10

An Introduction to Totally Disconnected Locally Compact Groups and Their Finiteness Conditions

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10.1 Introduction

The class of locally compact groups generalises discrete and Lie groups. Locally compact groups came to light in the first half of the 20th century, and since then they have played a central role among topological groups. The 20th century has been characterised by intense activity on the structure theory of many algebraic objects, e.g., simple finite groups, fields and rings with ascending chain condition. What about the general structure of locally compact groups? A basic strategy to understand the structure of a locally compact group G is to split it into smaller groups: let G_0 be the largest connected subset of G containing the identity element, which is a closed subgroup (see Fact 10.1), and produce the short exact sequence

$$1 \to G_0 \to G \to G/G_0 \to 1,$$

where G_0 is a **connected** locally compact group and G/G_0 is a **totally disconnected** locally compact group (i.e., the connected components of G/G_0 are reduced to singletons). Therefore, G can be regarded as an extension of its connected component by the totally disconnected piece G/G_0. It follows that questions about the structure of locally compact groups may be dealt with by treating separately the cases where G is connected and where G is totally disconnected and then combining the two answers. With the solution of Hilbert's fifth problem, our understanding of connected locally compact groups has significantly increased: they can be approximated by Lie groups (see Theorem 10.8). Therefore, the contemporary structure problem on locally compact groups concerns the class of totally disconnected locally compact groups.

The first part of the notes introduces the objects under investigation: topological groups that are locally compact and totally disconnected (TDLC-groups, for short). In particular, it includes basic properties of topological groups, a

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proof of van Dantzig's theorem, which is the fundamental structure theorem of TDLC-groups, and several examples.

The study of this class of topological groups can be made more manageable by dividing the infinity of objects under investigation into classes of types with "similar structure:" we focus on TDLC-groups satisfying some finiteness conditions. The most common finiteness condition for (totally disconnected) locally compact groups is *compact generation* – i.e., the topological group is algebraically generated by a compact subset – which naturally generalises the notion of finite generation that has been widely (and fruitfully) used in group theory to study infinite groups. Since every TDLC-group is a directed union of compactly generated open subgroups (see Fact 10.19), one can confine the investigation to the compactly generated case without losing too much information (at least from a local perspective).

All compactly generated TDLC-groups fall in the class of automorphism groups of locally finite connected graphs (see § 10.3.3). Indeed, [1] proved that, by using van Dantzig's theorem, we can always construct a locally finite connected graph Γ on which the group G acts vertex-transitively and with compact open stabilisers, the so-called *Cayley–Abels graph of G* (see § 10.4.2). To be precise, every compactly generated TDLC-group comes with a family of Cayley–Abels graphs which satisfy the property of being quasi-isometric to each-other: the Cayley–Abels graph Γ of a compactly generated TDLC-group is unique up to quasi-isometry. Therefore, by analogy with the theory of finitely generated groups and their Cayley graphs, we can study a compactly generated TDLC-group as a geometric object and all the properties of Γ that are invariant up to quasi-isometry become *geometric group invariants*[1] of G. It leads to a new line of research in topological group theory, *geometric group theory for compactly generated TDLC-groups*, which can be traced back to the work of [31].

As a consequence, TDLC-groups need to be viewed simultaneously as geometric objects and topological objects. Therefore, the interaction between the local structure (i.e., the topological one) and the large-scale structure (i.e., the geometric one) becomes important. Since profinite groups are trivial as geometric groups and discrete groups are trivial as topological groups, it is not surprising that the profinite groups and the discrete groups constitute the atomic pieces in the theory of TDLC-groups and that we are curious to understand to what extent well-known results about the geometry of infinite discrete groups find an analogue in such a topological context.

[1] The number of ends, growth and hyperbolicity are examples of geometric group invariants of a TDLC-group G.

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The final part of the chapter attempts to introduce the reader to some homological finiteness conditions for TDLC-groups that generalise compact generation in higher dimensions. Since these notes are meant to be on the non-specialist level, we only provide definitions and references, letting the reader decide how deep to dig into the subject.

The conclusive part briefly introduces the seminal work of [51, 52] which was a fundamental breakthrough in the theory of TDLC-groups after several years of stillness.

Willis' theory gave start to the research interest we now benefit from.

Pre-requisites: The reader is supposed to have mastered linear algebra, point-set topology and basic notions from group theory. These notes aim to help beginning Ph.D. students taking the first steps towards the theory of TDLC-groups. Therefore, this text is (supposed to be) accessible to non-specialists and self-contained (most of the results are proved in these notes) but, as with any advanced topic, some of the results are stated without proofs. In such a case, references are provided. All the proofs provided in these notes are rather classical and can be found in the literature.

The interested reader can also have a look at a few conference proceedings that collect part of the progress made with the general theory of TDLC-groups and include some interesting open problems; see for example [14] and [57].

Notation: We denote by

– $\mathbb{N}$ the set of natural numbers $\{0,1,2,\ldots\}$,
– $\mathbb{Z}$ the ring of rational integers and $\mathbb{Q}$ the ring of rational numbers,
– $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers,
– $\mathbb{R}_+$ the subset of non-negative real numbers.
– If R is a commutative ring with unit, $R^\times$ stands for its multiplicative group of units. For example, $\mathbb{R}_+^\times$ is the group of positive real numbers.

10.2 Preliminaries on Topological Groups

A complete and detailed introduction to the theory of topological groups can be found in several textbooks; for example, [27]. For convenience, all topological spaces appearing below are assumed to satisfy the Hausdorff separation axiom.[2]

[2] Totally disconnected locally compact groups are necessarily Hausdorff.

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10.2.1 Warming-up

Definition 10.1 (Topological group) A **topological group** is a group $(G,\cdot)$ which is also a topological space such that the following maps are continuous:

- the group operation

$$\_\cdot\_ \colon G\times G\to G,\quad (x,y)\mapsto x\cdot y,\quad \forall x,y\in G,$$

  where $G\times G$ is endowed with the product topology;
- the inversion map

$$\_^{-1}\colon G\to G,\quad x\mapsto x^{-1},\quad \forall x\in G.$$

If the underlying group G is cyclic (resp., abelian, nilpotent, etc.), the topological group G is also called cyclic (resp. abelian, nilpotent, etc.). The additive notation $(G,+)$ can describe some topological groups but only in the case when the topological group is abelian.

Remark Similarly, one has **topological rings** and **topological fields**.[3]

Every abstract group G can be viewed as a topological group if given the discrete topology. In such a case, G is called a **discrete group** and, since these notes concern topological groups, we will often refer to (abstract) groups as discrete groups.

Exercise 10.2 Let G be a topological group. Prove that:

1. If a subgroup $H\leqslant G$ is open, then it is also closed.
2. If a subgroup contains an open set then it is open.
3. A connected subset $C\ni x$ is contained in the intersection of all clopen[4] subsets $O\ni x$ of G.
4. If G is connected, then the only open subgroup of G is G itself.
5. Every quotient map of a topological group is open.
6. For $H\leqslant G$ normal, the quotient group G/H is discrete iff H is open. In particular, the group G is discrete iff the point 1_G is isolated, i.e., the singleton $\{1_G\}$ is open.

Other (less trivial) examples of a topological group are provided by the additive group $(\mathbb{R},+)$ of the reals equipped with the usual topology, its subgroups $\mathbb{Z}$ and $\mathbb{Q}$ (with the subspace topology) and the quotient $\mathbb{T}:=\mathbb{R}/\mathbb{Z}$ (with the quotient topology). These extend to all powers $(\mathbb{R}^d,+)$, and also $(\mathbb{C},+)$, because it can be easily proved that products of topological groups are again topological

[3] By “field” we always mean “commutative field”.
[4] A set which is both closed and open.

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groups. Moreover, if R is a topological ring, then the ring $M(n,R)$ of all $n \times n$ matrices with entries in R is a topological ring if endowed with the product topology of $R^{n\times n}$.

Exercise 10.3 Let R be a commutative topological ring such that inversion is continuous on the set of invertible elements (for example, if R is a topological field). Prove that

1 the group $(GL(n,R),\cdot)$ of all invertible $n \times n$ matrices with entries in R is a topological group (notice that $GL(n,R)$ is a subset of $M(n,R)$ but not a subgroup; in this case Cramer's rule helps with the continuity of the inversion map);
2 $SL(n,R) = \{M \in GL(n,R) \mid \det(M) = 1\}$ is closed in $GL(n,R)$.

A topology τ on the group G such that the space (G,τ) is a topological group is called a **group topology** on G. Obviously, a topology τ on G is a group topology if, and only if, the map $G \times G \to G, \quad (x,y) \mapsto xy^{-1}$, is continuous for all $x,y \in G$. Notice that, for every $g \in G$, the **left translation** $x \mapsto gx$, the **right translation** $x \mapsto xg$, as well as the **conjugation** $x \mapsto gxg^{-1}$ are continuous; in other words, every topological group is a homogeneous topological space.

Exercise 10.4 Prove the assertions above.

As a consequence, the topology of G is determined by a neighbourhood basis[5] of the identity: a family $\{U_\alpha\}_{\alpha\in I}$ of arbitrarily small neighbourhoods of 1_G determines the family $\{gU_\alpha\}_{\alpha\in I}$ of arbitrarily small neighbourhoods of any other group element g.

Example 10.5 One can define group topologies on G by declaring well-behaved collections of subsets to be neighbourhood basis at 1_G (see [9]):

– the **profinite topology** is determined by the family of all normal subgroups of finite index of G;
– the **pro-p topology** is determined, for a prime p, by all normal subgroups of G of finite index that is a power of p;
– the **p-adic topology** is determined, for a prime p, by the family $\{U_n\}_{n\in\mathbb{N}}$ of normal subgroups of G, where U_n is generated by the powers $\{g^{p^n} \mid g \in G\}$.

Example 10.6 (Absolute values on fields) An **absolute value** on a field $\mathbb{K}$ is a function $|\_|\colon \mathbb{K} \to \mathbb{R}$ satisfying

[5] A family $\mathcal{B}$ of neighbourhoods of the point x is said to be a *neighbourhood basis* of x if for every neighbourhood U of x there exists $V \in \mathcal{B}$ contained in U.

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$|1|$. $|x| \geqslant 0$ for every $x \in \mathbb{K}$, and $|x| = 0$ if and only if $x = 0$,
$|2|$. $|xy| = |x||y|$, for all $x, y \in \mathbb{K}$,
$|3|$. $|x+y| \leqslant |x| + |y|$, for all $x, y \in \mathbb{K}$.

If the absolute value $|\_|$ satisfies the stronger condition

$|\bar{3}|$. $|x+y| \leqslant \max\{|x|, |y|\}$, for all $x, y \in \mathbb{K}$,

it is said to be **non-Archimedean**, otherwise it is **Archimedean**. For example, the usual absolute value on $\mathbb{R}$ is Archimedean.

It is clear that $d(x,y) = |x-y|$ gives $\mathbb{K}$ a structure of a metric space, and the topology for which

$$B_\varepsilon(0) \colon = \{x \in \mathbb{K} \mid |x| < \varepsilon\}, \quad \varepsilon > 0,$$

form a basis of neighbourhoods of 0 is a field topology.

