{1.18}$$

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Here ε is the mean free path, that is, the mean free-flight distance of a particle. It is clear that these scalings do not induce any modification in the boundary conditions, namely $f_\varepsilon^\pm$ still satisfy (1.15)—(1.16). We prescribe an initial condition

$$f_\varepsilon^\pm(0,x,z,\omega) = \phi(x) \tag{1.19}$$

which is compatible with the expected asymptotic dynamics.

Bardos-Colonna-Golse [2] show that there exists a positive matrix $D(a)$, such that for $\phi \in C^\infty\big(\mathbb{R}^d\big)$ and for any $\tau > 0$, the functions $f_\varepsilon^\pm$, which are the solutions of Equations (1.20)—(1.15)—(1.16), converge in $C^0\Big([0,\tau], w^* - L^\infty(\mathbb{R}^d \times \mathbb{T}^2)\Big)$, to the solution of the diffusion equation

$$\partial_t f = \frac{1}{2} hc\nabla_x\big(D(a)\nabla_x f\big), \qquad f(0,x) = \phi(x),$$

as $\varepsilon \to 0$, with the diffusion coefficient $D(a)$ given by

$$D(a) = \frac{1}{2}\langle a^2\rangle + \sum_{k\geqslant 1}\langle a\circ T^k \otimes a\rangle = \frac{1}{2}\lim_{N\to\infty}\left\langle\left(\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a\circ T^k\right)^{\otimes 2}\right\rangle \geqslant 0. \tag{1.20}$$

The series $\sum_{k\geqslant 1}\Big\|\langle a\circ T^k \otimes a\rangle\Big\| < +\infty$ for any norm $\|\cdot\|$ on $M_d(\mathbb{R})$, where

$$\langle F\rangle = \left(\frac{1}{4\pi^2}\int_{\mathbf{T}^2} F\omega\right) dw.$$

Here $A\otimes B := A^t.B$ and $A^{\otimes 2}$ denotes the $r\times r$ symmetric matrix $A\otimes A$.

We now present a simpler proof of [**2**, Theorem 3].

The model examined in [**2**] is purely non-collisional. In this paper we take into account the rare collisions between molecules, which have a regularising effect for the approximation. We start with the model of [**2**], with an additional collision operator. The time variable is rescaled as $t \to t/\varepsilon$ and the problem of interest, posed in its scaled form, becomes

$$\partial_t f_{\varepsilon,\lambda}^\pm + \frac{1}{\varepsilon}a(\omega)\cdot\nabla_x f_{\varepsilon,\lambda}^\pm \pm \frac{1}{\varepsilon^2}c\partial_z f_{\varepsilon,\lambda}^\pm + \frac{\lambda}{\varepsilon^2}\Big[f_{\varepsilon,\lambda}^\pm - \langle\!\langle f_{\varepsilon,\lambda}^\pm\rangle\!\rangle\Big] = 0\,, \tag{1.21}$$

with boundary conditions (1.15)—(1.16) and initial data (1.19), where $\langle\!\langle\cdot\rangle\!\rangle$ is given by

$$\langle\!\langle F\rangle\!\rangle = \frac{1}{4\pi^2}\int_{\mathbf{T}^2\times[0,h]} F(\omega,z)\,d\omega dz\,.$$

Here λ is a positive constant representing the inverse of the average collision time.

The following asymptotical regime is observed.

For all $\lambda > 0$, the densities $f_{\varepsilon,\lambda}^\pm$ converge strongly to the solution of the diffusion equation $\partial_t f_\lambda = D_\lambda\Delta f_\lambda$, *not uniformly (a priori)* in λ. The following two questions are obviously interesting.

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(Q1) Does $D_\lambda \to D$ *as* $\lambda \to 0$?

(Q2) Do $f^{\pm}_{\varepsilon,\lambda} \to f^{\pm}_{\lambda}$ *as* $\varepsilon \to 0$, uniformly in λ?

The preceeding discussions show that to obtain the diffusion approximation, we have to study the resolution of the identity of the operator A.

The first main result of this paper is the following.

THEOREM 1.1. 1. *The spectrum of the operator A lies on the imaginary axis $i\mathbb{R}$.*

2. *Let $(a_k)_{k\in\mathbf{Z}^2\setminus\{0\}}$ be a family of complex numbers satisfying*

$$a_{M^2k} = a_k e^{2i\lambda} \tag{1.22}$$

and define $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then the family $\left(\phi^\lambda_+, \phi^\lambda_-\right)$ defined by

$$\phi^\lambda_+(z,\omega) = \frac{1}{4\pi^2} e^{\lambda z} \sum_{k\neq 0} a_{M^{-1}k} e^{ik\cdot\omega} \tag{1.23}$$

and

$$\phi^\lambda_-(z,\omega) = \frac{1}{4\pi^2} e^{-\lambda z} \sum_{k\neq 0} a_k e^{ik\cdot\omega} \tag{1.24}$$

is a generalised eigenvector of A for the element of the spectrum $i\lambda$, $\lambda\in\mathbb{R}$.

3. *The family $\left(\phi^\lambda_{+,\gamma}, \phi^\lambda_{-,\gamma}\right)$ indexed by the orbits γ of $\mathbb{Z}^2\setminus\{0\}$ under the action by multiplication on the left by M^2, defined by*

$$\phi_{+,\gamma}(z,\omega) = \frac{1}{4\pi^2} \sum_{n\in\mathbf{Z}} e^{i\lambda z + 2i\lambda n + iM^{2n+1}k^*(\gamma)\cdot\omega} \tag{1.25}$$

and

$$\phi_{-,\gamma}(z,\omega) = \frac{1}{4\pi^2} \sum_{n\in\mathbf{Z}} e^{-i\lambda z + 2i\lambda n + iM^{2n}k^*(\gamma)\cdot\omega}, \tag{1.26}$$

is a family of a generalised eigenvectors of A for the element of the spectrum $i\lambda$.

