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FINITELY GENERATED SUBGROUPS AND THE CENTRE OF SOME FACTOR GROUPS OF FREE PRODUCTS

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Dedicated to Bernhard Neumann on his 90th birthday

For groups of the type $F/[R,R]$, where F is a free product, we prove a generalisation of a theorem of Karrass and Solitar on a finitely generated subgroup of a free product containing a nontrivial subnormal subgroup. We also describe the centre of the group $F/[R,R]$.

1. INTRODUCTION

We denote by

$$F = (\mathop{*}_{i\in I} A_i) * X \tag{1}$$

the free product of nontrivial groups A_i $(i \in I)$ and a free group X with basis $\{x_j \mid j \in J\}$ such that $|I| \geqslant 1$ and $|J| \geqslant 1$. Define the rank of the decomposition (1) to be rank $F = |I| + |J|$. Let R be a normal subgroup of F such that $R \cap A_i = 1$ $(i \in I)$. Let $A = F/R$; $G = F/[R,R]$, and $N = R/[R,R]$.

Karrass and Solitar [1] proved that if a finitely generated subgroup H of the free product of two nontrivial groups contains a nontrivial subnormal subgroup of the free product then H is of finite index. We prove a generalisation of the above result to groups of the type $F/[R,R]$ modulo the subgroup $R/[R,R]$.

THEOREM 1. *Let rank $F \geqslant 2$. Let H be a finitely generated subgroup of the group G and let C be a nontrivial subgroup of H with a subnormal series:*

$$G = G_1 \rhd G_2 \rhd \ldots \rhd G_m \rhd C. \tag{2}$$

Then

(1) *if $C \not\leqslant N$ then $|G : HN| < \infty$;*
(2) *if $C \leqslant N$ then $\left|G_m N : (HN \cap G_m N)\right| < \infty$.*

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Our second result describes the centre of the group $F/[R,R]$. This generalises a result of Auslander and Lyndon [**2**] for the case $F = X$ which states that if R is a nontrivial normal subgroup of a non-cyclic free group $F(= X)$ then the centre of $F/[R,R]$ is trivial if and only if F/R is infinite.

Theorem 2. *Let F be the free product (1) with rank $F \geqslant 2$ and $R \neq 1$. If either the group $A = F/R$ is infinite or, in the decomposition (1), the factor X is not present, then the centre of the group G is trivial. If, however, the group A is finite and X is nontrivial, then the centre of the group G is a free Abelian group of rank equal to the rank of the free group X.*

The proofs of these theorems essentially use a generalisation of Magnus and Shmel'kin embeddings for groups of the type $F/[R,R]$ which is defined and studied in [**3**].

2. Notation and preliminaries

We use the following notation for a given group G: $[x,y] = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$, $x^{\alpha_1 y_1 + \ldots + \alpha_s y_s} = (x^{\alpha_1})^{y_1} \ldots (x^{\alpha_s})^{y_s}$, for $x, y, y_1, \ldots, y_s \in G$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{Z}$. We denote by $\langle U \rangle$ the subgroup generated by the set U and define $[U,V] = \big\langle [u,v] \mid u \in U, v \in V \big\rangle$.

We assume for simplicity that the sets I and J of indices are finite, although all our proofs are valid without this assumption. Let $I = \{1, \ldots, n\}$, $J = \{n+1, \ldots, n+l\}$. Denote by $\overline{f}$ the canonical image in A of an element $f \in F$. As canonical epimorphisms $F \to A$, $F \to G$ yield embeddings of subgroups A_i $(i \in I)$, we identify these subgroups with their images in A and G respectively. Denote by π the canonical epimorphism $G \to A$.