Exercise 10.7 Let $\mathbb{K}$ be a field equipped with the absolute value $|\_|$. The field topology defined on $\mathbb{K}$ by $|\_|$ is discrete iff $|x| = 1$ for all $x \neq 0$.

Example 10.8 (The field of p-adic numbers) For a prime p, the **p-adic absolute value** on $\mathbb{Q}$ is defined as $|x|_p = p^{-n}$, for every $x \in \mathbb{Q}$, where n is the unique integer such that $x = p^n(\frac{a}{b})$ and neither of the integers a and b is divisible by p (with the convention, $|0|_p = 0$). It is an example of non-Archimedean absolute value. The **p-adic metric** $d_p(x,y) = |x-y|_p$ induces a field topology on $\mathbb{Q}$ which is called the **p-adic topology** of $\mathbb{Q}$; see Example 10.5. As with the topology on $\mathbb{Q}$ inherited from $\mathbb{R}$, the metric space $(\mathbb{Q}, d_p)$ is not complete (i.e., not every Cauchy sequence converges in $(\mathbb{Q}, d_p)$). Let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ with respect to the p-adic metric. The field $\mathbb{Q}_p$ has characteristic zero and it is called the **field of p-adic numbers**. The metric d_p, and so the p-adic topology, extends from $\mathbb{Q}$ to $\mathbb{Q}_p$ and yields a topological field; see [37, § 12.3.4].

The identity component of G. On a topological space X one defines the equivalence relation $\sim$ as follows: $x \sim y$ if there exists a connected subspace $C \subseteq X$ such that $x, y \in C$. Each equivalence class is a maximal connected subspace which is called a **connected component** of X.

Definition 10.9 Given a topological group G, the connected component containing the identity is the **identity component** of G, and it is denoted by G_0. Clearly, G_0 is the union of all connected subspaces of G containing 1_G, and the topological group G is connected if, and only if, $G = G_0$. A topological group is said to be **totally disconnected** if the identity is its own connected component, that is, $G_0 = \{1_G\}$.

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Notice that, for every $g \in G$, the set $gG_0 = G_0 g$ is nothing but the connected component containing g because continuous maps preserve connectedness and translations are continuous. As a consequence, one has the following important fact.

Fact 10.1 *Given a topological group, the identity component is a closed characteristic subgroup.*

Proof The subspace G_0 is closed since the closure of a connected subspace is still connected.

For every $x \in G_0$, the translate $x^{-1}G_0 \ni 1_G$ is connected. It follows that $x^{-1} \in x^{-1}G_0 \subseteq G_0$, i.e., G_0 is closed under taking inverses. On the other hand, for all $x, y \in G_0$, one has $xy \in xG_0$ but xG_0 is the connected component of x which is in the same connected component as 1_G (because $x \sim 1_G$), i.e., $xy \in xG_0 = G_0$ and G_0 is a subgroup.

Finally, G_0 is characteristic (and in particular normal) because continuous homomorphisms preserve connectedness. □

Proposition 10.2 *Let G be a topological group and let G_0 be the identity component. Then G/G_0 is a totally disconnected group.*

Proof Let $\pi\colon G \to H\colon = G/G_0$ be the quotient map. One has to prove that $H_0 = 1$, i.e., $\pi^{-1}(H_0) = G_0$. By the maximality of G_0, it suffices to prove that $\pi^{-1}(H_0) \supseteq G_0$ is connected. To this end, suppose by contradiction $\pi^{-1}(H_0)$ is a disjoint union $C_1 \cup C_2$ of non-empty closed subsets of G. Since G_0 is connected, for every $g \in \pi^{-1}(H_0)$, either $gG_0 \subseteq C_1$ or $gG_0 \subseteq C_2$. Therefore, C_1 and C_2 are unions of G_0-cosets. Consequently, H_0 is the disjoint union of non-empty closed subsets, a contradiction. □

10.2.2 Profinite Groups

For a complete introduction to the realm of profinite groups the reader is referred to [40, 56] and [14, Chapter 3]. We recall only the definition and provide some easy examples.

A **directed poset** $(I, \preceq)$ is a set I with a binary relation $\preceq$ satisfying:

(DP1) $i \preceq i$, for $i \in I$;
(DP2) $i \preceq j$ and $j \preceq k$ imply $i \preceq k$, for $i, j, k \in I$;
(DP3) $i \preceq j$ and $j \preceq i$ imply $i = j$, for $i, j \in I$; and
(DP4) if $i, j \in I$, there exists some $k \in I$ such that $i \preceq k$ and $j \preceq k$.

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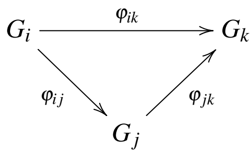
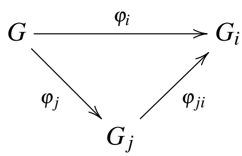
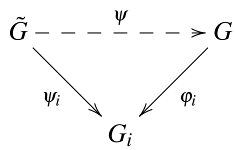
An **inverse system of topological groups over** I consists of a family $\{G_i \mid i \in I\}$ of topological groups together with continuous group morphisms $\varphi_{ij}\colon G_i \to G_j$, defined whenever $j \preceq i$, such that the diagram



commutes whenever $k \preceq j \preceq i$. In addition, we assume that φ_{ii} is the identity morphism for every $i \in I$. A projective system of topological groups is said to be surjective if every morphism φ_{ij} is surjective. A family of continuous group morphisms $\varphi_i\colon G \to G_i$ is said to be **compatible** with the inverse system (G_i, φ_{ij}, I) if, for every $i \preceq j$, the diagram



commutes. A topological group G together with a compatible family $\{\varphi_i\colon G \to G_i\}_{i\in I}$ of continuous morphisms is an **inverse limit** of the inverse system (G_i, φ_{ij}, I) if the following universal property is satisfied: for every topological group $\tilde{G}$ together with a compatible family $\{\psi_i\colon \tilde{G} \to G_i\}_{i\in I}$ of continuous group morphisms, there exists a unique continuous group morphism $\psi\colon \tilde{G} \to G$ such that the diagram



commutes for every $i \in I$. In such a case, we denote the inverse limit by

$$G = \varprojlim_{i\in I} (G_i, \varphi_{ij}),$$

and call the maps $\varphi_i\colon G \to G_i$ *projection morphisms*. If the family φ_{ij} is clear from the context, we use simply $G = \varprojlim_{i\in I} G_i$.

Proposition 10.3 ([40, Proposition 1.1.1]) *Let (G_i, φ_{ij}, I) be an inverse system of topological groups over a directed poset I. Then the following hold:*

(a) *there exists an inverse limit of the inverse system* (G_i, φ_{ij}, I)*;*
(b) *this limit is unique in the following sense: if* (G, φ_i) *and* (H, ψ_i) *are two limits of* (G_i, φ_{ij}, I)*, then there is a unique topological isomorphism* $\varphi\colon G \to H$ *such that* $\psi_i \varphi = \varphi_i$ *for each* $i \in I$*.*

In particular, the inverse limit (G, φ_i) can be constructed as follows:

- $G\colon = \{(g_i)_{i\in I} \in \prod_{i\in I} G_i \mid \varphi_{ij}(g_i) = g_j \text{ if } j \preceq i\}$;
- $\varphi_i\colon G \to G_i$ is the restriction of the projection $\prod_{i\in I} G_i \to G_i$;
- the group topology on G is the subspace topology inherited by the product topology of $\prod_{i\in I} G_i$ (and G turns out to be a closed subset).

Fact 10.4 ([40, Lemma 1.1.2]) *If* (G_i, φ_{ij}, I) *is an inverse system of topological groups, then* $\varprojlim_{i\in I} G_i$ *is a closed subgroup of the product* $\prod_{i\in I} G_i$*.*

Definition 10.10 A **profinite group** G is the inverse limit $\varprojlim_{i\in I} G_i$ of a surjective inverse system (G_i, φ_{ij}, I) of finite groups G_i, where each finite group G_i is assumed to have the discrete topology.

For a profinite group G, a neighbourhood basis at 1_G is $\{\ker(\varphi_i)\}_{i\in I}$, where $\varphi_i\colon G \to G_i$ are the canonical projection homomorphisms.

Fact 10.5 *A profinite group G is compact and totally disconnected.*

Proof It easily follows from the fact that G is a closed subset of the product of finite groups (see Fact 10.4). To see this, one needs to recall the following basic properties:

- closed subspaces and products of compact (resp. totally disconnected) spaces are compact (resp. totally disconnected);
- a finite group is a compact discrete group; in particular it is totally disconnected.

□

Example 10.11 (a) Let R be a profinite commutative ring with unit. Then the following groups (with topologies naturally induced from R) are profinite groups: $R^\times$, $GL(n,R)$ and $SL(n,R)$.
(b) Consider the natural numbers $I = \mathbb{N}$, with the usual partial ordering, and the group of integers $\mathbb{Z}$. Form the inverse system $\{\mathbb{Z}/n\mathbb{Z}, \varphi_{nm}\}$, where the map $\varphi_{nm}\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is the natural projection for $m \leqslant n$. The inverse limit produces the profinite group $\widehat{\mathbb{Z}}$, which can be identified with the set of equivalence classes of tuples of integers

$$\{\overline{(x_1, x_2, x_3, \dots)} \mid x_n \in \mathbb{Z}, \forall n \in \mathbb{Z}, \text{ and } x_m = x_n \bmod m \text{ whenever } m|n\}.$$

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Note that $\widehat{\mathbb{Z}}$ naturally inherits a structure of profinite ring from the finite rings $\mathbb{Z}/n\mathbb{Z}$. The ring $\widehat{\mathbb{Z}}$ is the **profinite completion** of $\mathbb{Z}$.

(c) Let p be a prime and form the profinite group defined by the inverse limit over the system $\{\mathbb{Z}/p^n\mathbb{Z}, \psi_{mn}\}$ given by the canonical projections. It is the **pro-p completion** of $\mathbb{Z}$ and it is topologically isomorphic to the ring of p-adic integers $\mathbb{Z}_p$: it suffices to prove that $\mathbb{Z}_p$ is the inverse limit of its quotients $\mathbb{Z}_p/p^n\mathbb{Z}_p$ (where the family $\{p^n\mathbb{Z}_p\}_{n\in\mathbb{N}}$ is the neighbourhood basis at 0 in the group of p-adic integers) and that each $\mathbb{Z}_p/p^n\mathbb{Z}_p$ is isomorphic to the finite group $\mathbb{Z}/p^n\mathbb{Z}$.

The set of the elements of $\mathbb{Z}_p$ can be then identified with the set of all equivalence classes of sequences $(a_1, a_2, a_3, \dots)$ of natural numbers such that $a_m = a_n (\operatorname{mod} p^m)$, whenever $m \leqslant n$.

Exercise 10.12 1 Let $\{G_i \mid i \in I\}$ be a collection of finite groups. Is the direct product a profinite group?

