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HAAGERUP PROPERTY FOR C^*-CROSSED PRODUCTS

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Abstract

Let Γ be a countable discrete group that acts on a unital C^*-algebra A through an action α. If A has a faithful α-invariant tracial state τ, then $\tau' = \tau \circ \mathcal{E}$ is a faithful tracial state of $A \rtimes_{\alpha,r} \Gamma$ where $\mathcal{E} : A \rtimes_{\alpha,r} \Gamma \to A$ is the canonical faithful conditional expectation. We show that $(A \rtimes_{\alpha,r} \Gamma, \tau')$ has the Haagerup property if and only if both (A, τ) and Γ have the Haagerup property. As a consequence, suppose that $(A \rtimes_{\alpha,r} \Gamma, \tau')$ has the Haagerup property where Γ has property T and A has strong property T. Then Γ is finite and A is residually finite-dimensional.

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1. Introduction

The Haagerup property was first defined for groups by Haagerup [4], as a weakened version of amenability. This concept was generalised to the context of von Neumann algebras by Choda [2] in order to distinguish particular group von Neumann algebras. Recently, Dong [3] introduced a notion of the Haagerup property for a pair consisting of a unital C^*-algebra and a faithful tracial state, by imitating the case of von Neumann algebras.

Definition 1.1 [3, Definition 2.3]. Let A be a unital C^*-algebra and τ a faithful tracial state on A. The pair (A, τ) is said to have the Haagerup property if there is a sequence $\{\Phi_n\}_{n\geq 1}$ of unital completely positive maps from A to itself satisfying the following conditions:

(1) each Φ_n decreases τ, that is, for all $a \in A^+$, $\tau(\Phi_n(a)) \leq \tau(a)$;
(2) for any $a \in A$, $\|\Phi_n(a) - a\|_{2,\tau} \to 0$ as $n \to +\infty$;
(3) each Φ_n is L_2-compact, that is, from the first condition, Φ_n extends to a compact bounded operator on its GNS space $L_2(A, \tau)$.

In [3], Dong explored the behaviour of the Haagerup property of a C^*-algebra in a C^*-dynamical system and obtained the following theorem.

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Theorem 1.2 [3, Theorem 2.5]. *Suppose that A is a unital separable C^*-algebra with a faithful tracial state τ and a τ-preserving action α of a countable discrete group Γ. If Γ is amenable and (A, τ) has the Haagerup property, then $(A \rtimes_{\alpha,r} \Gamma, \tau')$ also has the Haagerup property, where τ' is the induced trace of τ on $A \rtimes_{\alpha,r} \Gamma$.*

In [7], You defined the Haagerup property for an action $\alpha : \Gamma \curvearrowright A$ with respect to a faithful tracial state τ on A and gave the following theorem.

Theorem 1.3 [7, Theorem 1.4]. *Let Γ be a countable discrete group and α be an action of Γ on a unital C^*-algebra A such that α has the Haagerup property with respect to a faithful tracial state τ on A. Then the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$ has the Haagerup property with respect to the induced faithful tracial state τ' if A has the Haagerup property with respect to τ.*

Motivated by the above results, we give an equivalent description of the Haagerup property of unital C^*-algebra crossed products.

Theorem 1.4. *Let Γ be a countable discrete group that acts on a unital C^*-algebra A through an action α, τ be a faithful α-invariant tracial state of A, and τ' be the induced faithful tracial state of $A \rtimes_{\alpha,r} \Gamma$. Then $(A \rtimes_{\alpha,r} \Gamma, \tau')$ has the Haagerup property if and only if both (A, τ) and Γ have the Haagerup property.*

As a consequence, suppose that $(A \rtimes_{\alpha,r} \Gamma, \tau')$ has the Haagerup property, where Γ has property T and A has strong property T. Then Γ is finite and A is residually finite-dimensional.