Let M denote the group of matrices $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$, where T is a right A-module with a basis $\{t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+l}\}$. It is proved in [**3**] that the kernel of the homomorphism $\tau : F \to M$, defined by the mapping

$$a_i \to \begin{pmatrix} a_i & 0 \\ t_i(a_i - 1) & 1 \end{pmatrix}, \; x_j \to \begin{pmatrix} \overline{x}_j & 0 \\ t_j & 1 \end{pmatrix} \quad (a_i \in A_i,\ i \in I,\ j \in J),$$

is $[R,R]$. So we identify the groups $G = F/[R,R]$ and $F\tau$. We shall also need the following criterion from [**3**] for a matrix from M to belong to the group G:

$$\begin{pmatrix} a & 0 \\ t_1 u_1 + \ldots + t_n u_n + \ldots + t_{n+l} u_{n+l} & 1 \end{pmatrix} \in G \Leftrightarrow$$

$$u_1 \in (A_1 - 1) \cdot \mathbb{Z}A, \; \ldots, \; u_n \in (A_n - 1) \cdot \mathbb{Z}A,$$

$$u_1 + \ldots + u_n + (\overline{x}_{n+1} - 1)u_{n+1} + \ldots + (\overline{x}_{n+l} - 1)u_{n+l} = a - 1. \tag{3}$$

Lemma 1. *Let $c \in G \setminus N$, $a = c\pi$, $1 \neq t \in N$. Suppose that a is an element of prime order p and $t^{c-1} = 1$. Then $t \in \langle c^p \rangle \cdot N^{1+c+\ldots+c^{p-1}}$.*

Proof: As the elements of N are represented in M by unitriangular matrices and $N \lhd M$, we can identify N with the corresponding submodule of the module T. We shall use the language of modules and instead of t^{c-1} write $t(a-1)$. If we consider T as an $\langle a \rangle$-module then it is a free module with a basis consisting of elements of the form, for instance, $t_i y$ where $i \in I \cup J$ and y ranges over a fixed system of representatives of left cosets of the subgroup $\langle a \rangle$ in the group A. Hence, if $t(a-1) = 0$ then the element t is divisible by $1 + a + \ldots + a^{p-1}$ in the module T, but in the general case a quotient does not belong to N.

Let $f \in F$ and $f\tau = c$. Consider the subgroup $L = \langle f, R \rangle$ of the group F. Note that L/R is a subgroup of A and $L/[R,R]$ is a subgroup of G. By the Kurosh subgroup theorem the group L decomposes into a free product of some groups which are conjugates of A_i and a free group. As R has trivial intersection with any conjugate of a subgroup of A_i, we can replace the group F by the group L in our lemma.

So let $F = \langle f, R \rangle$. Then $A = F/R = \langle a \rangle$ is a cyclic group of prime order p. Consider an arbitrary element $h \in F \setminus R$ so that $\langle \overline{h} \rangle = A = \langle a \rangle$ and consequently, $(a-1) \cdot \mathbb{Z}A = (\overline{h} - 1) \cdot \mathbb{Z}A$ and $(1 + a + \ldots + a^{p-1}) \cdot \mathbb{Z}A = (1 + \overline{h} + \ldots + (\overline{h})^{p-1}) \cdot \mathbb{Z}A$. Then $t(a-1) = 0 \Leftrightarrow t(\overline{h} - 1) = 0$ and $N(1 + a + \ldots + a^{p-1}) = N\big(1 + \overline{h} + \ldots + (\overline{h})^{p-1}\big)$. Let $h = f^s r$, where $1 \leqslant s \leqslant p-1$, $r \in R$. We have $h^p = f^{sp} r^{(f^s)^{p-1} + \ldots + f^s + 1}$. It follows from this equation that

$$\big\langle (h\tau)^p \big\rangle \leqslant \langle c^p \rangle + N\big(1 + \overline{h} + \ldots + (\overline{h})^{p-1}\big) = \langle c^p \rangle + N(1 + a + \ldots + a^{p-1}).$$

Similarly, $\langle c^p \rangle \leqslant \big\langle (\overline{h})^p \big\rangle + N\big(1 + \overline{h} + \ldots + (\overline{h})^{p-1}\big)$, and hence

$$\big\langle (\overline{h})^p \big\rangle + N\big(1 + \overline{h} + \ldots + (\overline{h})^{p-1}\big) = \langle c^p \rangle + N(1 + a + \ldots + a^{p-1}).$$

Therefore, if required, in our lemma we can replace the element $c = f\tau$ by any element $h\tau$ $(h \in F \setminus R)$.