2 Consider the natural numbers $I = \mathbb{N}$, with the usual partial ordering, and form the constant inverse system $\{\mathbb{Z}, id\}$. Compute the limit.

3 Let G be a profinite group.

(a) A closed normal subgroup $H \leqslant G$ is open iff it has finite index.

(b) Every open subgroup H of G contains a subgroup H_G that is normal and open in G. (Hint: $H_G = \bigcap_{g\in G} gHg^{-1}$.)

10.2.3 Locally Compact Groups

An arbitrary topological space X is locally compact if every point admits a compact neighbourhood. A topological group G is **locally compact** if 1_G admits a compact neighbourhood. The additive group $(\mathbb{R}, +)$ with its usual topology is a locally compact, non-compact, abelian group. Clearly, the multiplicative group $(\mathbb{R}^\times, \cdot)$ is also locally compact (here $\mathbb{R}^\times$ carries the induced topology) and the same holds for the groups $(\mathbb{C}, +)$ and $(\mathbb{C}^\times, \cdot)$. On the other hand, $\mathbb{Q}$ is not locally compact with the topology inherited by $\mathbb{R}$ (see Proposition 10.7): local compactness is not inherited by subgroups; see Exercise 10.14.

Different examples of locally compact groups are discrete groups and profinite groups. If $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle group,[6] then Tychonov's theorem yields that every power $\mathbb{T}^I$ of $\mathbb{T}$ is again compact and, in particular, locally compact. This is actually the most general example of a compact abelian group: every compact abelian group is isomorphic to a closed subgroup of a power of $\mathbb{T}$; see [23, Corollary 11.2.2].

[6] The circle group is compact because it is the image of the compact subspace $[0,1] \subset \mathbb{R}$.

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Example 10.13 (Local fields) Non-discrete locally compact fields have been classified by [48]. A non-discrete locally compact field is either connected or totally disconnected. A non-discrete locally compact field is connected if and only if it is Archimedean (see Example 10.6), and then isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. Non-Archimedean locally compact fields are called **local fields**.[7] For further details see [37, § 12.3.4] and references there.

Exercise 10.14 1 A closed subgroup of a locally compact group is locally compact. The closure condition is necessary, see Proposition 10.7.
2 If R is a locally compact ring and n is a natural number, then $R^{n\times n}$ is a locally compact ring.
3 Every quotient of a locally compact group is locally compact.
4 The product of a finite family of locally compact groups is locally compact (for infinite products to be locally compact the condition "all but a finite number of factors are compact" is necessary).
5 If $\mathbb{K}$ is a topological field, then $GL(n,\mathbb{K})$ is open in $\mathbb{K}^{n\times n}$. Consequently, $(GL(n,\mathbb{K}),\cdot)$ is locally compact exactly if $\mathbb{K}$ is.
6 If $\mathbb{K}$ is a locally compact field, then $(SL(n,\mathbb{K}),\cdot)$ is locally compact.

Proposition 10.6 *A locally compact countable group is discrete.*

Proof Recall that a *Baire space* is a topological space with the property that for each countable collection of open dense sets $\{U_n\}_{n\in\mathbb{N}}$ their intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense. By the Baire category theorem – see [28, Theorem A] and references there – every locally compact group is a Baire space. Let G be a non-discrete locally compact group; in particular, each singleton $\{g\}$ is closed but not open in G. If G is countable, then

$$G = \{g_1,\dots,g_n,\dots\} = \bigcup_{n\in\mathbb{N}} (G\setminus\{g_n\})$$

but $\bigcap_{n\in\mathbb{N}}(G\setminus\{g_n\}) = \varnothing$, a contradiction. □

Proposition 10.7 ([27, Theorem 5.11]) *If a subgroup H of a topological group G is locally compact, then it is closed.*

Since the identity component G_0 is a closed normal subgroup of the locally compact group G (see Fact 10.1), one can form the quotient group G/G_0. By Exercise 10.14(3), G/G_0 is locally compact. Moreover, the locally compact group G/G_0 is totally disconnected by Proposition 10.2. Therefore, one has the short exact sequence

$$1 \to G_0 \to G \to G/G_0 \to 1 \tag{10.2.1}$$

[7] Some authors define a local field to be any commutative non-discrete locally compact field.

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and every locally compact group G is an extension of a TDLC-group G/G_0 by the connected locally compact group G_0, which can be approximated by Lie groups.

Theorem 10.8 *Every connected locally compact group is an inverse limit of Lie groups.*

The latter description is obtained as a consequence of the following fundamental result.

Theorem 10.9 (Gleason–Yamabe Theorem) *Let G be a locally compact group. For any open neighbourhood U of the identity there exists an open subgroup G' of G and a compact normal subgroup K of G' in U such that G'/K is isomorphic to a Lie group.*

We omit the proof of Gleason–Yamabe theorem but the reader is referred to [46, Theorem 1.1.13].

Proof of Theorem 10.8 Let G be connected and locally compact. In order to construct a projective system of Lie groups, let $\mathcal{N}$ be the set of all open neighbourhoods U of the identity 1_G. By Exercise 10.2(4), the only open subgroup G' of G is the whole group G. Therefore, for every $U \in \mathcal{N}$, there exists a compact normal subgroup K_U of G such that G/K_U is isomorphic to a Lie group. Let $\mathcal{K} = \{K_U \mid U \in \mathcal{N}\}$ be indexed by inverse inclusion. In particular, $(\mathcal{K}, \supseteq)$ is a directed poset; see [29, Glöckner's lemma, p. 148]. Whenever $M \supseteq N$ in $\mathcal{K}$, one has the continuous morphism $\varphi_{NM}\colon G/N \to G/M$ given by $gN \mapsto gM$ and $(G_N, \varphi_{NM}, \mathcal{K})$ is a surjective inverse system of Lie groups. By the universal property of the inverse limit, there exists a continuous morphism $\gamma\colon G \to \varprojlim(G_N, \varphi_{NM}, \mathcal{K})$; namely, $\gamma(g) = (gN)_{N\in\mathcal{K}}$. It is clear that the kernel of γ is given by $\bigcap\{N \mid N \in \mathcal{K}\}$ which is trivial since the elements of $\mathcal{K}$ can be arbitrarily small (see the Gleason–Yamabe theorem above). Thus one needs to prove that γ is open and surjective but it follows by [29, Theorem 1.33] and the fact that locally compact groups are always complete (see, for example, [29, Remark 1.31]). □

As [45] writes on his blog: *this theorem asserts the "mesoscopic" structure of a locally compact group (after restricting to an open subgroup G' to remove the macroscopic structure, and quotienting out by K to remove the microscopic structure) is always of Lie type.*

Remark (Hilbert's fifth problem) During the International Congress of Mathematicians (Paris, 1900) D. Hilbert presented a list of 23 open problems which turned out to be very influential for the mathematics of the 20th century. The

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fifth of these problems asked for a *topological description of Lie groups*: does every locally euclidean topological group admit a Lie group structure? Recall that a topological group is said to be *locally euclidean* if its identity element has a neighbourhood homeomorphic to an open subspace of $\mathbb{R}^n$. A positive solution to this problem was achieved in the early 1950s by [25], [35] and [58]. Hilbert's fifth problem motivated an enormous volume of work on locally compact groups that shed light on the structure of connected locally compact groups. An exposition on the celebrated solution of Hilbert's fifth problem can be found in [46].

Remark By Theorem 10.8, Lie group techniques may be used to analyse the structure of connected groups and their automorphisms. A canonical form for automorphisms of TDLC-groups has been developed in [51, 52]; see § 10.5.1.

10.3 Totally Disconnected Locally Compact Groups

10.3.1 van Dantzig's Theorem

By using arguments from [27], we prove van Dantzig's theorem – together with some consequences – which can be considered as the key theorem in the theory of TDLC-groups: the topology of a TDLC-group admits a well-behaved basis of identity neighbourhoods.

Theorem 10.10 (van Dantzig, 1936) *Let G be a TDLC-group. Then every neighbourhood of the identity contains a compact open subgroup.*

Proof We follow [23, Proof of Theorem 7.4.2.(b)]. By Vedenissov's Theorem (see [3, Theorem B.6.10]), there exists a neighbourhood basis $\mathcal{B}$ at the identity consisting of compact open sets. Let $K \in \mathcal{B}$. For each $x \in K$, there is an open set $U_x \ni 1_G$ with $xU_x \subseteq K$ (because left translation by x is continuous at 1_G) and an open set $V_x \ni 1_G$ with $V_xV_x \subseteq U_x$ (because the multiplication $\mu\colon G\times G \to G$ is continuous at $(1_G, 1_G)$). Since also the inversion map is continuous, the open set V_x can be chosen to be symmetric. For K is compact, there are $x_1,\ldots,x_n$ such that $K \subseteq x_1V_{x_1} \cup \ldots \cup x_nV_{x_n}$. Set $V = V_{x_1} \cap \ldots \cap V_{x_n}$ and notice that

$$KV \subseteq \Big(\bigcup_{i=1}^{n} x_iV_{x_i}\Big)V \subseteq \bigcup_{i=1}^{n} x_iV_{x_i}V_{x_i} \subseteq \bigcup_{i=1}^{n} x_iU_{x_i} \subseteq K.$$

The inclusion $V = 1_GV \subseteq KV \subseteq K$ implies that $VV \subseteq U$, $VVV \subseteq U$, etc. Since V is symmetric, the subgroup H generated by V is given by

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$$H = \bigcup_{n \in \mathbb{N}} \underbrace{V \cdots V}_{n} \subseteq K.$$

By Exercise 10.2, H is open (and so closed) and, for $H \subseteq K$, H is also compact. □

Hence, every TDLC-group contains arbitrarily small (compact) open subgroups. This is the opposite of what occurs in the connected case, where the only open subgroup is the whole group (see Exercise 10.2). This is also the opposite of what occurs in Lie Groups, which admit a neighbourhood of the identity that contains only the trivial subgroup.

10.3.2 Consequences of van Dantzig's Theorem

Here we collect a few long-known consequences of van Dantzig's theorem.

Proposition 10.11 *Given a locally compact group G, the identity component G_0 coincides with the intersection of all open subgroups of G.*

Proof To prove that G_0 is contained in the intersection it suffices to notice that every open subgroup H of G is a clopen set (see Exercise 10.2(3)).

For the reverse inclusion, we show that, for every $x \in G \setminus G_0$, there is an open subgroup H_x not containing x. By Proposition 10.2, the group G/G_0 is TDLC. Therefore, van Dantzig's theorem yields a neighbourhood basis at G_0 given by compact open subgroups of G/G_0. It follows that there is a compact open subgroup $K \subseteq G/G_0$ not containing xG_0. Given the quotient map $\pi_0 : G \to G/G_0$, we set $H_x = \pi_0^{-1}(K)$. □

Corollary 10.12 *If a topological group G admits a neighbourhood basis $\mathcal{B}$ at 1_G consisting of compact open subgroups, then G is TDLC.*

Proof The only part that needs some work is the total disconnectedness of G. By Proposition 10.11, $G_0 = \bigcap\{U \mid U \in \mathcal{B}\}$. But $\bigcap\{U \mid U \in \mathcal{B}\} = 1$ since we assume topological groups to be Hausdorff. □

In other words, van Dantzig's theorem characterizes TDLC-groups among topological (Hausdorff) groups: they are the ones whose compact open subgroups form a neighbourhood basis at 1_G. This property will reveal itself to be the most fruitful property of TDLC-groups.