4. *The decomposition of the identity associated with the operator A is*

$$dE(i\lambda) = \delta_{\lambda=0} + d\lambda \sum_{\gamma\in M^2\setminus\mathbf{Z}^2\setminus\{0\}} P_{\gamma,\lambda}, \tag{1.27}$$

where

$$P_{\gamma,\lambda}\left(f^+, f^-\right) = \left(\langle f^+|\phi^\lambda_{+,\gamma}\rangle\phi^\lambda_{+,\gamma}; \langle f^-|\phi^\lambda_{-,\gamma}\rangle\phi^\lambda_{-,\gamma}\right) \tag{1.28}$$

with the notation

$$\langle f \mid g\rangle = \frac{1}{4\pi^2}\int_0^h \int_{\mathbf{T}^2} f(z,\omega)\overline{g(z,\omega)}\, dz d\omega\,.$$

The reader can refer, for example, to [4] for a precise and yet elementary presentation of the notion of generalised eigenvectors.

The second main result of this paper is the following.

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Theorem 1.2. 1. *Let $a : \mathbb{T}^2 \to \mathbb{R}^d$ be in the class $C^3(\mathbb{T}^2)$ with mean value $\langle a \rangle = 0$ and initial data $\phi \in C^\infty(\mathbb{R}^d)$. Then for all $\lambda > 0$ the solutions $f^\pm_{\varepsilon,\lambda}$ of the system* (1.21)—(1.15)—(1.16)—(1.19) *converge strongly to the solution of the diffusion equation*

$$\partial_t f_\lambda - D_\lambda \Delta_x f_\lambda = 0, \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^d \tag{1.29}$$

$$f_0(0,x) = \phi(x), \quad x \in \mathbb{R}^d. \tag{1.30}$$

Furthermore

$$\|f^\pm_{\varepsilon,\lambda} - f_\lambda\|_{L^\infty\left(\mathbb{R}\times\mathbb{R}^d\times(0,h)\times\mathbb{T}^2\right)} = O(\varepsilon). \tag{1.31}$$

2. *Let $\phi \in C^\infty(\mathbb{R}^d)$ be initial data independent of the variables z and ω. Then the solutions $f^\pm_{\varepsilon,\lambda}$ of the system* (1.21)—(1.15)—(1.16)—(1.19) *and the solution f of the diffusion equation*

$$\partial_t f - D\Delta_x f = 0, \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}^d \tag{1.32}$$

$$f_0(0,x) = \phi(x), \quad x \in \mathbb{R}^d \tag{1.33}$$

satisfy

$$\lim_{(\lambda,\varepsilon)\to(0,0)} \|f^\pm_{\varepsilon,\lambda} - f\|_{L^\infty\left(\mathbb{R}\times\mathbb{R}^d\times(0,h)\times\mathbb{T}^2\right)} = 0\,. \tag{1.34}$$

The problem (1.21) is well-posed for every ε, both for $t > 0$ and for $t < 0$. Therefore, it is reversible in this sense. This is of course not true for the diffusion equation (1.29). However, the fact that the operator A is positive causes a difference between positive and negative time and this difference is magnified by the presence of the factor ε^{-1}. Therefore, we can associate a genuinely reversible problem at the "macroscopic limit" with an irreversible one.

In this work we shall not dwell on the existence and uniqueness proof of a solution for the Cauchy problem (1.21)—(1.15)—(1.16)—(1.19). A proof can be achieved by standard semigroup methods or by a characteristics method as in [**2**].

The paper is organised as follows. Section 2, containing the proof of Theorem 1.1, relies on spectral analysis of the operator A. The proof of Theorem 1.2 is carried out in Section 3 and a diffusion approximation obtained. The latter is a consequence of the different mixing properties inherited from the mapping T. The basic mixing properties of the map T are given in the Appendix.

2. Spectral Analysis and the Proof of Theorem 1.1.

The hyperbolic automorphism T of the torus (Arnold's cat map) defined by

$$T: \quad \mathbb{T}^2 \longrightarrow \mathbb{T}^2, \quad T\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \quad (\mathrm{mod}\ 2\pi) \tag{2.1}$$

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will be the only case treated here, though the method applies to any hyperbolic automorphism of $\mathbb{T}^n$. The map T is C^∞ and one-to-one and preserves the measure $d\omega_1 d\omega_2/4\pi^2$. Its inverse, also a C^∞ map, is given by

$$T^{-1}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \pmod{2\pi}. \tag{2.2}$$

We denote by A the unbounded operator on $L^2 \times L^2\big((0,h)\times\mathbb{T}^2\big)$ defined by

$$f = \begin{pmatrix} f^+\\ f^-\end{pmatrix}, \qquad Af = \begin{pmatrix}\partial_z & 0\\ 0 & -\partial_z\end{pmatrix}\begin{pmatrix} f^+\\ f^-\end{pmatrix} \tag{2.3}$$

with domain

$$D(A) = \Big\{\big(f^+, f^-\big) \in L^2 \times L^2\big((0,h)\times\mathbb{T}^2\big) \;\Big|\; Af \in L^2 \times L^2\big((0,h)\times\mathbb{T}^2\big) \text{ and} \tag{2.4}$$

$$f^+(0,\omega) = f^-(0,T\omega), \quad f^-(h,\omega) = f^+(h,T\omega), \quad \omega\in\mathbb{T}^2, \quad 0<z<h\Big\}. \tag{2.5}$$

In what follows, we illustrate the technique for finding the spectrum of the operator A.

PROOF OF THEOREM 1.1: The proof amounts to investigating the functions ϕ in the variable $z \in [0,h]$, with values in tempered distributions in the variable ω, not necessarily belonging to $D(A)$ but satisfying the boundary conditions prescribed in $D(A)$ such that $A\phi = \lambda\phi$.