(a) We first consider the case when $F \neq X$, that is, the factors A_i are present in the decomposition (1). Then every group A_i $(i \in I)$ must be cyclic of prime order p and its canonical image in A coincides with A. We may assume that $f = a \in A_1$. It is then possible to change the group X by multiplying its basis elements by suitable powers of the element a such that X is contained in R. Let $t = t_1 u_1 + \ldots + t_{n+l} u_{n+l}$, where $u_1, \ldots, u_{n+l} \in \mathbb{Z}A$. As $t(a-1) = 0$, every element u_i $(i = 1, \ldots, n+l)$ is divisible by $1 + a + \ldots + a^{p-1}$. Moreover, we know that

$$u_1 \in (A_1 - 1) \cdot \mathbb{Z}A = (A-1) \cdot \mathbb{Z}A, \ \ldots, \ u_n \in (A_n - 1) \cdot \mathbb{Z}A = (A-1) \cdot \mathbb{Z}A.$$

Therefore, the elements $u_1, \ldots, u_n$ are divisible by $a - 1$. But then $u_1 = 0, \ldots, u_n = 0$. Let

$$u_{n+1} = v_{n+1}(1 + a + \ldots + a^{p-1}), \ \ldots, \ u_{n+l} = v_{n+l}(1 + a + \ldots + a^{p-1}).$$

The element $t' = t_{n+1}v_{n+1} + \ldots + t_{n+l}v_{n+l}$ satisfies the criterion (3):

$$(\overline{x}_{n+1} - 1)v_{n+1} + \ldots + (\overline{x}_{n+l} - 1)v_{n+l} = 0 \cdot v_{n+1} + \ldots + 0 \cdot v_{n+l} = 0.$$

It follows that $t' \in N$ and consequently, $t \in N(1 + a + \ldots + a^{p-1})$.

(b) Next, we consider the case when $F = X = \langle x_1, \ldots, x_l \rangle$ is a free group. By changing the basis of F and the element f, if necessary, we may assume that $f = x_1$ and the remaining generators $x_2, \ldots, x_l \in R$. Then $R = \langle x_1^p, x_2, \ldots, x_l \rangle \cdot [F, F]$. Let $t = t_1u_1 + \ldots + t_lu_l$. It follows from the condition $t(a-1) = 0$ that

$$u_1 = k_1(1 + a + \ldots + a^{p-1}), \ \ldots, \ u_l = k_l(1 + a + \ldots + a^{p-1})$$

for some integers $k_1, \ldots, k_l$. Then we have

$$t_1u_1 = t_1k_1(1 + a + \ldots + a^{p-1}) = (x_1\tau)^{pk_1} = c^{pk_1},$$
$$t_2u_2 + \ldots + t_lu_l = (t_2k_2 + \ldots + t_lk_l)(1 + a + \ldots + a^{p-1}) \in N(1 + a + \ldots + a^{p-1}).$$

We conclude that $t \in \langle c^p \rangle + N(1 + a + \ldots + a^{p-1})$. This completes the proof of Lemma 1. $\square$