In general, total disconnectedness is not preserved under taking quotients. Thanks to van Dantzig's theorem this is not the case for locally compact groups.

Proposition 10.13 *The quotient of a TDLC-group by a closed normal subgroup is totally disconnected.*

Proof Let G be a TDLC-group and let N be a closed normal subgroup of G. It follows from van Dantzig's theorem that the collection of all compact open subgroups of G forms a neighbourhood basis at 1_G. Since quotient maps are open, the quotient G/N admits a neighbourhood basis at N formed by compact open subgroups that are the quotients of all compact open subgroups of G. Thus G/N is TDLC by Corollary 10.12. □

Exercise 10.15 Every locally compact group G contains an open subgroup H which is compact-by-connected.

Proposition 10.14 *A compact totally disconnected group is a projective limit of finite groups. In particular, a topological group is profinite if, and only if, it is compact and totally disconnected.*

Proof Let G be a compact totally disconnected group. The set $\mathcal{O}$ of all compact open subgroups of G forms a neighbourhood basis at 1_G. Since G is compact, every subgroup $H \in \mathcal{O}$ contains a subgroup which is both open and normal in G (see Exercise 3(2)). Thus, the family $\mathcal{N}\mathcal{O} = \{H \in \mathcal{O} \mid H \trianglelefteq G\}$ is a neighbourhood basis at 1_G. Therefore, the morphism $G \to \prod_{H \in \mathcal{N}\mathcal{O}} G/H$ of compact groups is injective and continuous, and provides a topological isomorphism from G to a closed subgroup of the latter product of finite groups. □

10.3.3 Examples of TDLC-groups

Many examples of a TDLC-group can be produced by means of van Dantzig's theorem.

- Discrete groups and profinite groups are rather trivial examples.
- (Local fields) Let $\mathbb{K}$ be a local field; see Example 10.13. Then $\mathbb{K}$ has a unique maximal compact subring

  $$\mathfrak{o}_{\mathbb{K}} = \{x \in \mathbb{K} \mid \{x^n \mid n \geqslant 1\} \text{ is relatively compact}\},$$

  which has a unique maximal ideal

  $$\mathfrak{p}_{\mathbb{K}} = \{x \in \mathbb{K} \mid \lim_{n \to \infty} x^n = 0\}.$$

  Both $\mathfrak{o}_{\mathbb{K}}$ and $\mathfrak{p}_{\mathbb{K}}$ are compact and open in $\mathbb{K}$. The ideal $\mathfrak{p}_{\mathbb{K}}$ is principal in $\mathfrak{o}_{\mathbb{K}}$: there is $\pi \in \mathbb{K}$ such that $\mathfrak{p}_{\mathbb{K}} = (\pi)$. The nested sequence of ideals

  $$(\pi) \supset \cdots \supset (\pi^n) \supset (\pi^{n+1}) \supset \cdots$$

  constitutes a basis of compact open subgroups at 0 in $\mathbb{K}$.
  Local fields fall into two families (see [37, § 12.3.4]):

  1 the fields of p-adic numbers, $\mathbb{Q}_p$ and their finite extensions, and

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2 the fields of formal Laurent series, $\mathbb{F}_q((t))$, over some finite field $\mathbb{F}_q$.

Note that $\mathbb{Q}_p$ admits $\{p^n\mathbb{Z}_p\}_{n\in\mathbb{N}}$ as a basis of compact open subgroups, where $\mathbb{Z}_p$ is the ring of p-adic integers, and $\mathbb{F}_q((t))$ has $\{t^n\mathbb{F}_q(t)\}_{n\in\mathbb{N}}$ as basis of compact open subgroups, where $\mathbb{F}_q(t)$ is the ring of formal Taylor series over $\mathbb{F}_q$.

- (Linear groups over local fields) Let $\mathbb{K}$ be a local field. For instance, the group $GL_n(\mathbb{K})$ is TDLC with the topology inherited by $\mathbb{K}^{n^2}$.
- (Lie groups over local fields) An exact analogue of Lie theory exists for analytic groups defined over local fields such as $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$; see [14, § 4] and [8].
- (Automorphism groups of connected locally finite graphs) A **graph** Γ is a pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is a set and $E\Gamma$ is a collection of unordered distinct pairs of elements from $V\Gamma$. The elements of $V\Gamma$ are called **vertices** and the elements of $E\Gamma$ are called **edges**. We will need a bit of terminology for graphs: two vertices v and u are said to be adjacent, if $\{v,u\}$ is an edge in Γ; a graph is **locally finite** if each vertex v has a finite number of adjacent vertices; a **path of length** n from v to u is a sequence $(v=v_0,v_1,\dots,v_n=u)$ of vertices, such that v_i and v_{i+1} are adjacent for $i=0,1,\dots,n-1$; a graph is **connected** if for any two vertices v and u there is a path from v to u in the graph. An **automorphism** of a graph Γ is a bijection $\varphi\colon V\Gamma\to V\Gamma$ such that $\{\varphi(v),\varphi(w)\}\in E\Gamma$ if and only if $\{v,w\}\in E\Gamma$. The collection of automorphisms forms a group under composition, and it is denoted by $\mathrm{Aut}(\Gamma)$. Let Γ be a connected graph and endow $\mathrm{Aut}(\Gamma)$ with the compact-open topology via considering $V\Gamma$ to be a discrete space. Namely, a basis of this topology is given by the sets

$$\Sigma_{v,w}=\{g\in\mathrm{Aut}(\Gamma)\mid g(v_i)=w_i\},$$

where $v=(v_1,\dots,v_n)$ and $w=(w_1,\dots,w_n)$ range over all finite[8] tuples of vertices of Γ. In particular, two automorphisms of Γ are "close" to each other if they agree on "many" vertices.

Exercise 10.16 The compact-open topology is a group topology on $\mathrm{Aut}(\Gamma)$ and it coincides with the pointwise convergence topology.

Remark The compact-open topology on $\mathrm{Aut}(\Gamma)$ also coincides with the **permutation topology** (cf. [30]).

Let $G=\mathrm{Aut}(\Gamma)$. The identity element of G belongs to an open set $\Sigma_{a,b}$ iff $a=b$. Consequently, the compact-open topology on G has a neighbourhood basis at

[8] Notice that the length of the tuples is arbitrary.

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the identity formed by the pointwise stabilisers of finite sets of vertices. Recall that, given a vertex $v \in \Gamma$, the set $G_{(v)} = \{g \in G \mid g(v) = v\}$ is a subgroup of G which is called the **stabiliser of** v. Given a finite set of vertices $\mathcal{V}$, the intersection $G_{(\mathcal{V})} = \bigcap_{v \in \mathcal{V}} G_{(v_i)}$ is the **pointwise stabiliser** of the set $\mathcal{V}$. Clearly, pointwise stabilisers of finite sets of vertices are open subgroups of G.

Proposition 10.15 *Let* Γ *be a connected locally finite graph. The pointwise stabilisers of finite sets of vertices are compact in the compact-open topology. In particular,* $G = \mathrm{Aut}(\Gamma)$ *is a TDLC-group.*

Sketch of the proof. It suffices to prove the claim for the stabiliser $G_{(v)}$ of an arbitrary vertex v. For $k > 0$, set

$$S_k(v) = \{w \in V\Gamma \mid d_\Gamma(v,w) \leqslant k\},$$

where $d_\Gamma(v,w)$ denotes the length of the shortest path from v to w. Since Γ is locally finite, every $S_k(v)$ is finite. Let $Sym(S_k(v))$ be endowed with the discrete topology. Clearly, the stabiliser $G_{(v)}$ permutes the elements in each $S_k(v)$ i.e., there exists a group homomorphism $\varphi_k \colon G_{(v)} \to Sym(S_k(v))$. The family $\{\varphi_k\}_{k>0}$ can then be used to construct an injective group homomorphism from $G_{(v)}$ to the profinite group $\prod_{k>0} Sym(S_k(v))$ which is continuous and closed. □

Remark The latter result does not state the “non-discreteness” of Aut(Γ). Indeed, it is a difficult task to determine whether a given group is discrete.

- (Neretin group of spheromorphisms of a d-regular tree) A **tree** is a connected graph without nontrivial cycles, where by nontrivial cycle we mean a path $(v_0, \dots, v_n)$ such that $n \geqslant 1$ and $v_0 = v_n$. A vertex $v \in VT$ of degree 1 (i.e., has a unique adjacent vertex) is called a **leaf**. For $d \in \mathbb{N}$, an **infinite** d**-regular tree** is an infinite tree whose vertices have degree $d+1$. A **finite** d**-regular tree** is a finite tree whose every vertex is either a leaf or has degree $d+1$, i.e., it is an **internal vertex**. A **rooted tree** is a tree with a distinguished vertex $o \in T$, called its **root**. If the tree is rooted then d-regularity requires the root to have degree d instead of $d+1$.
  An important property of a tree T is given by the fact that there exists a unique path connecting two vertices v and u. A **ray** in T is defined to be an infinite path, i.e., a sequence $(v_0, v_1, \dots)$ of distinct vertices of T such that the consecutive ones are adjacent. Two rays are said to be **asymptotic** if they have common tails.[9] Equivalence classes of asymptotic rays are called

[9] A tail of a sequence is a subsequence obtained after removing finitely many initial elements.

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the **ends** of T. All ends of T form the set ∂T which is called the **boundary** of the tree.

The construction of $\mathcal{N}_d$: Here we follow the construction and the arguments of [14, Chapter 8]. Let T be a d-regular tree. For every finite d-regular subtree $F \subseteq T$, denote by $T \setminus F$ the (no longer connected) graph obtained by removing from T all the edges and internal vertices of F. The connected components of $T \setminus F$ are rooted d-regular trees whose roots are the leaves of F. In particular, $T \setminus F$ is a rooted d-regular forest such that $\partial(T \setminus F) = \partial T$.[10]

Let $F_1, F_2 \subseteq T$ be two finite d-regular subtrees with the same number of leaves. Each forest isomorphism $\varphi\colon T \setminus F_1 \to T \setminus F_2$ induces a homeomorphism φ_* of ∂T, called *spheromorphism* of T. Clearly, different choices of subtrees F_1, F_2 can induce the same spheromorphism. Therefore, φ is just a representative of φ_*. This is important because, for each pair of spheromorphisms φ_* and ψ_*, we find *composable* representatives: we enlarge the finite trees in the representation of the spheromorphisms in order to make the isomorphisms φ and ψ share a target forest and a source forest. This procedure shows that the spheromorphism $\psi_* \circ \varphi_*$ is well defined (in particular, it coincides with the composition in $\mathrm{Homeo}(\partial T)$). Hence, the set of all spheromorphisms of a d-regular tree is a group, which is called a **Neretin group** and denoted by $\mathcal{N}_d$. The group of spheromorphisms of a d-regular tree has been introduced by [36] by analogy with the diffeomorphism group of a circle. Roughly speaking, a spheromorphism of ∂T is a transformation induced in the boundary ∂T by a piecewise tree automorphism; indeed, spheromorphisms are also known as *almost automorphisms of T*.