The first statement follows from the fact that A is skew-adjoint in $L^2\times L^2\big((0,h)\times\mathbb{T}^2\big)$.

The second part is obtained by an easy computation. In order to prove this, note first that, since the Fourier series of the function $f \in S'\big([0,h]\times\mathbb{T}^2\big)$ converges, we shall write $f^\pm$ in the form

$$f^\pm(z,\omega) = \sum_{k\in\mathbf{Z}^2} a_k^\pm(z) e^{ik\omega}, \tag{2.6}$$

where $(a_k)_{k\in\mathbf{Z}^2\setminus\{0\}}$ is a family of complex numbers. Since from (2.1) we have the relations

$$\begin{aligned} e^{ik\cdot T\omega} &= e^{ik\cdot M\omega}\\ &= e^{iMk\cdot\omega}, \end{aligned} \tag{2.7}$$

the condition $f^+(0,\omega) = f^-(0,T\omega)$, implies

$$\sum_{k\in\mathbf{Z}^2} a_k^+(0)e^{ik\omega} = \sum_{k\in\mathbf{Z}^2} a_k^-(0)e^{iMk\cdot\omega}, \tag{2.8}$$

whence

$$a_k^-(0) = a_{Mk}^+(0). \tag{2.9}$$

Similarly the condition $f^-(h,\omega) = f^+(h,T\omega)$, implies

$$\sum_{k\in\mathbf{Z}^2} a_k^-(h)e^{ik\omega} = \sum_{k\in\mathbf{Z}^2} a_k^+(h)e^{iMk\cdot\omega} \tag{2.10}$$

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and

$$a_k^+(h) = a_{Mk}^-(h). \tag{2.11}$$

On the other hand, the equation

$$\partial_z f^+ = \lambda f^+ \tag{2.12}$$

implies

$$a_k^+(h) = e^{\lambda h} a_k^+(0) \tag{2.13}$$

and from the equation

$$\partial_z f^- = -\lambda f^- \tag{2.14}$$

we get

$$a_k^-(h) = e^{-\lambda h} a_k^-(0). \tag{2.15}$$

From (2.8) we deduce that

$$a_k^-(h) = e^{-\lambda h} a_{Mk}^+(0). \tag{2.16}$$

Since (2.10) implies

$$a_k^-(h) = a_{M^{-1}k}^+(h), \tag{2.17}$$

we get from (2.12) that

$$a_{M^{-1}k}^+(h) = e^{\lambda h} a_{M^{-1}k}^+(0). \tag{2.18}$$

Hence, from (2.16) and (2.17), we have

$$a_{M^{-1}k}^+(0) = a_{Mk}^+(0) e^{-2\lambda h}. \tag{2.19}$$

Thus

$$a_k^+(0) = a_{M^2k}^+(0) e^{-2\lambda h}. \tag{2.20}$$

Note that the coefficients a_k of the function f do not grow exponentially for every k if and only if $f \in \mathcal{S}'\big([0,h] \times \mathbb{T}^2\big)$. From this we deduce that $\lambda \in i\mathbb{R}$. Thus, if we denote by $\sigma(A)$, (respectively, $\sigma_e(A)$) the spectrum (respectively, essential spectrum) of the operator A, we have $\sigma(A) = \sigma_e(A) = i\mathbb{R}$.

For the third statement, we reorganise (1.23)—(1.24) so that the sums over $k \in \mathbb{Z}^2 \setminus \{0\}$ are given by summing on the orbits on $\mathbb{Z}^2$ of the cyclic group generated by M^2. This enables us to eliminate the constraint (1.22) on the family (a_k).

Let $0 < z < h$, and δ_{z_0} be the distribution in the variable z_0; $\lambda = i\xi$, $\xi \in \mathbb{R}$, $z_0 \in (0,h)$. From (1.23)—(1.24), we get

$$\int e^{+i\xi z_0} \, \frac{1}{4\pi^2} \sum_{k \neq 0} e^{+i\xi z} \, a_{M^{-1}k} \, e^{ik.\omega} \, \frac{d\xi}{2\pi} = \delta_{z_0} \cdot \frac{1}{4\pi^2} \sum_{k \neq 0} a_{M^{-1}k} \, e^{ik.\omega}, \tag{2.21}$$

$$\int e^{-i\xi z_0} \, \frac{1}{4\pi^2} \sum_{k \neq 0} e^{-i\xi z} \, a_k \, e^{ik.\omega} \, \frac{d\xi}{2\pi} = \delta_{z_0} \cdot \frac{1}{4\pi^2} \sum_{k \neq 0} a_k \, e^{ik.\omega}. \tag{2.22}$$

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We now need to characterise the points on the orbits γ. In order to do so, we proceed as follows.

Let $\gamma = \{M^{2n}k \mid n \in \mathbb{Z}\}$ be an orbit of $\mathbb{Z}^2 \setminus \{0\}$. We choose a particular exponent n on γ in the following way: there exists a unique n_* such that $k = M^{2n_*}k(\gamma)$. Indeed, observe that the matrix M is hyperbolic with two real and distinct eigenvalues given by

$$\lambda_+ = 1 + \theta, \quad \lambda_- = \lambda_+^{-1}, \quad \text{with} \quad \theta = \frac{1+\sqrt{5}}{2} \tag{2.23}$$

and that the corresponding eigenvectors (related to the unstable and stable manifold) are

$$e_+ = \mu \begin{pmatrix} \theta \\ 1 \end{pmatrix}, \; e_- = \mu \begin{pmatrix} 1 \\ -\theta \end{pmatrix}, \tag{2.24}$$

with $\mu = (1+\theta^2)^{-1/2}$. The vectors (e_+, e_-) define an orthonormal basis. In this basis M is written in the form

$$M \sim \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \tag{2.25}$$

The expansion of $k \in \mathbb{Z}^2 \setminus \{0\}$ in the basis (e_+, e_-) gives $k = (k.e_+)e_+ + (k.e_-)e_-$ and

$$M^{2n}k = \lambda_+^{2n}(k \cdot e_+)e_+ + \lambda_-^{2n}(k \cdot e_-)e_-. \tag{2.26}$$