PROOF OF THEOREM 1. Based on the theorem of Karrass and Solitar [**1**], we assume that $R \neq 1$. For a given element $t' \in T$ denote by $\sigma(t')$ the support of t', that is, the set of all elements of A on which t' depends. Let $B = H\pi$. Because the group H is finitely generated, there is a finite system $\{y_1B, \ldots, y_sB\}$ of left cosets of the subgroup B in the group A such that for every matrix $\begin{pmatrix} b & 0 \\ t' & 1 \end{pmatrix} \in H$ the following inclusion holds:

$$\sigma(t') \subseteq \Sigma = y_1B \cup \ldots \cup y_sB. \tag{4}$$

(1) Suppose, by way of contradiction, that the index $|G : HN| = |A : B|$ is infinite. Let $c \in C \setminus N$ and $a = c\pi$. We can assume that the element a has either infinite order or its order is equal to a prime number p. As in Lemma 1 we identify N with a corresponding submodule of the module T. Let $0 \neq t \in N$, $\sigma(t) = \{z_1, \ldots, z_q\}$. There is an element $y \in A$ such that $y \notin z_j^{-1}y_iB$ ($j = 1, \ldots, q$; $i = 1, \ldots, s$). Then $\sigma(ty) \cap \Sigma = \emptyset$. Replacing t by ty we get $\sigma(t) \cap \Sigma = \emptyset$. It follows from the subnormality of the series (2) that $t(a-1)^m \in C$ so that $\sigma\big(t(a-1)^m\big) \cap \Sigma = \emptyset$. If $t(a-1)^m \neq 0$ we get a contradiction to inclusion (4).

So we may assume $t(a-1)^m = 0$. Now T is a free module over the group ring $\mathbb{Z}\langle a \rangle$. It then follows from the equation $t(a-1)^m = 0$, that the element a has finite order. We may assume that a has order a prime number p and that $t(a-1) = 0$. Using the criterion (3), if the element t is divisible in the module T by some natural number then the quotient also belongs to N. So, we can assume that the element t is not divisible by p. As $t(a-1) = 0$, by Lemma 1 the element t can be written in the form $t = (c^p)k + t'(1 + a + \ldots + a^{p-1})$, where $k \in \mathbb{Z}$, $t' \in N$. Let, for instance, $t = t_1(yu + \ldots) + \ldots$, where y is a representative of a left coset of $\langle a \rangle$ in A and the element $u \in \mathbb{Z}\langle a \rangle$ is not divisible by p. The coset yB differs from $y_1B, \ldots, y_sB$ and $\sigma(c^p) \subseteq \Sigma$. Then $t' = t_1(yv + \ldots) + \ldots$, where $v \in \mathbb{Z}\langle a \rangle$, $v(1 + a + \ldots + a^{p-1}) = u$. Since $(1 + a + \ldots + a^{p-1})^2$ is divisible by p, the element v is not divisible by $1 + a + \ldots + a^{p-1}$. Hence, $v(a-1) \neq 0$, so that $v(a-1)^m \neq 0$ and $\sigma\big(t'(a-1)^m\big) \not\subseteq \Sigma$. This is contrary to the condition $t'(a-1)^m \in C \leqslant H$.

(2) If $C \leqslant N$ we can identify C with the corresponding additive subgroup of the module T. Let $0 \neq t \in C$. Assume that the index

$$\big|G_mN : (G_mN \cap HN)\big| = \big|G_m\pi : (G_m\pi \cap B)\big|$$

is infinite. Then there is an element $a \in G_m\pi$ such that $\sigma(ta) \not\subseteq \Sigma$. Since C is normal in G_m it follows that $t(a-1) \in C$. On the other hand $\sigma\big(t(a-1)\big) \not\subseteq \Sigma$, contrary to (3). This completes the proof of Theorem 1. ∎

PROOF OF THEOREM 2.

LEMMA 2. *Let $C(G)$ be the centre of G. Then $C(G) \leqslant N$.*

PROOF: Suppose $c \in G \setminus N$. We shall prove that the element c does not commute with some element of N and so $c \notin C(G)$. Let $a = c\pi$. If $|a| = \infty$ then for every nontrivial element $t \in N$ we have $t(a-1) \neq 0$, that is, the elements t and c do not commute. So, we can assume that the order of c is finite and equal to a prime number p. Let $f \in F$ and $f\tau = c$. It follows from the conditions rank $F \geqslant 2$, $R > 1$, and $R \cap A_i = 1$ $(i \in I)$, that R is a free nonabelian group. Therefore, if we consider the Kurosh subgroup decomposition of the subgroup $\langle f, R \rangle$ of F into a free product, then its rank is at least 2, and we can assume in our lemma that $F = \langle f, R \rangle$. Then $F/A = A = \langle a \rangle$ is a cyclic group of prime order p.

