Forum of Mathematics, Sigma (2024), **Vol. 12**:e56 1–6
doi:10.1017/fms.2024.40

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RESEARCH ARTICLE

A short proof of the Hanlon-Hicks-Lazarev Theorem

Michael K. Brown [1] and Daniel Erman [2]

[1]Department of Mathematics and Statistics, Auburn University, 221 Parker Hall, Auburn, AL 36849, United States; E-mail: mkb0096@auburn.edu (corresponding author).
[2]Department of Mathematics, University of Hawai‘i at Mānoa, 2565 McCarthy Mall (Keller Hall 401A), Honolulu, HI 96822, United States; E-mail: erman@hawaii.edu.

Received: 2 January 2024; **Revised:** 13 March 2024; **Accepted:** 6 April 2024

2020 Mathematics Subject Classification: *Primary* – 13D02; *Secondary* – 14F08, 14M25

Abstract
We give a short new proof of a recent result of Hanlon-Hicks-Lazarev about toric varieties. As in their work, this leads to a proof of a conjecture of Berkesch-Erman-Smith on virtual resolutions and to a resolution of the diagonal in the simplicial case.

1. Main result

We give a short new proof of a recent result of Hanlon-Hicks-Lazarev about toric varieties and their multigraded Cox rings. Throughout, we let X be a simplicial, projective toric variety over an algebraically closed field k with $\mathrm{Cl}(X)$-graded Cox ring S. Our main result (Theorem 1.2) was first proven in [HHL], but our proof is independent from their methods. Our approach is more algebraic and simpler, while their approach is more explicit and connects to a wider range of topics, including symplectic geometry and homological mirror symmetry. See also the work of Favero-Huang [FH], which was completed simultaneously with [HHL] and whose main results coincide with some of Hanlon-Hicks-Lazarev's.

Our interest in these topics begins with a program to extend results on syzygies to multigraded or toric settings. The basic perspective, introduced by Berkesch-Erman-Smith in [BES20], is that many classical results about minimal free resolutions will have strong analogues in the toric setting, as long as one replaces minimal free resolutions with the more flexible notion of a virtual resolution.

Definition 1.1. Let M be a finitely generated $\mathrm{Cl}(X)$-graded S-module. A virtual resolution of M is a free complex $F_\bullet$ of S-modules such that there is a quasi-isomorphism $\widetilde{F_\bullet} \xrightarrow{\simeq} \widetilde{M}$ of complexes of $\mathcal{O}_X$-modules.[1]

The following is a consequence of Hanlon-Hicks-Lazarev's main result [HHL, Theorem A].

Theorem 1.2. *Let Y be a normal toric variety and $Y \hookrightarrow X$ a closed immersion that is a toric morphism [CLS, Definition 3.3.3]. Denote by I the defining ideal of $Y \hookrightarrow X$ (Definition 2.1). The S-module S/I admits a virtual resolution of length* $\mathrm{codim}(Y \subseteq X)$.

[1]If X is smooth, then $\widetilde{F_\bullet}$ consists of sums of line bundles and is sometimes called a line bundle resolution. See Remark 3.4 regarding the simplicial case.

And here is our short proof of Theorem 1.2. The proof relies on some elementary facts about toric varieties that we recall in Lemma 2.2 below.