Given an orbit γ, we have the relation

$$\frac{|M^n k \cdot e_+|}{|M^n k \cdot e_-|} = \lambda_+^{2n} \frac{|k \cdot e_+|}{|k \cdot e_-|}. \tag{2.27}$$

Take $k \in \gamma$ and define $n_*(k,\gamma)$ as the smallest integer $n \in \mathbb{Z}$ such that

$$\lambda_+^{2n} \frac{|k \cdot e_+|}{|k \cdot e_-|} \geqslant 1. \tag{2.28}$$

Finally, put $k^*(\gamma) = M^{n_*(k,\gamma)}k$, which is independent of the point k chosen on the orbit γ (see Figure 2).

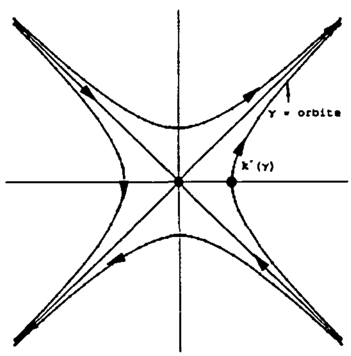

Figure 2

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The distance between γ and the origin is achieved at a unique point $k^*(\gamma)$ of γ. Hence if μ_γ is a family of scalars indexed by $M^2\backslash\mathbb{Z}^2 - \{0\}$, we can write

$$\phi_+^\lambda(z,\omega) = \frac{1}{4\pi^2} \sum_{M^2\backslash\mathbf{Z}^2-\{0\}\ni\gamma} \mu_\gamma \sum_{n\in\mathbf{Z}} e^{+\lambda z}\, e^{2\lambda nh}\, e^{iM^{2n+1}k^*(\gamma).\omega} \tag{2.29}$$

and

$$\phi_-^\lambda(z,\omega) = \frac{1}{4\pi^2} \sum_{M^2\backslash\mathbf{Z}^2-\{0\}\ni\gamma} \mu_\gamma \sum_{n\in\mathbf{Z}} e^{-\lambda z}\ \ e^{2\lambda nh}\, e^{iM^{2n}k^*(\gamma).\omega}. \tag{2.30}$$

Consequently a basis of generalised eigenfunctions takes the form

$$\phi_{+,\gamma}^\lambda(z,\omega) = \frac{1}{4\pi^2} \sum_{n\in\mathbf{Z}} e^{+\lambda z+2\lambda nh+iM^{2n+1}k^*(\gamma).\omega} \tag{2.31}$$

and

$$\phi_{-,\gamma}^\lambda(z,\omega) = \frac{1}{4\pi^2} \sum_{n\in\mathbf{Z}} e^{-\lambda z+2\lambda nh+iM^{2n}k^*(\gamma).\omega}, \tag{2.32}$$

where $\lambda \in \sigma(A) \subset i\mathbb{R}$, a spectrum with infinite multiplicity, and $\gamma \in M^2\backslash\mathbb{Z}^2 \setminus \{0\}$.

The proof of Statement 4 is nothing but the decomposition of $f^\pm$ on the family of functions $\left(\delta_z e^{ik\cdot\omega}\right)_{0<z<1;\ k\in\mathbf{Z}^2\backslash\{0\}}$ followed by an application of the Plancherel formula.

First we write $\phi_\pm^\lambda$ in the form of all Fourier modes which allows us to obtain expansions of all orbits. Take an orbit $\gamma \in M^2\backslash\mathbb{Z}^2 - \{0\}$ and set

$$f^+(z,\omega) = \frac{1}{4\pi^2} \int_0^h \sum_{k\neq 0} \widehat{f}_+(\xi,k)\ \delta_\xi(z)\ e^{ik.\omega} d\xi \tag{2.33}$$

and

$$f^-(z,\omega) = \frac{1}{4\pi^2} \int_0^h \sum_{k\neq 0} \widehat{f}_-(\xi,k)\ \delta_\xi(z)\ e^{ik.\omega} d\xi. \tag{2.34}$$

Integrating the functions $\phi_{\pm,\gamma}^\lambda$ over the spectrum $\sigma(A) \subset i\mathbb{R}$ leads to

$$\int e^{i\lambda\xi} \phi_{+,\gamma}^\lambda(z,\omega) d\xi = \frac{1}{4\pi^2} \sum_{n\in\mathbf{Z}} \delta(\xi - z + 2nh)\, e^{iM^{2n+1}k(\gamma).\omega} \tag{2.35}$$

and

$$\int e^{i\lambda\xi} \phi_{-,\gamma}^\lambda(z,\omega) d\xi = \frac{1}{4\pi^2} \sum_{n\in\mathbf{Z}} \delta(\xi - z + 2nh)\, e^{iM^{2n}k(\gamma).\omega}, \tag{2.36}$$

where $z \in (0,h)$, $\xi \in (0,h)$, implying $(\xi - z) \in (-h,h)$.

The cases $\xi \in (0,h)$ and $\xi \in \big(mh,(m+1)h\big)$, $m \in \mathbb{Z}$, have to be treated separately. THE CASE. $\xi \in (0,h)$: Write $\delta_{(\xi-z+2nh)} = \delta_{z-2nh}(\xi)$ and observe that $z - 2nh \in (0,h)$ implies $n = 0$. Then relations (1.25)-(1.26) for $\phi_\pm^\lambda$ are given by

$$\frac{1}{4\pi^2} \sum_{n\in\mathbf{Z}} \delta(\xi - z + 2nh)\, e^{iM^{2n+1}k^*(\gamma).\omega} = \frac{1}{4\pi^2}\ \ \delta_z(\xi)\, e^{iMk^*(\gamma).\omega}, \tag{2.37}$$

$$\frac{1}{4\pi^2} \sum_{n\in\mathbf{Z}} \delta(\xi - z + 2nh)\, e^{iM^{2n}k^*(\gamma).\omega} = \frac{1}{4\pi^2}\delta_z(\xi) e^{ik^*(\gamma).\omega}, \tag{2.38}$$

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but this case does not give all Fourier modes.