2. Main results

Throughout this section, let Γ be a countable discrete group that acts on a unital C^*-algebra A through an action α. We denote by $A \rtimes_{\alpha,r} \Gamma$ the reduced C^*-crossed product of (A, Γ, α), and identify $A \subseteq A \rtimes_{\alpha,r} \Gamma$ as well as $\Gamma \subseteq A \rtimes_{\alpha,r} \Gamma$ through their canonical embeddings. We fix a faithful α-invariant tracial state τ of A and write $\tau' = \tau \circ \mathcal{E}$ for the induced faithful tracial state of $A \rtimes_{\alpha,r} \Gamma$ where $\mathcal{E} : A \rtimes_{\alpha,r} \Gamma \to A$ is the canonical faithful conditional expectation. Since τ is a faithful tracial state on A, by the GNS construction, τ defines an A-Hilbert bimodule, denoted by $L_2(A, \tau)$. We also denote by $\|\cdot\|_{2,\tau}$ the associated Hilbert norm. Thus for each $a \in A$, we have

$$\|a\|_{2,\tau} = (\tau(a^*a))^{1/2} \leq \|a\|.$$

If $\phi : A \to A$ is a completely positive map such that $\tau \circ \phi \leq \tau$, then ϕ can be extended to a contraction $T_\phi : L_2(A, \tau) \to L_2(A, \tau)$. We say ϕ is L_2-compact if T_ϕ is a compact operator on $L_2(A, \tau)$. By considering rank-one operators, we see that ϕ is L_2-compact if and only if for any $\varepsilon > 0$, there exists a finite-rank map $Q : A \to A$ such that

$$\|\phi(x) - Q(x)\|_{2,\tau} \leq \varepsilon \|x\|_{2,\tau}$$

for all $x \in A$.

Theorem 2.1. *The following statements are equivalent.*

(1) *$(A \rtimes_{\alpha,r} \Gamma, \tau')$ has the Haagerup property.*
(2) *Both (A, τ) and Γ have the Haagerup property.*
(3) *α has the Haagerup property with respect to τ and (A, τ) has the Haagerup property.*

Proof. (1) $\Rightarrow$ (2). Suppose that $(A \rtimes_{\alpha,r} \Gamma, \tau')$ has the Haagerup property. Then there is a sequence $\{\Phi_n\}_{n\geq 1}$ of unital completely positive maps from $A \rtimes_{\alpha,r} \Gamma$ to itself satisfying the conditions of Definition 1.1. For each n, let

$$\phi_n(a) = \mathcal{E} \circ \Phi_n(a)$$

for all $a \in A$. Then ϕ_n is a unital completely positive map from A to itself and

$$\tau \circ \phi_n(a) = \tau \circ \mathcal{E} \circ \Phi_n(a) = \tau' \circ \Phi_n(a) \leq \tau'(a) = \tau(a)$$

for all $a \in A^+$. As n tends to infinity, we have

$$\begin{aligned} \|\phi_n(a) - a\|_{2,\tau}^2 &= \tau((\phi_n(a) - a)^*(\phi_n(a) - a)) \\ &= \tau((\mathcal{E} \circ \Phi_n(a) - \mathcal{E}(a))^*(\mathcal{E} \circ \Phi_n(a) - \mathcal{E}(a))) \\ &= \tau((\mathcal{E}(\Phi_n(a) - a))^*\mathcal{E}(\Phi_n(a) - a)) \\ &\leq \tau \circ \mathcal{E}((\Phi_n(a) - a)^*(\Phi_n(a) - a)) \\ &= \|\Phi_n(a) - a\|_{2,\tau'}^2 \to 0 \end{aligned}$$

for any $a \in A$. Since Φ_n is L_2-compact, for any $\varepsilon > 0$, there exists a finite-rank map $Q : A \rtimes_{\alpha,r} \Gamma \to A \rtimes_{\alpha,r} \Gamma$ such that

$$\|\Phi_n(x) - Q(x)\|_{2,\tau'} \leq \varepsilon \|x\|_{2,\tau'}$$

for all $x \in A \rtimes_{\alpha,r} \Gamma$. Hence, we have

$$\begin{aligned} \|\phi_n(a) - \mathcal{E} \circ Q(a)\|_{2,\tau} &= \|\mathcal{E} \circ \Phi_n(a) - \mathcal{E} \circ Q(a)\|_{2,\tau} \\ &\leq \|\Phi_n(a) - Q(a)\|_{2,\tau'} \\ &\leq \varepsilon \|a\|_{2,\tau'} = \varepsilon \|a\|_{2,\tau} \end{aligned}$$

for all $a \in A$. So ϕ_n is L_2-compact. Hence, (A, τ) has the Haagerup property.