Proof of Theorem 1.2. The Cox ring S of X is positively $\mathrm{Cl}(X)$-graded [BE21, Definition A.1, Example A.2], and so we may consider $\mathrm{Cl}(X)$-graded minimal free resolutions of $\mathrm{Cl}(X)$-graded S-modules. Let R be the normalization of S/I and $F_\bullet$ the minimal free resolution of R as an S-module. Since Y is normal, $\widetilde{R} = \mathcal{O}_Y$ as a sheaf on X, and so $F_\bullet$ is a virtual resolution of S/I_Y. By Lemma 2.2(1) and [CLS, Theorem 1.1.17 and Proposition 1.3.8], the ring R is a product of affine semigroup rings of the same dimension. Hochster's Theorem therefore implies that each component of R is a Cohen-Macaulay ring [Hoc72, Theorem 1]. It follows that R is also a Cohen-Macaulay S/I-module: indeed, we have $\dim(R) = \dim(S/I)$, and since R is a finitely generated S/I-module [Eis, Theorem 4.14], any system of parameters on S/I is a system of parameters on each component of R and hence a regular sequence. The length of $F_\bullet$ is the projective dimension of R, which, by the Auslander-Buchsbaum formula [Eis, Theorem 19.9], is equal to $\mathrm{depth}_S(S) - \mathrm{depth}_S(R) = \dim(S) - \dim(S/I)$ (while the version of the Auslander-Buchsbaum formula we cite pertains to local rings, the desired result for the polynomial ring S follows by [BH, Proposition 1.5.15]). Lemma 2.2(2) therefore implies that the length of F is equal to $\mathrm{codim}(Y \subseteq X)$. □

We now describe applications of Theorem 1.2 and their history. For a fuller discussion, see [HHL, §1]. We start with a special case, first proven by Hanlon-Hicks-Lazarev:

Theorem 1.3 ([HHL] Corollary B). *The coordinate ring of the diagonal embedding $X \subseteq X \times X$ admits a virtual resolution of length* $\dim X$.

Special cases of Theorem 1.3 were studied in [BE21, BS22, Can03], and [BPS01, And] study closely related questions. It was known that this result would immediately yield proofs of two conjectures that also had received independent interest. The first conjecture is due to Berkesch-Erman-Smith [BES20, Question 1.3] and was proven by Hanlon-Hicks-Lazarev:

Theorem 1.4 ([HHL] Corollary C). *Any module M as in Definition 1.1 has a virtual resolution of length* $\leq \dim X$.

Hilbert's Syzygy Theorem gives a bound of $\dim S = \dim X + \mathrm{rank}\,\mathrm{Cl}(X)$; Theorem 1.4 implies that the added flexibility of virtual resolutions allows for significantly shorter resolutions, especially when $\mathrm{rank}\,\mathrm{Cl}(X)$ is large. See [BES20, HNVT22, BKLY21] and elsewhere for many examples of this phenomenon. Prior to [HHL], Theorem 1.4 had been proven in several special cases: when $\mathrm{rank}\,\mathrm{Pic}(X) = 1$, it essentially follows from Hilbert's Syzygy Theorem; for products of projective spaces, it was shown in [BES20, Theorem 1.2] (see also [EES15, Corollary 1.14]); Yang proved it for any monomial ideal in the Cox ring of a smooth toric variety [Yan21]; and Brown-Sayrafi proved it for smooth projective toric varieties of Picard rank 2 [BS22].

The second conjecture, due to Orlov, is the special case of [Orl09, Conjecture 10] for toric varieties. This was first proven by Favero-Huang in [FH, Theorem 1.2], and independently and essentially simultaneously in [HHL, Corollary E].

Theorem 1.5. *The Rouquier dimension of $D^b(X)$ equals* $\dim X$.

Special cases of Theorem 1.5 had been established in [BC23, BF12, BDM19, BFK19] before Favero-Huang and Hanlon-Hicks-Lazarev proved it in general. The full version of Orlov's Conjecture states that Theorem 1.5 extends to any smooth quasi-projective variety; see [BC23, §1.2] for a list of known cases of this conjecture.

Theorem 1.2 easily implies Theorems 1.3, 1.4 and 1.5. To prove Theorem 1.3, observe that the diagonal $X \subseteq X \times X$ satisfies the conditions of Theorem 1.2. To prove Theorem 1.4, one can simply follow the method of [BES20, Proof of Theorem 1.2]. For Theorem 1.5, one can use standard techniques on derived categories; see, for example, the proof of [HHL, Corollary E].