THE CASE. $\xi \in \big(mh,(m+1)h\big)$: Observe that $(z-2nh) \in \big(mh,(m+1)h\big)$ entails $2n+m=0$. It follows that the relations

$$\frac{1}{4\pi^2}\sum_{n\in\mathbf{Z}}\delta(\xi-z+2nh)\,e^{iM^{2n+1}k^*(\gamma).\omega}=\frac{1}{4\pi^2}\,\delta_z(\xi)\,e^{iM^{-m+1}k(\gamma).\omega} \tag{2.39}$$

and

$$\frac{1}{4\pi^2}\sum_{n\in\mathbf{Z}}\delta(\xi-z+2nh)\,e^{iM^{2n}k^*(\gamma).\omega}=\frac{1}{4\pi^2}\,\delta_z(\xi)\,e^{iM^{-m}k(\gamma).\omega} \tag{2.40}$$

give us all Fourier modes.

We can therefore write down the spectral measure of the operator A. Since $\xi\in(0,h)$ and $\xi-z\in(-h,h)$ imply $n=0$ for all $z\in(0,h)$, we get

$$\frac{1}{4\pi^2}\delta(\xi-z+2nh)\,e^{iM^{2n+1}k(\gamma)}=\frac{1}{4\pi^2}\sum_{n\in\mathbf{Z}}\delta(\xi)\,e^{iMk^*(\gamma)\cdot\omega}$$

and

$$\frac{1}{4\pi^2}\delta(\xi-z+2nh)\,e^{iM^{2n}k^*(\gamma).\omega}=\frac{1}{4\pi^2}\delta(\xi)\,e^{ik^*(\gamma)\cdot\omega}.$$

Finally we obtain

$$\begin{aligned} f^+(z,\omega)&=\frac{1}{4\pi^2}\int_0^h\sum_\gamma\hat f_+\,e^{iMk^*(\gamma).\omega}d\xi\\ &=\int_{0<\xi<h}\sum_\gamma\int_{\lambda\in\mathbf{R}}e^{i\lambda\xi}\phi^\lambda_{+,\gamma}(z,\omega)d\lambda\;\hat f_+d\xi \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} f^-(z,\omega)&=\frac{1}{4\pi^2}\int_0^h\sum_\gamma\hat f_-\;\delta_\xi(z)\;e^{ik^*(\gamma).\omega}d\xi\\ &=\int_{0<\xi<h}\sum_\gamma\int_{\lambda\in\mathbf{R}}e^{i\lambda\xi}\phi^\lambda_{-,\gamma}(z,\omega)d\lambda\,\hat f_-d\xi. \end{aligned} \tag{2.42}$$

This proves statement 4. and completes the proof of Theorem 1.1. □

3. THE PROOF OF THEOREM 1.2.

The proof of Theorem 1.2 is based on the following fundamental lemma.

LEMMA 3.1. *The diffusion coefficient D_λ defined in Theorem 1.2 has the form*

$$D_\lambda=\Big\langle\Big\langle a(\omega)\psi_\lambda(z,\omega)\Big\rangle\Big\rangle\cdot I, \tag{3.1}$$

where ψ_λ is the solution of the equation $(\lambda I-\lambda\Pi+A)\psi_\lambda+a(\omega)=0$. Futhermore

$$D_\lambda\to D\qquad\text{as}\qquad\lambda\to0. \tag{3.2}$$

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PROOF OF LEMMA 3.1: To prove (3.1), we seek the solution as an expansion of the form

$$f_{\varepsilon,\lambda}(t,x,z,\omega) = f_0(t,x,z) + \varepsilon f_1(t,x,z,\omega) + \varepsilon^2 f_2(t,x,z,\omega) + r_\varepsilon(t,x,z,\omega) \tag{3.3}$$

as in [3], where $f_i \equiv f_i(t,x,z,\omega)$ are functions defined on $\mathbb{R}^+ \times \mathbb{R}^d \times (0,h) \times \mathbb{T}^2$ that we substitute into (1.21). The identification of successive powers of ε leads to

$$\varepsilon^{-2}: \quad Af_0 + \lambda\Big(f_0 - \langle\!\langle f_0 \rangle\!\rangle\Big) = 0, \tag{3.4}$$

$$\varepsilon^{-1}: \quad a(\omega)\cdot\nabla_x f_0 + Af_1 + \lambda\Big(f_1 - \langle\!\langle f_1 \rangle\!\rangle\Big) = 0, \tag{3.5}$$

$$\varepsilon^{0}: \quad \partial_t f_0 + a(\omega)\cdot\nabla_x f_1 + Af_2 + \lambda\Big(f_2 - \langle\!\langle f_2 \rangle\!\rangle\Big) = 0. \tag{3.6}$$

