Proceedings of the Edinburgh Mathematical Society (1991) **34**, 313–320 ©

INVARIANTS AND EXAMPLES OF GROUP ACTIONS ON TREES AND LENGTH FUNCTIONS

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(Received 18th October 1989)

An action of a group G on a tree, and an associated Lyndon length function l, give rise to a hyperbolic length function L and a normal subgroup K having bounded action. The Theorem in Section 1 shows that for two Lyndon length functions l, l' to arise from the same action of G on some tree, $L=L'$ and $K=K'$. Moreover for L non-abelian $L=L'$ implies $K=K'$. That this is not so for abelian L is shown in Section 2 where two examples of Lyndon length functions l, l' on an H.N.N. group are given, with their associated actions on trees, for which $L=L'$ is abelian but $K \neq K'$.

1980 *Mathematics subject classification* (1985 Revision): 20F32, 20E06.

Introduction

A group G acting as a group of isometries on an $\mathbb{R}$-tree T has an associated hyperbolic length function $L: G \to \mathbb{R}$ and, for each point $u \in T$, a Lyndon length function $l_u: G \to \mathbb{R}$. It is shown in [**8**] that G has a maximal normal subgroup K which has bounded action on T. Both L and K are determined by the Lyndon lengths l_u and these results are brought together in the theorem in Section 1 where conditions are given for two Lyndon lengths to arise from an action of G on some tree. A result of [**1**] shows that a non-abelian L determines a Lyndon length function, and also the normal subgroup K. In Section 2 properties of two examples of Lyndon length functions on an H.N.N. group, and the associated group actions on trees, are given. The examples illustrate that an abelian L may not determined a Lyndon length function nor the maximal normal subgroup K associated with an action on a tree.

1. Invariants of actions

Let a group G act as a group of isometries on a metric tree ($\mathbb{R}$-tree) T. The notation used follows that of [**7**], where metric trees are defined, and frequent reference will also be made to [**1**], where properties of more general Λ-trees are established (here Λ denotes an ordered abelian group). We note that L and l used in [**1**] have been interchanged.

A function $l: G \to \mathbb{R}$ is called an abstract Lyndon length function if it satisfies the following axioms for all $x, y, z \in G$:

A1′ $l(1)=0$;

A2 $l(x)=l(x^{-1})$;

A4 $c(x,y)<c(x,z)$ implies $c(x,y)=c(y,z)$,

where $c(x,y)=\frac{1}{2}(l(x)+l(y)-l(xy^{-1}))$.

For each point $u\in T$ a function $l_u: G\to\mathbb{R}$ is defined by $l_u(x)=d(u,xu)$, where d is the metric on T. It is clear that l_u satisfies the axioms listed above and so is a Lyndon length function. The hyperbolic length function $L: G\to\mathbb{R}$ is defined by $L(x)=\inf\{d(u,xu);\ u\in T\}$. The following property is part of Corollary 6.13 of [**1**].

Lemma. *For any $u\in T$, $L(x)=\max(0, l_u(x^2)-l_u(x))$.*

The subset N of G consists of the elements $x\in G$ with $L(x)=0$. In Section 7 of [**1**], the hyperbolic length L is defined to be *abelian* if $L(xy)\leqq L(x)+L(y)$ for all $x,y\in G$. We note that if L is abelian then N is a subgroup of G. If not, there exists $x,y\in N$ with $xy\notin N$, in which case $L(x)=L(y)=0$ and $L(xy)>0$. In Theorem 7.8 of [**1**], for L non-abelian, a formula is given expressing the Lyndon length l_u in terms of L, for some $u\in T$.

By Corollary 2.4 of [**8**], for $N\neq G$, there is a maximal normal subgroup K of G having bounded action on T; and for any $u\in T$, $K=\text{core } H_u$ where

$$H_u=\{a\in G; l_u(ax)=l_u(x) \text{ for all } x\notin N\}.$$

It follows that $K\subseteq N$.