Fact 10.16 *The set of all spheromorphisms is a subgroup of the homeomorphism group of the boundary ∂T. Moreover, every automorphism φ of T induces an isomorphism $T \setminus F \to T \setminus \varphi(F)$ of forests which is independent on the finite d-regular subtree F. Thus,* $\mathrm{Aut}(T)$ *can be regarded as a subgroup of $\mathcal{N}_d$.*

Remark Let φ_* be a spheromorphism of T_d and suppose that φ_* admits a representative $\varphi\colon T_d \setminus F \to T_d \setminus F$ that leaves the trees of $T_d \setminus F$ in place. Then φ can be extended to an automorphism of the tree T_d which belongs to

[10] Since a forest is a disjoint union of trees the notion of boundary can be easily extended to forests.

the pointwise stabiliser of the finite tree F. As a consequence, φ_* belongs to the image of $\mathrm{Aut}(T_d)$ in $\mathcal{N}_d$.

In order to topologize the group $\mathcal{N}_d$, the first attempt is to endow $\mathrm{Homeo}(\partial T)$ with the compact-open topology and then give $\mathcal{N}_d$ the subspace topology. Unfortunately, the resulting topological group is not locally compact: $\mathcal{N}_d$ is not closed in $\mathrm{Homeo}(\partial T)$ with respect to the compact-open topology (see Proposition 10.7).

Instead of restricting a topology from a larger topological group, we could try to "copy and paste around" a topology coming from an abstract subgroup which is also a topological group.

Lemma 10.17 ([14, Lemma 8.4, p. 137]) *Suppose that an abstract group G contains a topological group H as a subgroup. If, for all open subsets $U \subseteq H$ and $g, g' \in G$, the intersection $gUg' \cap H$ is open in H, then G admits a unique group topology such that the inclusion $H \to G$ is continuous and open.*

Theorem 10.18 *Neretin group $\mathcal{N}_d$ admits a unique group topology such that the natural embedding $\mathrm{Aut}(T_d) \to \mathcal{N}_d$ is continuous and open. With this topology, $\mathcal{N}_d$ is a TDLC-group.*

Proof By the lemma above, one needs to show that for every open $U \subseteq \mathrm{Aut}(T_d)$ and all $\varphi_*, \psi_* \in \mathcal{N}_d$, the subset $\mathrm{Aut}(T_d) \cap \varphi_* U \psi_*$ is open in $\mathrm{Aut}(T_d)$. A sub-basis of the compact-open topology on $\mathrm{Aut}(T_d)$ is given by vertex stabilisers and therefore one has to show the claim only for the sets in the sub-basis.

To this end, let v be a vertex of Γ. Let S be a sufficiently large sphere centred at v such that the spheromorphisms φ_* and ψ_* admit representatives $\varphi\colon T_d \setminus F_1 \to T_d \setminus S$ and $\psi\colon T_d \setminus S \to T_d \setminus F_2$. Denote by $G_{(S)}$ the pointwise stabiliser of S. Since $G_{(S)}$ is an open subgroup of $\mathrm{Aut}(T_d)$ contained in $G_{(v)}$, there exist finitely many elements $g_1, \ldots, g_n \in G_{(v)}$ such that $G_{(v)} = \bigsqcup_{i=1}^n g_i G_{(S)}$. Therefore,

$$\psi_* G_{(v)} \varphi_* = \bigsqcup_{i=1}^{n} \psi_* g_i G_{(S)} \varphi_* = \bigsqcup_{i=1}^{n} \psi_* g_i \varphi_* (\varphi_*^{-1} G_{(S)} \varphi_*).$$

By Remark 10.3.3, $\varphi_*^{-1} G_{(S)} \varphi_*$ coincides with the pointwise stabiliser of the finite tree F_1 and, therefore, it is contained in the image of $\mathrm{Aut}(T_d)$ in $\mathcal{N}_d$. It then follows that $\psi_* G_{(v)} \varphi_* \cap \mathrm{Aut}(T_d)$ is open in $\mathrm{Aut}(T_d)$ (it is the union of translates of the open subgroup $G_{(F_1)}$). □

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- (Fundamental groups of graphs of profinite groups) Bass–Serre theory carries over to the realm of TDLC-groups. Let $(\mathcal{G}, \Lambda)$ be a finite graph of profinite groups; see [17]. The fundamental group $\pi_1(\mathcal{G}, \Lambda)$ can be endowed easily with a TDLC-group topology.
- (Topological semi-direct products) Let G and H be topological groups. Suppose that G acts on H continuously, i.e., there is a group action of G on H such that the map $\alpha\colon G \times H \to H$ defined by the action is continuous. The **topological semi-direct product** is the abstract semi-direct product $H \rtimes G$ endowed with the product topology.

 Exercise 10.17 Let G and H be TDLC-groups such that G acts continuously on H. The topological semi-direct product $H \rtimes G$ is a TDLC-group.

- (Powers of topological groups) Let G be a locally compact group. For I infinite, the power G^I fails to be locally compact as soon as G is non-compact. To deal with this issue, given a compact open subgroup U in G, one defines the **semi-restricted power**

$$G^{I,U} = \{(g_i)_{i \in I} \in G^I \mid g_i \in U \text{ for all but finitely many } i \in I\}.$$

 There is a unique group topology on $G^{I,U}$ that makes the embedding of U^I a topological isomorphism onto an open subgroup. Moreover, such a group topology is locally compact; see [20, Proposition 2.4]. TDLC-groups are full of compact open subgroups, and therefore they are amenable to this construction; in particular, semi-restricted powers of TDLC-groups are again TDLC-groups.

10.4 Finiteness Properties for TDLC-groups

10.4.1 Compact Generation and Presentation

There are several finiteness conditions that a TDLC-group can satisfy. At this early stage, we are interested in two finiteness conditions that naturally generalise the notions of finite generation and finite presentation in the context of locally compact groups.

Definition 10.18 A locally compact group G is said to be

(CG) **compactly generated** if it has a compact generating set S.
(CP) **compactly presented** if it has a presentation $\langle S \mid R\rangle$ as an abstract group with the generating set S compact in G and the relators in R of bounded length.

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It is straightforward that (CP) implies (CG). The converse is not true; see for instance [21, Example 8.A.28]. The notion of compact presentation was introduced in 1964 by Kneser but it has attracted much less attention than compact generation until recently: for example, [21] proved that compact presentation is equivalent to a metric condition and [17] provided an equivalent notion of compact presentation via generalised presentations which is based on van Dantzig's theorem and the notion of a fundamental group of finite graphs of profinite groups.[11]

Example 10.19 1 Every profinite group is (trivially) compactly generated and compactly presented.

2 Every compactly generated abelian TDLC-group is topologically isomorphic to $\mathbb{Z}^n \times K$, where $n \in \mathbb{N}$ and K is a compact abelian group; see [23, Theorem 12.5.5]. In particular, it is compactly presented.

3 The field of p-adic numbers $\mathbb{Q}_p$ is not compactly generated because it is the ascending union of nested compact open subgroups, i.e., $\mathbb{Q}_p = \bigcup_{n\in\mathbb{Z}} p^n\mathbb{Z}_p$.

4 The automorphism group of a d-regular tree is compactly generated; see Corollary 10.21.

5 The special linear group $\mathrm{SL}_2(\mathbb{Q}_p)$ is compactly generated. Indeed, by Ihara's Theorem [43, p. 143, Corollary 1 to Theorem 3], we can decompose $\mathrm{SL}_2(\mathbb{Q}_p)$ into the amalgamated free product

$$\mathrm{SL}_2(\mathbb{Z}_p) *_I \mathrm{SL}_2(\mathbb{Z}_p),$$

where I is a compact open subgroup.

6 More generally, the fundamental group of a finite graph of profinite groups satisfies (CP); see [17, § 5.8].

7 The Neretin groups are compactly presented; see [12] and [32].

Fact 10.19 *Every locally compact group is a directed union of compactly generated open subgroups.*

Proof It suffices to notice that, for any $g \in G$ and any compact open neighbourhood V_g of g, the subgroup $\bigcup_{n>0}(V_g \cup V_g^{-1})^n$ is open in G and compactly generated. □

10.4.2 The Cayley–Abels Graphs

Recall that a **group** G **acts on a graph** Γ if the set of vertices $V\Gamma$ is a G-set and, for every $g \in G$, $\{gv, gw\} \in E\Gamma$ if and only if $\{v, w\} \in E\Gamma$. The group G

[11] Which plays a role in the geometric group theory of TDLC-groups that can be compared with the role played by free groups in the discrete case.

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acts **vertex-transitively** on Γ if $V\Gamma$ is a transitive G-set. Given a vertex $v \in V\Gamma$, the set $G_{(v)} = \{g \in G \mid gv = v\}$ is the **vertex stabiliser** of v in G.

Definition 10.20 For a TDLC-group G, a locally finite connected graph Γ on which G acts vertex-transitively with compact open vertex stabilisers, is called a **Cayley–Abels graph of** G.

Exercise 10.21 Let Γ be a Cayley–Abels graph of G. Let $V\Gamma$ be endowed with the discrete topology. Prove that the map $G \times V\Gamma \to V\Gamma$ is continuous; that is, a TDLC-group G always acts continuously on its Cayley–Abels graphs. Moreover, prove that, as far as G is non-discrete, the action of G on Γ is never free.[12]

Proposition 10.20 *Let G be a TDLC-group. If G has a Cayley–Abels graph, then G is compactly generated.*

Proof Let Γ be a Cayley–Abels graph of G and $v \in V\Gamma$. Since Γ is locally finite, one has $\mathrm{star}(v) = \{v_1, \ldots, v_n\}$. Since G acts on Γ vertex-transitively, for every $i = 1, \ldots, n$, there is a $g_i \in G$ such that $v_i = g_i v$. We claim that, for every $g \in G$, there is an $h \in \langle g_1, \ldots, g_n\rangle$ such that $gv = hv$. This implies that $h^{-1}g \in G_{(v)}$, which is compact and open by hypothesis. In other words, $G = \langle G_{(v)}, g_1, \ldots, g_n\rangle$.

Let us prove the claim: for every $g \in G$ there is a path in Γ connecting v and gv because Γ is connected. We proceed by induction on the length of the path. For $k = 0$ there is nothing to prove. Suppose the hypothesis for k and prove it for $k+1$. For Γ is vertex-transitive, a path of length $k+1$ connecting v and gv is given by a $(k+1)$-tuple $(v, \gamma_1 v, \ldots, \gamma_k v, gv)$ with $\gamma_1, \ldots, \gamma_k \in G$. By the inductive hypothesis, there is $h \in \langle g_1, \ldots, g_n\rangle$ such that $\gamma_k v = hv$. Therefore, the group element h^{-1} maps the edge $\{\gamma_k v, gv\}$ to the edge $\{v, h^{-1}gv\}$. In other words, $h^{-1}gv$ is adjacent to v, i.e., $h^{-1}gv = v_j = g_j v$ for some $j \in \{1, \ldots, n\}$ and the claim holds. ☐

Corollary 10.21 *$Aut(\mathcal{T}_d)$ is compactly generated for every $d \in \mathbb{N}$.*

Now, we do the converse: we start with a compactly generated TDLC-group G and we construct a (family of) Cayley–Abels graph(s) of G. In particular, we show that, for every compact open subgroup U of G, there is a Cayley–Abels graph admitting U as stabiliser of some vertex.