The first equation is solved by taking $f_0 \equiv f_0(t,x)$ independent of z and ω. This suggests looking for f_1 in the form

$$f_1(t,x,z,\omega) = c\psi_\lambda(z,\omega)\cdot\nabla_x f_0(t,x), \tag{3.7}$$

where the function ψ_λ satisfies the equation

$$(\lambda I - \lambda\Pi + A)\psi_\lambda + a(\omega) = 0. \tag{3.8}$$

Observe now that $\lambda\Pi$ is compact, as a finite rank projector (of dimension 1). We have that $\lambda I + A$ is invertible and $\langle a \rangle = 0$. This implies that the solution $\psi_\lambda = -(\lambda I + A)^{-1}a(\omega)$ is admissible. Thus (3.8) possesses a unique solution ψ_λ such that

$$\psi_\lambda = -(\lambda I - \lambda\Pi + A)^{-1}a(\omega). \tag{3.9}$$

Observe next that (3.6) can be solved for f_2 if and only if $\partial_t f_0 + a(\omega)\cdot\nabla_x f_1$ is orthogonal to $\operatorname{Ker} A$, that is, to constants, so the solution f_0 must satisfy

$$\partial_t f_0 + c^2\frac{\partial}{\partial x}\Big\langle\!\Big\langle a(\omega)\psi_\lambda(z,w)\Big\rangle\!\Big\rangle\frac{\partial f_0}{\partial x} = 0. \tag{3.10}$$

Assume that f_0 satisfies the initial data and boundary conditions (1.15)—(1.16). We substitute for ψ_λ in (3.10) to get

$$\partial_t f_0 + c^2\Big\langle\!\Big\langle a(\omega)(\lambda I - \lambda\Pi + A)^{-1}\Big\rangle\!\Big\rangle\Delta f_0 = 0. \tag{3.11}$$

Substituting (3.9) into (3.10) provides the diffusion coefficient

$$D_\lambda = \Big\langle\!\Big\langle a(\omega)(\lambda I - \lambda\Pi + A)^{-1}a(\omega)\Big\rangle\!\Big\rangle. \tag{3.12}$$

Limit of D_λ when $\lambda \to 0$: The proof applies the Fourier inversion theorem connected with the function $e^{-\lambda|x|}$. Observe that, since

$$\int_{\mathbf{R}} e^{-izx}e^{-\lambda|x|}\,dx = \frac{2\lambda}{\lambda^2+z^2},$$

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the Fourier inversion formula gives

$$e^{-\lambda|x|} = \frac{1}{\pi}\int_{\mathbf{R}} \frac{e^{izx}}{\lambda^2+z^2}\,dz = \frac{1}{\pi}\int_{\mathbf{R}} \frac{e^{-izx}}{\lambda^2+z^2}\,dz.$$

Denote by dE the spectral measure of the operator A and write

$$D_\lambda = \int_{Sp(iA)\subset\mathbb{R}} \frac{1}{\lambda+iz}\,\Big\langle dE(iz)a(\omega), a(\omega)\Big\rangle. \tag{3.13}$$

Set

$$d\mu(z) = \Big\langle dE(iz)a(\omega), a(\omega)\Big\rangle \tag{3.14}$$

to obtain

$$D_\lambda = \Re e\int_{Sp(iA)\subset\mathbb{R}} \frac{d\mu(z)}{\lambda+iz} = \int_{\mathbf{R}} \frac{\lambda d\mu(z)}{\lambda^2+z^2}, \tag{3.15}$$

where $\Re e(z)$ denotes the real part of z. Set

$$\widehat{\mu}(\xi) = \int e^{-i\xi z}d\mu(z).$$

Using Plancherel's Theorem and the Fourier inversion theorem gives

$$\begin{aligned} D_\lambda &= \frac{1}{2}\int \widehat{\mu}(\xi)\,e^{-\lambda|\xi|}d\xi \\ &= \frac{1}{2}\int \langle e^{-\xi A}a, a\rangle\, e^{-\lambda|\xi|}d\xi. \end{aligned} \tag{3.16}$$

Since T is a mixing transformation (as detailed in the Appendix), taking the limit as λ tends to 0 yields

$$\lim_{\lambda\to 0} D_\lambda = \frac{1}{2}\int \widehat{\mu}(\xi)\,d\xi. \tag{3.17}$$

Finally, applying Wiener's lemma to D_λ in a neighbourhood of 0 and letting λ tend to zero yields

$$D_\lambda = \int_{\mathbf{R}} \frac{\lambda}{\lambda^2+z^2}\,d\mu(z) \to D = \frac{1}{2}\frac{d\mu}{dx}\Big|_{x=0} \tag{3.18}$$

The proof of the Lemma 3.1 is now complete. ∎

We are now ready for the proof of Theorem 1.2. Set

$$r_{\varepsilon,\lambda}(t,x,z,\omega) = f_{\varepsilon,\lambda}(t,x,z,\omega) - f_0(t,x,z,\omega) - \varepsilon f_1(t,x,z,\omega) - \varepsilon^2 f_2(t,x,z,\omega) \tag{3.19}$$

and substitute (3.19) into (1.21). Taking into account the hierarchy of (3.4)—(3.6), we get

$$\partial_t r_\varepsilon + \frac{1}{\varepsilon}a(\omega)\cdot\nabla_x r_\varepsilon - \frac{1}{\varepsilon^2}c\partial_z r_\varepsilon + \frac{\lambda}{\varepsilon^2}(I-\Pi)r_\varepsilon = -\varepsilon\partial_t f_1 + a(\omega)\cdot\nabla_x f_2 + \varepsilon\partial_t f_2\,, \tag{3.20}$$

$$\begin{aligned} r_{\varepsilon,\lambda}(0,x,z,\omega) &= \phi(x) \\ &= \varepsilon f_1(0,x,z,\omega) - \varepsilon f_2(0,x,z,\omega). \end{aligned} \tag{3.21}$$