For $l: G\to\mathbb{R}$ an abstract Lyndon length function the associated hyperbolic length function $L: G\to\mathbb{R}$ is defined by $L(x)=\max(0, l(x^2)-l(x))$ and, for $N\neq G$, the maximal normal subgroup K of G with bounded action is defined by $K=\text{core } H$ where $H=\{a\in G;\ l(ax)=l(x) \text{ for all } x\notin N\}$. By the lemma the condition $N\neq G$ is equivalent to $L\neq 0$ on G (i.e. there is some $x\in G$ with $L(x)>0$). For a Lyndon length function $l': G\to\mathbb{R}$, L' and K' are similarly defined.

Theorem. *Let $l, l': G\to\mathbb{R}$ be Lyndon length functions with $L, L'\neq 0$. If there exists an action of G on a tree T, and points $v, w\in T$ with $l=l_v$, $l'=l_w$ then*

(i) $L=L'$, *and* (ii) $K=K'$.

Moreover if L is non-abelian then (i) *implies* (ii).

Proof. By the lemma and Corollary 2.4 of [**8**], L and K are invariants of the action of G on T, and are determined by the Lyndon length function at any point of T. It follows that $L=L'$ and $K=K'$.

If $L=L'$ is non-abelian then by Theorem 7.8 of [**1**] a Lyndon length function l_u is

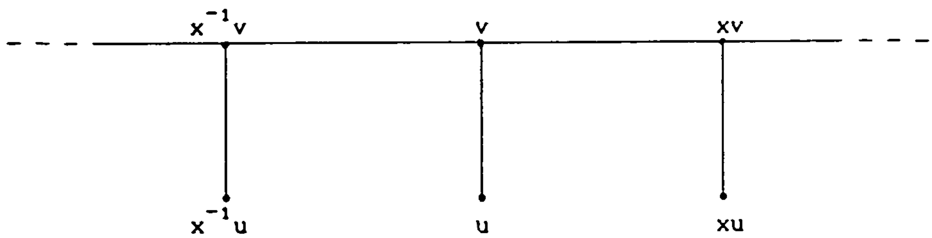

FIGURE 1

determined by L, and l_u arises at some point u on any tree on which G acts having hyperbolic length $L=L'$. Hence $K=K'=\text{core } H_u$.

For an abstract Lyndon length function $l:G\to\mathbb{R}$ Chiswell [2] defines an associated tree T, with a distinguished point $u\in T$, and an action of G on T such that $l=l_u$. We recall some details of the construction. The points of T are equivalence classes $[(x,r)]$ for all $x\in G$, $0\leqq r\leqq l(x)$, and $u=[(x,0)]$. Under the action of G the image $xu=[(x,l(x))]$. For $v,w\in T$, $[v,w]$ denotes the segment of T from v to w. For $x,y\in G$ then $[u,xu]\cap[u,yu]=[u,v]$ where the distance $d(u,v)=c(x^{-1},y^{-1})=\frac{1}{2}(l(x)+l(y)-l(x^{-1}y))$.

For an action of a group G on a tree T each element $x\notin N$ has a unique *axis* in T. The axis $=\{v\in T;\ L(x)=l_v(x)\}$, and is an isometric image of $\mathbb{R}$ on which x acts by translation through $L(x)$. The existence of axes is established in Theorem 6.6 of [1] and Theorem II.2.3 of [5], where it is also shown that the axis $=\bigcup_{n\in\mathbb{Z}}[x^nv,x^{n+1}v]$, where for any $u\in T$, $[u,xu]\cap[u,x^{-1}u]=[u,v]$. If T arises by Chiswell's construction from a length l then $l(x)=d(u,xu)$, and the part of $[u,xu]$ intersecting the axis for x, that is $[v,xv]$, is the middle section of length $L(x)$.

2. Two examples

The theorem raises the question of whether condition (i) implies condition (ii) if L is abelian. The following examples of Lyndon lengths l, l' on a group G with $L=L'$, but $K\neq K'$, show that this is not true in general.

Let G be the group with generators t, g_i, for $i\in\mathbb{Z}$ and relations $t^{-1}g_i\,t=g_{i+1}$, for $i\in\mathbb{Z}$. Let N be the subgroup generated by the generators g_i, for $i\in\mathbb{Z}$. Then N is a normal subgroup of G, and G is an H.N.N. extension with base N and single stable letter t. Any element x of G can be uniquely written as $x=at^r$ for some $a\in N$, $r\in\mathbb{Z}$.