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Our proof of Theorem 1.2 is quite simple, perhaps embarrassingly so given the prior partial results on these questions cited above. It is not yet clear how to compare our resolutions to those obtained in [HHL], but we believe that the two constructions agree in the case of Theorem 1.3. Their work gives a creative perspective on building these resolutions, drawing motivation from the symplectic side of the mirror symmetry functor and involving a wide array of ideas.[2] The resolutions they obtain are quite explicit; indeed, their resolution of the diagonal yields a canonical generating set for the derived category of any normal toric variety, proving a claim of Bondal [HHL, Corollary D]. However, some algebraic aspects of their constructions are harder to determine. For instance, if $F_\bullet$ is the free complex of S-modules corresponding to one of their resolutions, their work implies that the modules $H_i(F_\bullet)$ correspond to the zero sheaf on X for all $i > 0$, but it is not clear whether $H_i(F_\bullet)$ equals the zero module on the nose (i.e., it is not clear if $F_\bullet$ is acyclic as a complex of S-modules). The S-module that arises as $H_0(F_\bullet)$ is also unclear. By comparison, the complexes that arise in our construction are always acyclic, and they resolve normalizations of coordinate rings. However, we are not able to give as explicit of a description of the terms. It would be very interesting to better compare these complexes, and to compare them with those in [BE21, BS22]. Favero-Huang's approach [FH] can almost certainly yield all of the above results as well, and it would be interesting to compare to those resolutions too.

Remark 1.6. As our resolutions from Theorem 1.2 rely only on standard algebraic constructions, they can be directly computed in *Macaulay2* [M2]. The constructions in [HHL] are explicit, but due to their novelty, computing them in practice requires more effort. Of course, if one could show that the two constructions coincide, this would shed more light on both.

2. Some elementary facts about toric varieties

Definition 2.1. Let X, Y and S be as in Theorem 1.2, $B \subseteq S$ the irrelevant ideal of X, and Z the closure in $\operatorname{Spec}(S)$ of the inverse image of Y under the canonical surjection $\pi\colon \operatorname{Spec}(S) \setminus V(B) \to X$. The defining ideal of Y in X is the radical ideal $I \subseteq S$ corresponding to the closed subset $Z \subseteq \operatorname{Spec}(S)$.

Lemma 2.2. *Let Z and I be as in Definition 2.1.*

1. *The irreducible components of Z are affine toric varieties of the same dimension. Furthermore, if the divisor class group* $\operatorname{Cl}(X)$ *is torsion-free, then Z is irreducible.*
2. *We have* $\dim(S) - \dim(S/I) = \operatorname{codim}(Y \subseteq X)$.

Proof. Since $Y \hookrightarrow X$ is a toric morphism, it induces an embedding $T_Y \hookrightarrow T_X$ on tori and hence a surjection $p\colon M_X \twoheadrightarrow M_Y$ of lattices. Taking the pushout of the surjection p and the canonical map $M_X \to \mathbb{Z}^{\dim S}$ yields the morphism

$$\begin{tikzcd} 0 \arrow[r] & M_X \arrow[r] \arrow[d, "p"] & \mathbb{Z}^{\dim S} \arrow[r] \arrow[d, "q"] & \operatorname{Cl}(X) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & M_Y \arrow[r] & M' \arrow[r] & \operatorname{Cl}(X) \arrow[r] & 0 \end{tikzcd} \tag{2.3}$$

of exact sequences. The abelian group M' is isomorphic to $\mathbb{Z}^r \oplus A$, where r is defined to be $\dim(S) - \dim(X) + \dim(Y)$, and A is some finite abelian group. We observe that I coincides with the radical of $J := \ker(S \hookrightarrow k[\mathbb{Z}^{\dim S}] \xrightarrow{q} k[M'])$; note that $k[M']$ need not be reduced when $\operatorname{char}(k) \neq 0$, since M' may have torsion, and so J need not be radical. Let us verify that $I = \operatorname{rad}(J)$: since p is surjective, the Snake Lemma implies that q is surjective, and so J is the defining ideal of the closure of $\operatorname{Spec}(k[M'])$ in

[2] In a different direction, we refer to Borisov's work [Bor00] for an alternative proof of Hochster's Theorem [Hoc72, Theorem 1] – the main ingredient of our proof of Theorem 1.2 – and an explanation of how the techniques used there relate to mirror symmetry.