Let U be a compact open subgroup of G. For every symmetric subset $S = S^{-1} \subseteq G \setminus U$ define the graph $\Gamma^G_{U,S}$ by setting

$$V\Gamma^G_{U,S} = \{gU \mid g \in G\}, \quad \text{and} \quad E\Gamma^G_{U,S} = \{\{gU, gsU\} \mid g \in G, s \in S\}.$$

Clearly, G acts transitively on the set of vertices of $\Gamma^G_{U,S}$.

[12] A groups acts *freely* on a set if point stabilisers are trivial.

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Proposition 10.22 *With the above notation, the following hold:*

1 if S is a finite set, then $\Gamma^G_{U,S}$ is locally finite;
2 $\Gamma^G_{U,S}$ is connected if, and only if, $G = \langle S \cup U \rangle$.

If G is compactly generated, there exists a Cayley–Abels graph of G.

Proof 1. Since the action is vertex-transitive, it suffices to prove that the vertex U has finitely many neighbours if S is finite. Since $U = xU$, for every $x \in U$, one has that the set $\{xU, xsU\}$ is an edge for every $s \in S$, and therefore the set of all neighbours of U coincides with $\{xsU \mid x \in U, s \in S\}$. In order to determine the cardinality of such a set, one must count the number of left cosets of U that are necessary to cover each double coset UsU. But this number is finite because U is open and the double coset UsU is compact (since U is compact). Therefore, if S is finite, the graph $\Gamma^G_{U,S}$ is locally finite.

2. Suppose the graph $\Gamma^G_{U,S}$ is connected. For every $g \in G$, there is a path $p = (v_0, \ldots, v_n)$ connecting the vertex U to the vertex gU. In particular, the vertices of the path p can be written as

$$v_0 = U, \quad v_1 = u_1 s_1 U, \quad \ldots, \quad v_n = u_1 s_1 \cdots u_n s_n U,$$

where each $u_i \in U$ and each $s_j \in S$. Since $u_1 s_1 \cdots u_n s_n U = gU$, it follows that g belongs to the subgroup generated by $U \cup S$.

Conversely, suppose that $G = \langle U \cup S \rangle$. Let gU and hU be any two vertices of the graph. The group element $g^{-1}h$ can then be written as a word $u_1 s_1 \cdots u_n s_n u_{n+1}$ such that each $u_i \in U$ and each $s_j \in S$. Thus, the sequence of vertices

$$(gU, gu_1 s_1 U, gu_1 s_1 u_2 s_2 U, \cdots, gu_1 s_1 \cdots u_n s_n u_{n+1} U = hU)$$

is a path in $\Gamma^G_{U,S}$ connecting gU and hU. □

Remark The first (technical) construction of the Cayley–Abels graph is due to [1]. A less technical approach to Cayley–Abels graphs was provided in [31], where the Cayley–Abels graphs were at the time called *rough Cayley graphs*. Today the widely accepted nomenclature is "Cayley–Abels graph".

Proposition 10.23 *Let G be a compactly generated TDLC-group with Cayley–Abels graph Γ. The group homomorphism $\psi\colon G \to \mathrm{Aut}(\Gamma)$ defined by the action of G on Γ is continuous, the kernel of ψ is compact and the image of ψ is closed.*

Proof A basis of the compact-open topology of $\mathrm{Aut}(\Gamma)$ is given by the family of pointwise stabilisers in $\mathrm{Aut}(\Gamma)$ of finite sets of vertices. The pre-image of each of these sets is the intersection of finitely many open subgroups of G (that

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are the stabilisers $G_{(v)}$ of the vertices v in the finite set). Since each stabiliser $G_{(v)}$ is open, ψ is continuous.

The kernel of ψ is closed because it is the pre-image of the closed set $\{1\}$ and then it is compact since $\ker(\psi) \subseteq G_{(v)}$, for any $v \in V\Gamma$.

To prove that the image $\psi(G)$ is closed, it suffices to see that $\psi(G) \cap H$ is closed for every H in the basis of the compact-open topology of $\mathrm{Aut}(\Gamma)$. Since every such H is the intersection of finitely many vertex stabilisers in $\mathrm{Aut}(\Gamma)$, we only need to prove that $\psi(G) \cap H$ is closed whenever H is the stabiliser in $\mathrm{Aut}(\Gamma)$ of a single vertex v. In such a case, $\psi(G) \cap H$ coincides with $\psi(G_{(v)})$, which is compact because ψ is continuous and $G_{(v)}$ is compact. In particular, $\psi(G) \cap H$ is closed. □

Corollary 10.24 *Compactly generated TDLC-groups are second countable modulo a compact normal subgroup.*

The representation $\psi\colon G \to \mathrm{Aut}(\Gamma)$ is called the **Cayley–Abels representation of** G.

Remark One can say more on the image $\psi(G)$ of the Cayley–Abels representation: $\psi(G)$ is a cocompact subgroup of $\mathrm{Aut}(\Gamma)$, see [50, Lemma 3.12].

Exercise 10.22 Suppose G is a compactly generated TDLC-group and Γ is a Cayley–Abels graph of G. Show that a closed subgroup $K \leqslant G$ is compact if and only if, for all $v \in V\Gamma$, $Kv := \{kv \mid k \in K\}$ is finite.

The geometric structure of compactly generated TDLC-groups:

Definition 10.23 ([26]) Two metric spaces (X, d_X) and (Y, d_Y) are said to be **quasi-isometric** if there is a map $\varphi\colon X \to Y$ and constants $a \geqslant 1$ and $b \geqslant 0$ such that, for all $x_1, x_2 \in X$, one has

$$\frac{1}{a} d_X(x_1, x_2) - \frac{b}{a} \leqslant d_Y(\varphi(x_1), \varphi(x_2)) \leqslant a d_X(x_1, x_2) + ab,$$

and, for all $y \in Y$,

$$d_Y(y, \varphi(X)) \leqslant b.$$

Such a map φ is called a **quasi-isometry**. Moreover, being quasi-isometric is an equivalence relation on the class of metric spaces.

Every connected graph Γ can be regarded as a metric space: two vertices v and w are points at distance 1 if, and only if, there is an edge connecting v and w. In other words, we endow the set $V\Gamma$ with the path-length metric $d_\Gamma\colon V\Gamma \times V\Gamma \to \mathbb{N}$ defined as follows:

$$d_\Gamma(v, w) = \min\{\text{length of } p \mid p \text{ path connecting } v \text{ and } w\}, \quad v, w \in V\Gamma.$$

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Theorem 10.25 ([1], [31, Theorem 2]) *Let G be a compactly generated TDLC-group. Any two Cayley–Abels graphs of G are quasi-isometric.*

The quasi-isometric invariance allows us to define *geometric invariants* of a compactly generated TDLC-group G by considering quasi-isometric invariants of a Cayley–Abels graph associated to G. For example, one can give the following definitions (that are long-known for finitely generated discrete groups) for a compactly generated TDLC-group G:

(Hyp) G is said to be **hyperbolic** if some (and hence any) Cayley–Abels graph of G is hyperbolic.

(Ends) The **number of ends** of G is defined to be the number of ends of some (and hence any) Cayley–Abels graph of G.

The class of hyperbolic TDLC-groups is a rich source of compactly presented TDLC-groups; see [21]. Indeed, geometric invariants often reflect structural properties of the group: for instance, an analogue of the famous Stallings decomposition theorem is available in the context of TDLC-groups; see [1, Struktursatz 5.7 and Korollar 5.8], [31, Theorem 1.3] and [15].

10.4.3 Finiteness Conditions in Higher Dimension

For discrete groups, finite generation is the first in two sequences of increasingly stronger properties: the homological finiteness conditions, the **types** $(\mathbf{FP}_n)_{n\in\mathbb{N}}$ over a commutative unital ring R and the homotopical finiteness conditions, the **types** $(\mathbf{F}_n)_{n\in\mathbb{N}}$. We recall the definitions here, but the reader is referred to [11, Chapter VIII] for details:

(FP$_n$) A discrete group G is **of type FP$_n$** $(0 \leqslant n < \infty)$ over R if there is a projective resolution $\{P_i\}$ of the trivial module R over $R[G]$ such that each projective $R[G]$-module P_i is finitely generated for $i \leqslant n$. (If $R = \mathbb{Z}$, the reference to the ring R usually drops.)

(F$_n$) A discrete group G is **of type F$_n$** $(0 \leqslant n < \infty)$ if there exists a contractible G-CW-complex with finite cell stabilisers and such that G acts on the n-skeleta with finitely many orbits. See [33, Lemma 4.1].

We will refer to the properties above as the classical finiteness properties for discrete groups. These properties are known to satisfy the following:

– A discrete group G is of type F_1 if, and only if, it is finitely generated if, and only if, it is of type FP_1 over R.
– A discrete group G is of type F_2 if, and only if, it is finitely presented but being of type FP_2 is strictly weaker than finite presentation; see [7].

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- For each $n \geqslant 1$, a discrete group of type F_n is of type FP_n over R but the converse is not true (the converse becomes true if the group is supposed to be finitely presented and $R = \mathbb{Z}$).
- Being of type FP_n over R (resp. of type F_n) is a geometric property of the finitely generated group, that is, it is invariant up to quasi-isometry.

A first attempt at generalising these properties to the realm of locally compact groups is due to Abels and Tiemeyer [3]. They introduced **compactness properties** for locally compact groups – we avoid here the (very technical) definitions – which are two sequences $(\mathbf{CP}_n)_{n\geqslant 0}$ and $(\mathbf{C}_n)_{n\geqslant 0}$ of increasingly stronger properties satisfying:

- for all $n \geqslant 1$, a discrete group is of type (CP_n) (resp. C_n) if and only if it is of type FP_n (resp. F_n);
- a locally compact group is of type C_1 if, and only if, it is compactly generated if, and only if, it is of type CP_1;
- a locally compact group is of type C_2 if, and only if, it is compactly presented but being of type CP_2 is strictly weaker than compact presentation;
- for each $n \geqslant 1$, a locally compact group of type C_n is also of type CP_n but the converse is not true;
- being of type CP_n (resp. C_n) is invariant "up to compactness": the compactness properties remain unchanged by passing to a cocompact[13] subgroup or by taking the quotient by a compact normal subgroup. Such an invariance is weaker than the invariance up to quasi-isometry among the class of compactly generated locally compact groups.