Define $l:G\to\mathbb{R}$ by $l(at^r)=|r|$. It is easily shown that l is a Lyndon length function. In fact, in the notation of [6], l is an extension of $l_1=0$ on N by l_2 on G/N, an infinite cyclic group, where $l_2(t^rN)=|r|$.

For the identity element $1\in N$, define $m(1)=-\infty$. For $a\neq 1$ in N, if $a=g_{i_1}^{\varepsilon_1}g_{i_2}^{\varepsilon_2}\cdots g_{i_k}^{\varepsilon_k}$, $\varepsilon_j=\pm 1$, in reduced form, define $m(a)=2\max(i_1,i_2,\dots,i_k)$. For all $a\in N$ and $r\in\mathbb{Z}$ define $a_r=t^{-r}at^r$. If $a=g_{i_1}^{\varepsilon_1}g_{i_2}^{\varepsilon_2}\cdots g_{i_k}^{\varepsilon_k}$, in reduced form, then by repeated application of the relations, $a_r=g_{i_1+r}^{\varepsilon_1}g_{i_2+r}^{\varepsilon_2}\cdots g_{i_k+r}^{\varepsilon_k}$, and so $m(a_r)=m(a)+2r$.

https://doi.org/10.1017/S0013091500007197 Published online by Cambridge University Press

Define $l': G \to \mathbb{R}$ by $l'(at^r) = \max\{m(a)+r, |r|\}$. It is shown in Theorem 2 of [3] that l' is a Lyndon length function, and that N has unbounded lengths.

Proposition 1. *The Lyndon length functions* l, $l': G \to \mathbb{R}$ *have* $L(at^r) = L'(at^r) = |r|$ *with* $K = N$ *and* $K' = \{1\}$.

Proof. For $x = at^r$ then $x^2 = at^r at^r = aa_{-r}t^{2r}$. Thus $l(x^2) = |2r|$ and $l(x) = |r|$, and hence

$$l(x^2) - l(x) = |2r| - |r| = |r| \geqq 0$$

by the lemma $L(x) = \max(0, l(x^2) - l(x)) = |r|$.

From the definition of the function m above, if $r \geqq 0$ then $m(aa_{-r}) = m(a)$ and if $r \leqq 0$ then $m(aa_{-r}) = m(a_{-r}) = m(a) - 2r$. Thus for $r \geqq 0$, $l'(x) = \max(m(a)+r, r)$ and

$$l'(x^2) = \max(m(aa_{-r}) + 2r, 2r) = \max(m(a) + 2r, 2r).$$

The maximum will be achieved in the same position for $l'(x^2)$ and $l'(x)$ and so $l'(x^2) - l'(x) = r = (m(a)+2r) - (m(a)+r) = 2r - r$. For $r \leqq 0$, $l'(x) = \max(m(a)+r, -r)$ and $l'(x^2) = \max(m(aa_{-r}) + 2r, -2r) = \max((m(a)-2r)+2r, -2r) = \max(m(a), -2r)$. The maximum will be achieved in the same position for $l'(x^2)$ and $l'(x)$ and so $l'(x^2) - l'(x) = -r = m(a) - (m(a)+r) = -2r - (-r)$. We therefore have $L'(x) = |r|$.

The length function l has zero length for all elements of N and so $K = N$.

For $a \in N$, $l'(t^{-r}at^r) = l'(a_r) = \max(m(a_r), 0) = \max(m(a) + 2r, 0)$. So for $a \neq 1$ the lengths $l'(t^{-r}at^r)$ are unbounded for $r > 0$. Thus any non-trivial normal subgroup of G, contained in N, will have unbounded lengths under l'. It follows that $K' = \{1\}$.

For an action of a group on a tree giving a non-abelian hyperbolic length function Theorem 7.8 of [1] determines a Lyndon length function of the action solely in terms of the hyperbolic length function. Since the maximal normal subgroup having bounded action on a tree is an invariant of the action, and since Proposition 1 gives $K \neq K'$ it follows that the Lyndon length functions l and l' cannot arise from the same action of G on some tree. A corresponding result to Theorem 7.8 for abelian hyperbolic length functions is therefore not possible.