For each n, let

$$\varphi_n(g) = \tau'(\Phi_n(g)g^{-1})$$

for all $g \in \Gamma$. Clearly $\varphi_n(e) = 1$ and for all $g_1, \ldots, g_m \in \Gamma$ and all $c_1, \ldots, c_m \in \mathbb{C}$, the positivity of τ' yields

$$\begin{aligned} \sum_{i,j=1}^{m} c_i \bar{c}_j \varphi_n(g_j^{-1} g_i) &= \sum_{i,j=1}^{m} c_i \bar{c}_j \tau'((\Phi_n(g_j^{-1} g_i))(g_j^{-1} g_i)^{-1}) \\ &= \sum_{i,j=1}^{m} \tau'(\bar{c}_j g_j \Phi_n(g_j^{-1} g_i) c_i g_i^{-1}) \\ &= \tau'\Big(\sum_{i,j=1}^{m} \bar{c}_j g_j \Phi_n(g_j^{-1} g_i) c_i g_i^{-1}\Big) \geq 0. \end{aligned}$$

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Hence φ_n is positive definite on Γ. Moreover, as $n \to +\infty$,

$$|\varphi_n(g) - 1| = |\tau'(\Phi_n(g)g^{-1}) - 1| = |\tau'(\Phi_n(g)g^{-1}) - \tau'(gg^{-1})|$$
$$= |\tau'((\Phi_n(g) - g)g^{-1})| \leq \|\Phi_n(g) - g\|_{2,\tau'} \to 0$$

for any $g \in \Gamma$. It follows from the compactness of Φ_n that

$$\limsup_{g\to\infty} |\varphi_n(g)| = \limsup_{g\to\infty} |\tau'(\Phi_n(g)g^{-1})| \leq \limsup_{g\to\infty} \|\Phi_n(g)\|_{2,\tau'} \to 0.$$

This proves the Haagerup property of Γ.

(2) $\Rightarrow$ (3). It follows from [7, Remark 1.5] that if Γ has the Haagerup property, then α has the Haagerup property with respect to any faithful tracial state on A.

(3) $\Rightarrow$ (1). This follows from Theorem 1.3. □

Corollary 2.2. *Suppose that Γ has property T and A has strong property T (in the sense of [5]). If $(A \rtimes_{\alpha,r} \Gamma, \tau')$ has the Haagerup property, then Γ is finite and A is residually finite-dimensional.*

Proof. Suppose that $(A \rtimes_{\alpha,r} \Gamma, \tau')$ has the Haagerup property. It follows from Theorem 2.1 that both (A, τ) and Γ have the Haagerup property. Hence Γ is finite. It follows from [6, Theorem 4.7] that A is residually finite-dimensional. □

Let $U(A)$ be the unitary group of a unital C^*-algebra A and $\varphi : \Gamma \to U(A)$ be a group homomorphism. Define an action α^φ of Γ on A by

$$\alpha_s^\varphi(a) = \varphi(s)a\varphi(s)^*$$

for all $s \in \Gamma$ and $a \in A$. If A has a faithful tracial state τ (for example, $A = M_n(\mathbb{C})$), then τ is α^φ-invariant. The next result follows from Theorem 2.1.

Corollary 2.3. *If (A, τ) has the Haagerup property, then Γ has the Haagerup property if and only if $(A \rtimes_{\alpha^\varphi,r} \Gamma, \tau')$ has the Haagerup property.*

Remark 2.4. Suppose that G is a discrete group that acts on a unital C^*-algebra B through an action β. Using [1, Theorem 4.2.6, Propositions 6.3.3 and 6.3.4], we can obtain the following result. If $B \rtimes_{\beta,r} G$ has a tracial state, then $B \rtimes_{\beta,r} G$ is nuclear if and only if B is nuclear and G is amenable.

Suppose that X is a compact space with an action $\tilde{\beta}$ of a discrete group G by homeomorphisms. Let β be the induced action of G on $C(X)$, that is, β is defined by $\beta_s(f)(x) = f(\tilde{\beta}_{s^{-1}}(x))$. Using the above remark, we can obtain the following result.

Corollary 2.5. *If X has a fixed point, then $C(X) \rtimes_{\beta,r} G$ is nuclear if and only if G is amenable.*

Proof. Let x_0 be the fixed point. Define $\tau : C(X) \to \mathbb{C}$ by

$$\tau(f) = f(x_0)$$

for all $f \in C(X)$. Hence

$$\tau(\beta_s(f)) = \beta_s(f)(x_0) = f(\tilde{\beta}_{s^{-1}}(x_0)) = f(x_0) = \tau(f)$$

for all $s \in G$ and $f \in C(X)$. Therefore, $C(X) \rtimes_{\beta,r} G$ has a tracial state. □

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