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Since ϕ is a regular function, the right-hand sides of (3.20) and (3.21) are continuous. In particular, we have

$$(3.22)\qquad \left\|\partial_t f_1 + a(\omega)\cdot\nabla_x f_2 + \varepsilon\partial_t f_2\right\|_{L^\infty} \leqslant C_\lambda, \quad \text{and} \quad \left\|r_\varepsilon(0,x,z,\omega)\right\| \leqslant C\varepsilon,$$

where C_λ is a constant depending on ψ_λ (through f_1) and on the initial data $\phi(x)$. By the Maximum Principle

$$(3.23)\qquad \|\psi_\lambda\|_{L^\infty\left(\mathbf{R}\times\mathbb{R}^d\times(0,h)\times\mathbf{T}^2\right)} \leqslant \frac{1}{\lambda}\left\|a(\omega)\right\|_{L^\infty},$$

which tends to ∞ when λ tends to 0. From (3.7) it follows that

$$(3.24)\quad \|f_1\|_{L^\infty(\mathbf{R}\times\mathbb{R}^d\times(0,h)\times\mathbf{T}^2)} \leqslant C_\lambda\|\psi_\lambda\|_{L^\infty\left(\mathbf{R}\times\mathbb{R}^d\times(0,h)\times\mathbf{T}^2\right)}\|\varphi\|_{L^\infty\left(\mathbf{R}\times\mathbb{R}^d\times(0,h)\times\mathbf{T}^2\right)}.$$

Moreover the inequality

$$(3.25)\qquad \|f_{\varepsilon,\lambda} - f_0^\lambda\|_{L^\infty\left(\mathbf{R}\times\mathbb{R}^d\times(0,h)\times\mathbf{T}^2\right)} \leqslant \|r_\varepsilon\|_{L^\infty\left(\mathbf{R}\times\mathbf{R}^d\times(0,h)\times\mathbf{T}^2\right)} + \varepsilon\|f_1\|_{L^\infty\left(\mathbf{R}\times\mathbf{R}^d\times(0,h)\times\mathbb{T}^2\right)}$$

holds. This yields

$$(3.26)\qquad \|f_{\varepsilon,\lambda} - f_\lambda\|_{L^\infty\left(\mathbf{R}\times\mathbb{R}^d\times(0,h)\times\mathbf{T}^2\right)} \leqslant C_\lambda\varepsilon.$$

Since $D_\lambda \to D$, we can also estimate $f_\lambda - f$ for $\lambda \to 0$. We proceed as follows: subtracting (1.29) from (1.32) yields

$$(3.27)\qquad \partial_t(f_\lambda - f) = D_\lambda\Delta(f_\lambda - f) + (D_\lambda - D)\Delta f.$$

It follows from (3.2) that $(D_\lambda - D)\Delta f \to 0$. Equation (3.26) is then integrated with respect to time and the Maximum Principle yields

$$(3.28)\qquad \|D_\lambda\int_0^t \Delta(f_\lambda - f)\,ds\|_{L^\infty\left(\mathbf{R}\times\mathbb{R}^d\times(0,h)\times\mathbf{T}^2\right)} = O(\lambda).$$

We deduce that $f^\pm_{\varepsilon,\lambda}$ converge strongly to f_λ. Let $\chi \in C_0^\infty\left(\mathbb{R}^+\times\mathbb{R}^d\right)$ be a test function and $f(t,x)$ the weak limit of a subsequence of the family $f^\pm_{\varepsilon,\lambda}$ in $L^\infty\left(\mathbb{R}^+\times\mathbb{R}^d\right)$. (For work with differential equations it is convenient to restrict the term test function to functions $\chi(x)$ which are continuous, have continuous derivatives of all orders and vanish identically outside some finite interval. For example, the function $\chi(x) = \exp(-x^{-2})\exp\left[-(x-a)^{-2}\right]$, $0 \leqslant x \leqslant a$; $\chi(x) = 0$, $x \leqslant 0$ or $a \leqslant x$, is a test function.) This subsequence satisfies

$$(3.29)\quad \left|\int_{\mathbf{R}^+\times\mathbf{R}^d}\left(f^\pm_{\varepsilon,\lambda} - f^\pm_\varepsilon\right)\chi(t,x)\,dxdt\right| \leqslant \left|\int_{\mathbf{R}^+\times\mathbf{R}^d}\left(f^\pm_{\varepsilon,\lambda} - f_\lambda\right)\chi(t,x)\,dxdt\right| + \left|\int_{\mathbf{R}^+\times\mathbf{R}^d}(f_\lambda - f)\,\chi(t,x)\,dxdt\right| + \left|\int_{\mathbf{R}^+\times\mathbf{R}^d}\left(f^\pm_\varepsilon - f\right)\chi(t,x)\,dxdt\right|$$

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for all $\varepsilon > 0$. Using (1.31) yields

$$\left|\int_{\mathbb{R}^+\times\mathbb{R}^d}\left(f^{\pm}_{\varepsilon,\lambda}-f_\lambda\right)\chi(t,x)\,dtdx\right| \leqslant C_\lambda\,\varepsilon. \tag{3.30}$$

By letting λ tend to 0 in the second term of (3.29) and using Lemma 3.1, we derive

$$\left|\int_{\mathbb{R}^+\times\mathbb{R}^d}(f_\lambda - f)\,\chi(t,x)\,dtdx\right| \to 0 \quad \text{as} \quad \lambda\to 0. \tag{3.31}$$

We may proceed as in [**2**, Theorem 3] to estimate the third term of the right-hand side of (3.29) and let $\varepsilon\to 0$. The convergence of $f^{\pm}_{\varepsilon,\lambda}$ to f_λ (uniformly in λ) is obtained by observing that for fixed $\varepsilon > \varepsilon_0$

$$\left|\int_{\mathbb{R}^+\times\mathbb{R}^d}\left(f^{\pm}_{\varepsilon,\lambda}-f^{\pm}_{\varepsilon}\right)\chi(t,x)\,dtdx\right| = C_\lambda\,\varepsilon_0, \tag{3.32}$$

since

$$\left\|f^{\pm}_{\varepsilon,\lambda}-f^{\pm}_{\varepsilon}\right\|_{L^\infty\left(\mathbb{R}\times\mathbb{R}^d\times(0,h)\times\mathbb{T}^2\right)} = O(\lambda). \tag{3.33}$$