For the example of the Lyndon length function $l: G \to \mathbb{R}$ it can be easily shown that Chiswell's construction gives a tree T which is an isometric copy of $\mathbb{R}$ on which at^r acts by translation through r. T is the axis for all elements not in N. This is an example of the case $(\cap A_s \neq \phi)$ described in part (b) of Theorem 7.5 of [1].

Chiswell's construction for $l': G \to \mathbb{R}$ gives a tree T', with base point u, and an action of G on T' such that $l' = l_u$. This is the situation described in Theorem 5 of [4], where N has unbounded lengths and so, by Theorem 3.2 of [7], fixes no point of T'. It is also described in case (ε) of part (b) of Theorem 7.5 of [1], where it is shown that G fixes a unique end of T'. We illustrate this result by considering the axes in T'. Elements $x, y \notin N$ are said to be *cyclically related* if there exists r, s with $x^r = y^s$. An element has the same axis as a power and so cyclically related elements have identical axes. A *half-*

line in T' is an isometric image of $(-\infty,0]$. For $t\in G$, $l'(t)=L'(t)=1$ and so the axis for t consists of $\bigcup_{n\in\mathbf{Z}}[u,t^nu]$. By the *left* of this axis we mean the direction from u to t_u^{-n}, for $n>0$.

Proposition 2. *Under the action of G on T' two non cyclically related elements, not in N, have axes that intersect in a half-line. In particular the axes for t and at^r $(r>0)$ agree from the left to the point $t^{-(m(a)/2)}u$.*

Proof. We first consider the axes for t and at^r, with $r>0$. For $k>0$, $(at^r)^k=at^rat^r\dots at^r=aa_{-r}a_{-2r}\dots a_{-(k-1)r}t^{kr}=bt^s$, writing $b=aa_{-r}\dots a_{-(k-1)r}$ and $s=kr$. Since $r>0$, $m(b)=m(a)$. Let $x=t^n$ and $y=(at^r)^k=bt^s$ where $n>0$ will be taken to be as large as we like. The Lyndon lengths $l'(x)=n$ and

$$l'(y)=\max\begin{cases}m(b)+s\\ s\end{cases}=\max\begin{cases}m(a)+s\\ s\end{cases}.$$

Following Chiswell's construction we compare the segments $[u,x^{-1}u]$ and $[u,y^{-1}u]$, and the segments $[u,xu]$ and $[u,yu]$.

The segments $[u,x^{-1}u]$ and $[u,y^{-1}u]$ coincide to a distance $c(x,y)$ from u where

$$\begin{aligned}2c(x,y)&=l'(x)+l'(y)-l'(xy^{-1})\\&=l'(x)+l'(y)-l'(yx^{-1})\\&=l'(x)+l'(y)-l'(bt^{s-n})\\&=n+\max\begin{cases}m(b)+s\\ s\end{cases}-\max\begin{cases}m(b)+s-n\\ n-s\end{cases}\\&=n+\max\begin{cases}m(b)+s-(n-s)\\ s\end{cases}\\&=\max\begin{cases}m(b)+2s\\ 2s\end{cases}=\max\begin{cases}m(a)+2s\\ 2s\end{cases}\quad\text{for } n \text{ sufficiently large.}\end{aligned}$$

For $m(a)<0$, $l'(y)=L'(y)=s$ and so $[u,y^{-1}u]$ is contained in the axis for y, which is also the axis for at^r. Since $c(x,y)=s$, $[u,y^{-1}u]$ is contained in $[u,x^{-1}u]$, and hence, since the above holds for any $k>0$, the axis for at^r agrees from the left with the axis for t at least as far as the point u. For $m(a)>0$, $l'(y)=m(a)+s$ and $L'(y)=s$, and so the part of the axis of y in $[u,y^{-1}u]$ is the middle section of length s. Since $c(x,y)=(m(a)/2)+s$ this lies in $[u,x^{-1}u]$. Since the section of $[u,y^{-1}u]$ to distance $m(a)/2$ from u is not in the axis

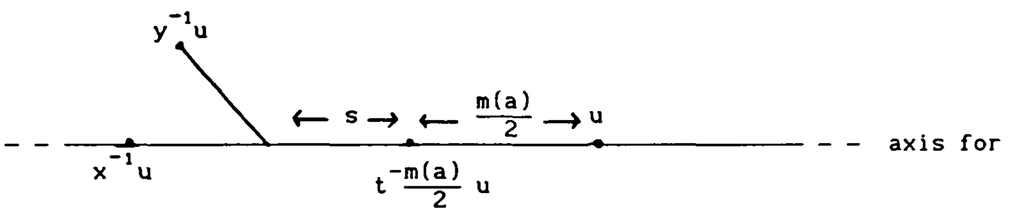

FIGURE 2

for y it follows that the axis for at^r coincides with that for t from the left to a distance $m(a)/2$ to the left of u, that is to the point $t^{-(m(a)/2)}u$.