(a) Let the groups A_i be present in the decomposition (1), that is, $F \neq X$. In this case we can assume that $f = c = a \in A_1$ and $X \leqslant R$. Then each A_i must be cyclic of order p and its canonical image in A must coincide with A. Let $n \geqslant 2$ and $A_1 = \langle a_1 \rangle$, $A_2 = \langle a_2 \rangle$, $\bar{a}_1 = \bar{a}_2 = a$. Consider the element $t = [a_1\tau, a_2\tau] = (t_1 - t_2)(a-1)^2$. We have $t(a-1) \neq 0$. If $n = 1$, that is, $F = A_1 * X$, then rank $X \geqslant 1$. Let $t = x_2\tau = t_2$. We have again $t(a-1) \neq 0$.

(b) Let $F = \langle x_1, \ldots, x_l \rangle$ be a free group. Then it is possible to assume that $f = x_1$ and $x_2, \ldots, x_l \in R$. Let $t = x_2\tau = t_2$. Then $t(a-1) \neq 0$. Lemma 2 is proved. ∎

LEMMA 3. *Let A be an infinite group. Then $C(G) = 1$.*

PROOF: Based on the previous lemma, it is sufficient to prove that for every nontrivial element $t \in N$ there is an element $a \in A$ such that $t(a-1) \neq 0$. As A is infinite we can choose an element a such that $\sigma(t) \neq \sigma(ta)$. Then $t(a-1) \neq 0$. Lemma 3 is proved. ∎

Now we assume that the group A is finite and let d denote the sum of all elements of A.

Lemma 4. *If A is a finite group, then $C(G) = Td \cap N$.*

Proof: It is obvious that Td is contained in the centre of the group M, therefore $C(G) \geqslant Td \cap N$. Assume that some element $t \in T$ does not belong to Td. Let, for example, $t = t_1(k_1 b_1 + k_2 b_2 + \ldots) + \ldots$, where $k_1, k_2, \ldots$ are integers, $b_1, b_2, \ldots$ are different elements of A and $k_1 \neq k_2$. If $a = b_1^{-1} b_2$, then $ta \neq t$. This means that the element t does not centralise the group G. Hence, $C(G) \leqslant Td \cap N$. This proves Lemma 4. ∎

The proof of Theorem 2 follows from Lemma 3 and the following lemma.

Lemma 5. *Let A be a finite group. Then the centre of the group G coincides with an additive subgroup of the module T generated by the elements $t_{n+1}d, \ldots, t_{n+l}d$.*

Proof: Let

$$t = t_1 u_1 + \ldots + t_n u_n + t_{n+1} u_{n+1} + \ldots + t_{n+l} u_{n+l} \in C(G) = Td \cap N.$$

As every element u_i $(i = 1, \ldots, n+l)$ is divisible by d, it is possible to represent it in the form $u_i = k_i d$, $k_i \in \mathbb{Z}$. It follows from the criterion (3) that $u_1, \ldots, u_n \in (A-1) \cdot \mathbb{Z}A$. Then $u_1 = \ldots = u_n = 0$. So $C(G)$ is contained in the additive subgroup of the module T generated by the elements $t_{n+1}d, \ldots, t_{n+l}d$. On the other hand, every element $t_j d$ $(j \in J)$ which centralises G and belongs to G satisfies the criterion (3): $(\overline{x}_j - 1)d = 0$. This completes the proof of Lemma 5 and, in turn, the proof of Theorem 2. ∎

References

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