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Spec(S). Diagram (2.3) induces the following morphism of short exact sequences of algebraic groups:

$$\begin{tikzcd} 0 & T_X \arrow[l] & \mathrm{Spec}(k[\mathbb{Z}^{\dim S}]) \arrow[l, "\alpha"'] & \ker(\alpha) \arrow[l] & 0 \arrow[l] \\ 0 & T_Y \arrow[l] \arrow[u] & \mathrm{Spec}(k[M']) \arrow[l, "\beta"'] \arrow[u] & \ker(\beta) \arrow[l] \arrow[u, "\cong"'] & 0. \arrow[l] \end{tikzcd}$$

It follows that $\alpha^{-1}(T_Y)$ is equal to the image of Spec$(k[M'])$ in Spec$(k[\mathbb{Z}^{\dim S}])$. Since Z is equal to the closure of $\alpha^{-1}(T_Y)$ in Spec(S), we conclude that $I = \mathrm{rad}(J)$.

Writing $R = k[\mathbb{Z}^r]$ and $A = \bigoplus_{i=1}^t \mathbb{Z}/(n_i)$, we have

$$k[M'] \cong R[z_1, \ldots, z_t]/(z_1^{n_1} - 1, \ldots, z_t^{n_t} - 1).$$

The quotient of $k[M']$ by its nilradical is therefore a product of copies of R, and so I is a finite intersection of prime ideals arising as kernels of ring homomorphisms $S \to R$. It therefore follows from [CLS, Proposition 1.1.8] that the irreducible components of Z are affine toric varieties of dimension r. If Cl(X) is torsion-free, then the bottom row of Diagram (2.3) splits, and so $A = 0$, which means I is prime. This proves (1). As for (2), we have shown that $\dim(Z) = r$, which is precisely $\dim(S) - \mathrm{codim}(Y \subseteq X)$. □

3. Examples

Example 3.1. Let $X = \mathbb{P}^n$ and $T = k[x_0, \ldots, x_n, y_0, \ldots, y_n]$, the Cox ring of $X \times X$. Let $I_\Delta \subseteq T$ be the defining ideal (Definition 2.1) of the diagonal $X \subseteq X \times X$ (i.e., the ideal corresponding to the closure of the set of points in Spec(T) of the form $(x_0, \ldots, x_n, tx_0, \ldots, tx_n)$, where $t \in k^*$). One easily checks that I_Δ is the kernel of the map $S \to k[x_0, \ldots, x_n, y_0, \ldots, y_n, t]$ given by $x_i \mapsto x_i$ and $y_i \mapsto tx_i$, and so T/I_Δ is isomorphic to the normal semigroup ring $k[x_0, \ldots, x_n, tx_0, \ldots, tx_n]$. The ideal I_Δ is generated by the 2×2 minors of the matrix $\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \end{pmatrix}$. More specifically, these minors vanish on Δ, and since this is a generic matrix, the ideal of 2×2 minors is prime of codimension n. As T/I_Δ is already normal, the virtual resolution of T/I_Δ arising from Theorem 1.2 is just the minimal free resolution of T/I_Δ, which is given by the Eagon-Northcott complex on this matrix.