For $m(a)<0$, when $l'(y)=s$, consider the segments $[u,xu]$ and $[u,yu]$ which coincide up to a distance $c(x^{-1},y^{-1})$ from u, where

$$2c(x^{-1},y^{-1})=l'(x)+l'(y)-l'(x^{-1}y)=l'(x)+l'(y)-l'(t^{-n}bt^s)$$

$$=l'(x)+l'(y)-l'(b_nt^{s-n})$$

$$=n+s-\max\begin{cases}m(b_n)+s-n=n+s\\ n-s\end{cases}-\max\begin{cases}(m(b)+2n)+s-n\\ n-s\end{cases}$$

$$=n+s-\max\begin{cases}m(b)+n+s\\ n-s\end{cases}\quad \text{for } n \text{ sufficiently large.}$$

For k, and hence s, sufficiently large $2c(x^{-1},y^{-1})=n+s-(m(b)+n+s)=-m(b)=-m(a)$. $[u,yu]$ lies in the axis for y, which is the axis for at^r, and this coincides with $[u,xu]$ to a distance $c(x^{-1},y^{-1})=-(m(a)/2)$ from u. Thus the axis for at^r coincides with the axis for t from the left to a distance $-(m(a)/2)$ to the right of u, that is to the point $t^{-(m(a)/2)}u$.

Now consider the axes for two general elements at^r, bt^r, taking r, $s>0$ since the axis for an element is the axis for its inverse. Since their axes agree from the left with the axis for t their intersection must agree from the left at least as far as the point $t^{-c}u$ where $c=\max(m(a)/2,m(b)/2)$. The two axes must therefore either be identical or intersect in a half-line. If $m(a)\neq m(b)$ then the two axes will separate at the point $t^{-c}u$ and so intersect in a half-line. An element has the same axis as a power and so powers can be taken to equate powers of t. It suffices therefore to consider elements $x=at^r$ and $y=bt^r$, and it remains to consider the case $m(a)=m(b)$. For x and y non cyclically related $a\neq b$. Moreover, since taking powers of x and y gives similar elements, it can be assumed that r is as large as we please. Thus

$$l'(x)=l'(y)=\max\begin{cases}m(a)+r\\ r\end{cases}\quad\text{and}\quad L'(x)=L'(y)=r.$$

Consider the segments $[u, xu]$ and $[u, yu]$ which coincide to a distance $c(x^{-1}, y^{-1})$ where

$$
\begin{aligned}
2c(x^{-1}, y^{-1}) &= l'(x) + l'(y) - l'(x^{-1}y) \\
&= l'(x) + l'(y) - l'(t^{-r}a^{-1}bt^{r}) \\
&= l'(x) + l'(y) - l'(a_r^{-1}b_r) \\
&= 2\max\begin{cases} m(a)+r \\ r \end{cases} - \max\begin{cases} m(a_r^{-1}b_r) \\ 0 \end{cases} \\
&= 2\max\begin{cases} m(a)+r \\ r \end{cases} - \max\begin{cases} m(a^{-1}b)+2r \\ 0 \end{cases} \\
&= 2\max\begin{cases} m(a)+r \\ r \end{cases} - (m(a^{-1}b)+2r) \\
&= \max\begin{cases} 2m(a)-m(a^{-1}b) \\ -m(a^{-1}b) \end{cases} \quad \text{for } r \text{ sufficiently large.}
\end{aligned}
$$

The parts of the axes for x and y within $[u, xu]$ and $[u, yu]$ respectively extended to a distance $(m(a)/2)+r$ from u. For r sufficiently large this will be greater than $c(x^{-1}, y^{-1})$, which is independent of r, and so the two axes will diverge at this point. The two axes therefore intersect in a half-line.

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