For the (more amenable) class of TDLC-groups, a different approach to finiteness conditions was recently introduced and investigated in [17] and [16]:

<u>(FP_n)</u> A TDLC-group G is **of type $\mathbf{FP}_n$** ($0 \leqslant n < \infty$) over R if there is a resolution $\{P_i\}$ of the trivial module R over $R[G]$ such that each P_i is a permutation $R[G]$-module[14] with compact open stabilisers and finitely many orbits for $i \leqslant n$.

<u>(F_n)</u> A TDLC-group G is **of type $\mathbf{F}_n$** ($0 \leqslant n < \infty$) if there exists a contractible G-CW-complex X with compact open cell stabilisers such that G acts on the n-skeleta of X with finitely many orbits.

[13] A closed subgroup H is *cocompact* if the quotient G/H, equivalently $H\backslash G$, equipped with the quotient topology is compact.
[14] A *permutation $R[G]$-module* is a module $R[\Omega]$ freely R-generated by a G-set Ω.

These finiteness conditions for TDLC-groups satisfy the following:

– For $R = \mathbb{Q}$, permutation $\mathbb{Q}[G]$-modules with compact open stabilisers are projective objects in the category ${}_{\mathbb{Q}[G]}\mathbf{dis}$ of rational discrete $\mathbb{Q}[G]$-modules[15] (see [17]) and so we recover the homological flavour of the classical definition. In particular, whenever G is discrete, the new definition reduces to the classical notion of type FP_n over $\mathbb{Q}$.
– For all $n \geqslant 1$, a discrete group is of type F_n in the category of TDLC-groups if, and only if, it is of type F_n in the classical sense because requiring compact open discrete stabilisers reduces to finite stabilisers.
– A TDLC-group is of type F_1 if, and only if, it is compactly generated if, and only if, it is of type FP_1 over R (see [16, Proposition 3.4] and [17, Proposition 5.3]).
– A TDLC-group is of type F_2 if and only if it is compactly presented by [16, Proposition 3.4] but being of type FP_2 is strictly weaker than compact presentation.
– For each $n \geqslant 1$, a TDLC-group of type F_n is also of type FP_n over R (see [16, Fact 2.7]) but the converse is not true (the converse becomes true if the group is supposed to be compactly presented and R is replaced by $\mathbb{Z}$).
– Being of type FP_n over R (resp. F_n) is a geometric property (see [16, Corollary 5.7]).

Remark All the finiteness conditions above can be extended to $n = \infty$. [42] showed that Neretin groups are of type F_∞ which, in particular, implies the fact – observed already in [17] – that these groups are of type FP_∞ over $\mathbb{Q}$.

Example 10.24 Hyperbolic TDLC-groups are of type F_n for some finite n. It is a classical result that, for a large enough constant d, the (topological realisation of the) Rips complex $P_d(\Gamma)$ is contractible whenever Γ is a hyperbolic (locally finite) graph.

[16] showed that it is possible to introduce further two sequences $(\textbf{types } \mathbf{KP}_n)_{n\geqslant 0}$ and $(\textbf{types } \mathbf{K}_n)_{n\geqslant 0}$ of increasingly stronger compactness properties. The concrete motivation comes from the fact that permutation $R[G]$-modules with compact open stabilisers fail to be projective over the ring $R = \mathbb{Z}$ in the abelian category ${}_{\mathbb{Z}[G]}\mathrm{dis}$. In such a case, being of type FP_n lack of a description based on the existence of partial projective resolutions of the trivial module $\mathbb{Z}$ of finite type. The strategy to get back such a description is to embed the category ${}_{\mathbb{Z}[G]}\mathrm{dis}$ into a quasi-abelian category ${}_{\mathbb{Z}[G]}\mathrm{top}$ where the discrete

[15] An $R[G]$-module M is said to be *discrete* if the action $G \times M \to M$ is continuous when M carries the discrete topology.

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permutation $\mathbb{Z}[G]$-modules with compact stabilisers go back to being projective again. The objects of ${}_{\mathbb{Z}[G]}\mathrm{top}$ are the so-called *k-$\mathbb{Z}[G]$-modules*, i.e., module objects in the category of k-spaces over the k-$\mathbb{Z}$-algebra $\mathbb{Z}[G]$. The reader is referred to [22] for the definition and the background on this category. Since all locally compact Hausdorff spaces are k-spaces, every TDLC group is automatically a k-group and one can take advantage of the homological machinery developed in [22] to define the following finiteness conditions:

(KP_n) A TDLC-group G is **of type $\mathbf{KP}_n$** ($0 \leqslant n < \infty$) if there is a projective resolution $\{P_i\}$ of the trivial module $\mathbb{Z}$ in ${}_{\mathbb{Z}[G]}\mathrm{top}$ such that each P_i is a free k-$\mathbb{Z}[G]$-module on a compact space for $i \leqslant n$.

(K_n) A TDLC-group G is **of type $\mathbf{K}_n$** ($0 \leqslant n < \infty$) if there exists a contractible G-KW-complex[16] X with cocompact n-skeleta, in the compact Hausdorff model structure on G-k-spaces.

Remark In [16] the sequence $(\text{types } \mathrm{KP}_n)_{n \geqslant 0}$ is defined over an arbitrary ring R.

The latter properties relate to types FP_n and types F_n as follows.

- A TDLC group G is of type FP_n over $\mathbb{Z}$ if, and only if, it is of type KP_n (see [16, Theorem 3.10]).
- If G has type F_n then it has type K_n (see [16, Theorem 3.23]).

Open Problem Despite the abundance of finiteness properties that are available in the TDLC context, the theory of finiteness conditions for TDLC-groups is still much less developed than the one for discrete groups. Moreover, very little is known about the relation (if one exists) among the properties of different sequences $\mathrm{CP}_n, \mathrm{FP}_n$ and KP_n (resp. $\mathrm{C}_n, \mathrm{F}_n$ and K_n). For example, is it true that type K_n implies type F_n?

Open Problem It would be relevant to find an example of a TDLC-group of type FP_2 over $\mathbb{Q}$ which is not compactly presented and is "sufficiently" non-discrete (for example, it is not quasi-isometric to a discrete group). Unfortunately, the strategy developed in [7] does not seem to have a TDLC-analogue.

Open Problem For a TDLC-group G one can introduce several homological invariants. Which homological invariants are geometric? In 1991 Gromov asked whether the cohomological dimension of discrete groups is a geometric invariant. Under additional finiteness assumptions, [24] proved that quasi-isometric groups have the same cohomological dimension and [41] removed

[16] The definition of such an object can be found in [22].

the finiteness assumptions in the case of rational cohomological dimension. This yields the natural question: is the rational discrete cohomological dimension defined in [17] a geometric invariant of a TDLC-group?

10.5 Willis' Theory of TDLC-groups: a Sketch

As mentioned at the end of the introduction, Willis' theory of the scale function was a fundamental breakthrough in the theory of TDLC-groups after several years of stillness and it motivated a vast research interest in the theory of TDLC-groups, which led to a systematic study of the class of compactly generated, topologically simple TDLC-groups that are non-discrete. In this section we confine ourselves to a brief introduction of the scale function and to some comments on (topologically) simple TDLC-groups. Nevertheless, the student that is approaching the study of TDLC-groups is encouraged to dig further into this fundamental subject.

10.5.1 Scale Function and Tidy Subgroups

Let $\alpha \in \mathrm{Aut}(G)$ and U be a compact open subgroup of G. Then

$$[\alpha(U) \colon U \cap \alpha(U)] < \infty$$

because $U \cap \alpha(U)$ is open in the compact set U. The **scale of** α is

$$s(\alpha) = \inf\{[\alpha(U) \colon U \cap \alpha(U)] \mid U \text{ compact open subgroup of } G\}.$$

A subgroup U is **tidy** for α if the infimum is attained at U. Tidy subgroups for α always exists since $s(\alpha)$ is the minimum of a subset of $\mathbb{N}$. Every tidy subgroup U can be decomposed as the product of a subgroup where α expands and a subgroup where α shrinks:

$$\text{if } U_{\pm} \colon = \textstyle\bigcap_{k>0} \alpha^{\pm k}(U), \text{ then } U = U_+ U_-.$$

By construction, U_+ and U_- are closed subgroups, $\alpha(U_+) \geqslant U_+$ and, similarly, $\alpha(U_-) \leqslant U_-$. Moreover, it can be shown that $s(\alpha)$ represents the factor by which α expands U_+, i.e., $s(\alpha) = [\alpha(U_+) \colon U_+]$.

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A striking result in the theory of the scale is the existence of an algorithm, the so-called **tidying procedure**, which produces a tidy subgroup when the input is an arbitrary compact open subgroup.

Definition 10.25 The **scale function** of G is defined as the map

$$s\colon G \to \mathbb{Z}^+, \quad x \mapsto s(\alpha_x), \quad x \in G,$$

where α_x denotes the inner automorphism $y \mapsto xyx^{-1}$.

The scale function s is known to satisfy the following properties (see [51, 52]):

(s1) s is continuous if $\mathbb{Z}^+$ carries the discrete topology;
(s2) $s(x) = 1 = s(x^{-1})$ if and only if there is a compact open subgroup U such that $xUx^{-1} = U$;
(s3) $s(x^n) = s(x)^n$, for every $x \in G$ and $n \geqslant 0$;
(s4) $\Delta(x) = s(x)/s(x^{-1})$, where $\Delta\colon G \to \mathbb{Q}^+$ is the modular function;
(s5) $s(\alpha(x)) = s(x)$ for every $x \in G$ and $\alpha \in \mathrm{Aut}(G)$.

The scale function encodes structural information of the group G. For a summary on the scale function (which in particular highlights the fact that tidy subgroups for automorphisms of TDLC-groups are analogues of the Jordan canonical form of linear transformations) the reader is referred to [55] and references there.

Remark In recent years, Willis' theory has been investigated from different points of view bringing new approaches to the scale function of a TDLC-group. For example, [34] offers an interpretation of the fundamental ingredients of Willis' theory (that are tidy subgroups and scale function) in the setting of permutation group theory. Another example is given by the work initiated in [6], where Willis' topological dynamics of automorphisms has been reformulated in terms of the long-known theory of topological entropy.

10.5.2 Comments on Simple TDLC-groups

Simple groups play an important role in group theory as the "indecomposable factors". Indeed several types of simple groups have been completely classified; for instance, the simple finite groups and the simple connected Lie groups. Long-known classes of simple TDLC-groups are the class of simple Lie groups over local fields (see [10]) and a class of automorphism groups of trees (see [47]).

In the realm of simple TDLC-groups, a distinction between topological simplicity (i.e., every closed normal subgroup is trivial) and abstract simplicity (i.e., the underlying abstract group is simple) is necessary. Examples show that a topologically simple TDLC-group can fail to be abstractly simple, see [54], but no example is known of a topologically simple compactly generated TDLC-group that fails to be abstractly simple. Among compactly generated TDLC-groups, [44] showed that there are $2^{\aleph_0}$ non-isomorphic compactly generated abstractly simple TDLC-groups.

Scale function: The theory of the scale produces invariants that could be important tools in the classification. For example:

- The set of values of the scale function: if G is compactly generated, the set of prime divisors of the values of the scale is finite (see [53]); this set could distinguish between compactly generated simple TDLC-groups.
- The flat-rank: a notion of rank for TDLC-groups which is defined via a notion of distance on the space of compact open subgroups (see [5]).