Example 3.2. Let X be the weighted projective space $\mathbb{P}(1,1,2)$ and T the Cox ring $k[x_0, x_1, x_2, y_0, y_1, y_2]$ of $X \times X$. By a calculation similar to Example 3.1, the ring T/I_Δ is isomorphic to the semigroup ring $k[x_0, x_1, x_2, tx_0, tx_1, t^2x_2]$, which is not normal because tx_2 lies in the fraction field and satisfies the integral equation $(tx_2)^2 - x_2 \cdot (t^2x_2) = 0$. Let R be the normalization of T/I_Δ. A presentation matrix for R as a T-module is given as follows, where the rows correspond to the generators 1 and tx_2:

$$\begin{array}{c} 1 \\ tx_2 \end{array} \begin{pmatrix} x_1y_0 - x_0y_1 & x_2y_0 & x_2y_1 & x_0y_2 & x_1y_2 \\ 0 & -x_0 & -x_1 & -y_0 & -y_1 \end{pmatrix}.$$

The free resolution of R as a T-module is given by

$$\begin{matrix} T \\ \oplus \\ T(-1,-1) \end{matrix} \xleftarrow{\begin{bmatrix} x_1y_0 - x_0y_1 & x_2y_0 & x_2y_1 & x_0y_2 & x_1y_2 \\ 0 & -x_0 & -x_1 & -y_0 & -y_1 \end{bmatrix}} \begin{matrix} T(-1,-1) \\ \oplus \\ T(-2,-1)^2 \\ \oplus \\ T(-1,-2)^2 \end{matrix} \xleftarrow{\begin{bmatrix} -x_2 & 0 & -y_2 \\ x_1 & -y_1 & 0 \\ -x_0 & y_0 & 0 \\ 0 & -x_1 & -y_1 \\ 0 & x_0 & y_0 \end{bmatrix}} \begin{matrix} T(-3,-1) \\ \oplus \\ T(-2,-2) \\ \oplus \\ T(-1,-3) \end{matrix} \leftarrow 0. \tag{3.3}$$

Additionally, we have the short exact sequence $0 \to T/I_\Delta \to R \to Q \to 0$, and $Q = tx_2 \cdot k[x_2, y_2]$. One can directly compute that the sheaf $\widetilde{Q}$ corresponding to Q is the zero sheaf on $X \times X$. In fact, since

Q is annihilated by x_0, x_1, y_0 and y_1, we can reduce to checking that $\widetilde{Q}$ is also zero on the affine patch $D(x_2y_2)$. The global sections of $\widetilde{Q}$ on this patch are $Q[x_2^{-1}, y_2^{-1}]_{(0,0)} = 0$, and thus $\widetilde{Q} = 0$, as desired.

Remark 3.4. Since $\mathcal{O}(-1)$ and $\mathcal{O}(-3)$ are not vector bundles on $\mathbb{P}(1,1,2)$, the resolution (3.3) does not induce a locally free resolution of the diagonal. Indeed, virtual resolutions are not guaranteed to induce locally free resolutions of $\mathcal{O}_X$-modules unless X is smooth. Alternatively, as in [HHL], one could consider the corresponding toric stack.

Remark 3.5. In many of the prior known cases of Theorem 1.4, a slightly stronger result was proven. Namely, it was shown that for any such M, there exists another module M' satisfying $\widetilde{M} = \widetilde{M'}$ and $\operatorname{pdim}(M') \le \dim X$; see [EES15, BHS, Yan21]. It would be interesting to determine if this was true in general.

Acknowledgements. We are very grateful to Andrew Hanlon, Jeff Hicks and Oleg Lazarev for patiently talking to us about their work and for several inspiring conversations. We only found this approach because of our efforts to understand their beautiful results. We also thank Christine Berkesch, Lauren Cranton Heller, Mahrud Sayrafi and Jay Yang for helpful comments and discussions. Finally, we thank the anonymous referee for many helpful suggestions.

Competing interest. The authors have no competing interest to declare.

Financial support. The second author was supported by NSF grant DMS-2200469.

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