The proof of Theorem 1.2. is complete. ∎

The diagram

$$\begin{tikzcd}
f^{\pm}_{\varepsilon,\lambda} \arrow[r, "\textit{strong convergence}", "\varepsilon\to 0"'] & f_\lambda \arrow[d, "\substack{\textit{strong convergence}\\ \lambda\to 0}"] \\
f^{\pm}_{\varepsilon} \arrow[r, "\textit{weak convergence}", "\varepsilon\to 0"'] & f
\end{tikzcd} \tag{3.34}$$

summarises the proof of Theorem 1.2. The upper horizontal arrow shows that, with an additional collision operator in the system (1.21)—(1.15)—(1.16)—(1.19), the densities of particles $f^{\pm}_{\varepsilon,\lambda}$ converge uniformly in λ to a solution f_λ to (1.29). The vertical arrow indicates that the diffusion coefficient $D_\lambda\to D$ as $\lambda\to 0$, so f_λ converges to f, a solution of the diffusion equation (1.32). The lower horizontal arrow is a result of [**2**, Theorem 3].

4. Appendix

It is illuminating to see some basic mixing properties of the map T.

Proposition 4.1. *Let $0\leqslant\chi(R)$ be a decreasing positive function tending to 0 as R tends to infinity. Introduce the class of functions*

$$H_\chi = \left\{ f\in L^2\left(\mathbb{T}^2\right) \text{ such that } \sum_{|k_1|,|k_2|>R}\left|\hat{f}(k)\right|^2 \leqslant \chi(R)^2\|f\|_2^2\right\}. \tag{4.1}$$

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Then for any pair $(f,g) \in H_\chi$ with mean value $\langle f\rangle = \langle g\rangle = 0$, we have

$$\left|\langle f\circ T^n \cdot g\rangle\right| \leqslant \frac{1}{2\pi^2}\|f\|_2\|g\|_2\chi\left(\left(C_0\frac{3+\sqrt{5}}{2}\right)^{n/2}\right), \quad \textit{with } C_0 = \frac{1+\sqrt{5}}{\sqrt{2}}. \tag{4.2}$$

PROOF OF PROPOSITION 4.1: From the Plancherel formula we have

$$\langle f\circ T^n \cdot g\rangle = \frac{1}{4\pi^2}\sum_{k\neq 0}\hat{f}\left(M^{-n}k\right)\hat{g}(-k) \tag{4.3}$$

for any pair $(f,g) \in L^2(\mathbb{T}^2)$ with mean values $\langle f\rangle = \langle g\rangle = 0$. For any $R > 0$ decompose the above sum in two parts corresponding to K_R and K_R^c, with K_R (see Figure 3) given by

$$K_R = \left\{k \in \mathbb{Z}^2 \text{ such that } \sup\left(|k_1|, |k_2|\right) \leqslant R\right\}.$$

Since g belongs to the class H_χ, the Cauchy-Schwartz inequality yields the estimate

$$\left|\sum_{k\in K_R^c}\hat{f}\left(M^{-n}k\right)\hat{g}(-k)\right| \leqslant \|f\|_2\|g\|_2\chi(R). \tag{4.4}$$

If p and q are two integers such that $(p,q) = 1$ and θ is defined by (2.22), we have

$$\inf_{(p,q)=1}\left|\theta - \frac{p}{q}\right| \geqslant \frac{1}{\left(1+\sqrt{5}\right)q^2}. \tag{4.5}$$

Indeed, the minimal polynomial of θ over $\mathbb{Q}$ is $P(X) = X^2 - X - 1 = (X-\theta)(X+\theta^{-1})$ and

$$\left|\theta - \frac{p}{q}\right|\left|P'(\xi)\right| = \left|P\left(\frac{p}{q}\right)\right| = \frac{|p^2 - qp - q^2|}{q^2} \geqslant \frac{1}{q^2}. \tag{4.6}$$

For $k \in K_R$, introduce the decomposition $k = (k\cdot e_+)e_+ + (k\cdot e_-)e_-$. With (4.5) we get

$$|k\cdot e_-| \geqslant \theta^{-1}|k|^{-1} \geqslant \left(\sqrt{2}R\theta\right)^{-1},$$

whence

$$|M^{-n}k| \geqslant \frac{\lambda_+^n}{\sqrt{2}R\theta}.$$

Since $f \in H_\chi$ and χ is nonincreasing, this implies

$$\left|\sum_{k\in K_R-\{0\}}\hat{f}\left(M^{-n}k\right)\hat{g}(-k)\right| \leqslant \|f\|_2\|g\|_2\chi\left(\frac{\lambda_+^n}{\sqrt{2}R\theta}\right). \tag{4.7}$$

Relation (4.2) is obtained by choosing $R = \lambda_+^{n/2}$ in (4.4) and 4.7). This proves in particular that the series $\sum_{k\geqslant 1}\langle a\circ T^k \otimes a\rangle$ in the definition of $D(a)$ is absolutely convergent.

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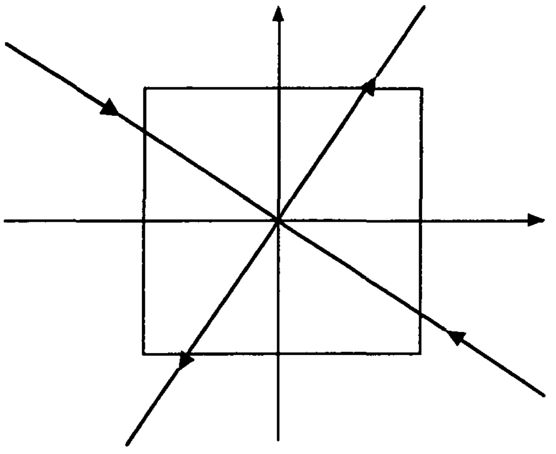

Figure 3

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