Remark In some cases, the flat-rank can be related to a cohomological invariant as shown in [17].

Local-to-Global principle: Relates the global properties of a compactly generated topologically simple TDLC-group with the structural properties of its compact open subgroups. This approach was initiated by

- [54] who showed that simplicity affects the local structure of the compactly generated TDLC-group: if G is compactly generated and topologically simple, then no compact open subgroup is solvable;
- [4] who addressed the question of which profinite groups can occur as compact open subgroups of compactly generated topologically simple TDLC-groups.

Decomposition theory: Includes methods for "breaking" a given TDLC-group into smaller (simple) pieces; see [13]. This approach has been successful for several classes of groups; for example, finite groups, profinite groups and algebraic groups. Therefore, one would hope to obtain analogous results for TDLC-groups.

General decomposition results have been obtained by [38, 39] who made use of the theory of *elementary groups* introduced by [49]. Elementary groups are TDLC-groups that are built out of discrete and compact pieces.

Geometrisation: Cayley–Abels graphs allow the study of compactly generated TDLC-groups from a geometric perspective. Often a TDLC-group admits other types of geometric objects, e.g., *buildings*, that one can consider. For example, semisimple Lie groups over a local field act on *affine buildings*, and also on *spherical buildings* (e.g., the affine building of $SL_2(\mathbb{Q}_p)$ is a tree and the spherical building is its boundary). Buildings are able to determine some properties of the group they are associated to. For instance, in some cases, they facilitate the computation of the Euler characteristic, which turns out to have a curious connection with the value of a zeta-function in -1 (see [19]).

Since profinite groups are trivial as geometric objects, the geometric behaviour of compactly generated TDLC-groups is often related to the geometric behaviour of discrete groups. There is an ongoing programme of studying geometric properties of TDLC-groups by analogy with discrete groups. The aim is to understand to what extent long-known results on discrete groups find an analogue in the framework of TDLC-groups; see, for example, [2, 18, 21, 31].

Bibliography

[1] Abels, H. 1973/74. Specker-Kompaktifizierungen von lokal kompakten topologischen Gruppen. *Math. Z.*, **135**, 325–361.

[2] Arora, S., Castellano, I., Corob Cook, G. and Martínez-Pedroza, E. 2021. Subgroups, hyperbolicity and cohomological dimension for totally disconnected locally compact groups. *Journal of Topology and Analysis*, 1–27.

[3] Außenhofer, L., Dikranjan, D. and Giordano Bruno, A. 2021. *Topological Groups and the Pontryagin-van Kampen Duality: An Introduction*. Vol. 83. Walter de Gruyter GmbH & Co KG.

[4] Barnea, Y., Ershov, M. and Weigel, Th. 2011. Abstract commensurators of profinite groups. *Trans. Amer. Math. Soc.*, 5381–5417.

[5] Baumgartner, U., Rémy, B. and Willis, G. A. 2007. Flat rank of automorphism groups of buildings. *Transformation Groups*, **12**(3), 413–436.

[6] Berlai, F., Dikranjan, D. and Giordano Bruno, A. 2013. Scale function vs topological entropy. *Topology Appl.*, 2314–2334.

[7] Bestvina, M. and Brady, N. 1997. Morse theory and finiteness properties of groups. *Invent. Math.*, 445–470.

[8] Bode, A. and Dupré, N. 2021. Locally analytic representations of p-adic groups. www.icms.org.uk/seminars/2020/lms-autumn-algebra-school-2020

[9] Bourbaki, N. 1998. *General topology. Chapters 5–10*. Springer-Verlag, Berlin.

[10] Bourbaki, N. 2002. *Lie groups and Lie algebras. Chapters 4–6*. Springer-Verlag, Berlin.

[11] Brown, K. S. 1994. *Cohomology of groups*. Springer-Verlag, New York.

[12] Caprace, P.-E. and De Medts, T. 2011. Simple locally compact groups acting on trees and their germs of automorphisms. *Transform. Groups*, 375–411.

[13] Caprace, P.-E. and Monod, N. 2011. Decomposing locally compact groups into simple pieces. *Math. Proc. Cambridge Philos. Soc.*, 97–128.

[14] Caprace, P.-E. and Monod, N. 2018. Future directions in locally compact groups: a tentative problem list. In: *New directions in locally compact groups*. Cambridge Univ. Press, Cambridge.

[15] Castellano, I. 2020. Rational discrete first degree cohomology for totally disconnected locally compact groups. *Math. Proc. Cambridge Philos. Soc.*, 361–377.

[16] Castellano, I. and Corob Cook, G. 2020. Finiteness properties of totally disconnected locally compact groups. *J. Algebra*, 54–97.

[17] Castellano, I. and Weigel, Th. 2016. Rational discrete cohomology for totally disconnected locally compact groups. *J. Algebra*, 101–159.

[18] Castellano, I., Marchionna, B. and Weigel, Th. 2022. A Stallings-Swan theorem for unimodular totally disconnected locally compact groups. *arXiv preprint arXiv:2201.10847*.

[19] Castellano, I., Chinello, G. and Weigel, Th. In preparation. The Hattori-Stallings rank, the Euler characteristic and the ζ-functions of a totally disconnected locally compact group.

[20] Cornulier, Y. 2019. Locally compact wreath products. *Journal of the Australian Mathematical Society*, **107**(1), 26–52.

[21] Cornulier, Y. and de la Harpe, P. 2016. *Metric geometry of locally compact groups*. EMS Tracts in Mathematics, vol. 25. European Mathematical Society (EMS), Zürich. Winner of the 2016 EMS Monograph Award.

[22] Corob Cook, G. 2017. Homotopical algebra in categories with enough projectives. *arXiv preprint arXiv:1703.00569*.

[23] Dikranjan, D. 2018. Introduction to topological groups. http://users.dimi.uniud.it/~dikran.dikranjan/ITG.pdf.

[24] Gersten, S. 1993. Quasi-isometry invariance of cohomological dimension.

[25] Gleason, A. M. 1952. Groups without small subgroups. *Annals of mathematics*, 193–212.

[26] Gromov, M. 1984. Infinite groups as geometric objects. Pages 385–392 of: *Proceedings of the International Congress of Mathematicians*, vol. 1.

[27] Hewitt, E. and Ross, K. A. 2012. *Abstract Harmonic Analysis: Volume I Structure of Topological Groups Integration Theory Group Representations*. Springer Science and Business Media.

[28] Hofmann, K. H. 1980. A note on Baire spaces and continuous lattices. *Bulletin of the Australian Mathematical Society*, **21**(2), 265–279.

[29] Hofmann, K. H. and Morris, S. A. 2007. *The Lie theory of connected pro-Lie groups: a structure theory for pro-Lie algebras, pro-Lie groups, and connected locally compact groups*. Vol. 2. European Mathematical Society.

[30] Karrass, A. and Solitar, D. 1956. Some remarks on the infinite symmetric groups. *Math. Z.*

[31] Krön, B. and Möller, R. G. 2008. Analogues of Cayley graphs for topological groups. *Math. Z.*, 637–675.

[32] Le Boudec, A. 2017. Compact presentability of tree almost automorphism groups. *Annales de l'Institut Fourier*.

[33] Luc̆k, Wolfgang. 2000. The type of the classifying space for a family of subgroups. *Journal of Pure and Applied Algebra*, **149**(2), 177–203. Elsevier.

[34] Möller, R. G. 2002. Structure Theory of Totally Disconnected Locally Compact Groups via Graphs and Permutations. *Canadian Journal of Mathematics*.

[35] Montgomery, D. and Zippin, L. 1952. Small subgroups of finite-dimensional groups. *Proceedings of the National Academy of Sciences of the United States of America*, **38**(5), 440.

[36] Neretin, Y. A. 1992. On combinatorial analogs of the group of diffeomorphisms of the circle. *Izvestiya Rossiiskoi Akademii Nauk.*

[37] Palmer, T. W. 1994. *Banach Algebras and the General Theory of *-Algebras: Volume 2,*-Algebras*. Cambridge University Press.

[38] Reid, C. D. and Wesolek, P. R. 2018a. Dense normal subgroups and chief factors in locally compact groups. *Proceedings of the London Mathematical Society*.

[39] Reid, C. D. and Wesolek, P. R. 2018b. The essentially chief series of a compactly generated locally compact group. *Mathematische Annalen*.

[40] Ribes, L. and Zalesskii, P. 2000. Profinite groups. Pages 19–77 of: *Profinite Groups*. Springer.

[41] Sauer, R. 2003. Homological Invariants and Quasi-Isometry. *Geometric & Functional Analysis GAFA*, **16**, 476–515.

[42] Sauer, R. and Thumann, W. 2017. Topological models of finite type for tree almost automorphism groups. *International Mathematics Research Notices*, **2017**(23), 7292–7320.

[43] Serre, Jean-Pierre. 1980. SL 2. Pages 69–136 of: *Trees*. Springer.

[44] Smith, S. M. 2017. A product for permutation groups and topological groups. *Duke Mathematical Journal*.

[45] Tao, T. 2011. What's new. *https://terrytao.wordpress.com/2011/10/08/254a-notes-5-the-structure-of-locally-compact-groups-and-hilberts-fifth-problem/*.

[46] Tao, T. 2014. *Hilbert's fifth problem and related topics*. American Mathematical Soc.

[47] Tits, J. 1970. Sur le groupe des automorphismes d'un arbre. In: *Essays on topology and related topics*. Springer.

[48] Van Dantzig, D. 1931. *Studien over topologische algebra*. Ph.D. thesis, HJ Paris Amsterdam.

[49] Wesolek, P. R. 2015. Elementary totally disconnected locally compact groups. *Proceedings of the London Mathematical Society*, **110**(6), 1387–1434.

[50] Wesolek, P. R. 2018. *An introduction to totally disconnected locally compact groups*. Available from https://zerodimensional.group/reading_group/190227_michal_ferov.pdf

[51] Willis, G. A. 1994. The structure of totally disconnected, locally compact groups. *Mathematische Annalen*, **300**, 341–363.

[52] Willis, G. A. 2001a. Further properties of the scale function on a totally disconnected group. *Journal of Algebra*.

[53] Willis, G. A. 2001b. The number of prime factors of the scale function on a compactly generated group is finite. *Bulletin of the London Mathematical Society*, **33**(2), 168–174.

[54] Willis, G. A. 2007. Compact open subgroups in simple totally disconnected groups. *Journal of Algebra*, **312**, 405–417.

[55] Willis, G. A. 2008. *A canonical form for automorphisms of totally disconnected locally compact groups*. De Gruyter.

[56] Wilson, J. S. 1998. *Profinite groups*. Clarendon Press.
[57] Wood, D. R., De Gier, J., Praeger, C. E. and Tao, T. 2018. *2016 MATRIX Annals*. Springer.
[58] Yamabe, H. 1953. A generalization of a theorem of Gleason. *Annals of Mathematics*, 351–365.

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