GENERALIZED THEORY OF SHELL FLASH AND ACCRETING WHITE DWARFS*

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ABSTRACT

Shell flashes take place both in deep interior of red giant stars and near surface of accreting white dwarfs. Theories of shell flashes have been thus far presented piece by piece in different papers. It is the purpose of the present review to construct and generalize them in order to reach better understanding. A non-linear yet almost analytical theory is presented which treats the development of the shell flash in finite amplitude. Recurrence of the shell flashes is also shown to be well understood as a non-linear oscillation in dissipative system which tends to be its limit cycle. As a result strength of the peak energy-generation and recurrence time of the shell flashes are related with mass of the accreting white dwarfs, accretion rate, etc.

1. INTRODUCTION

Schwarzschild and Härm (1965) discovered first that helium burning in a thin shell is unstable even when electrons are non-degenerate therein. They have advanced a linearized theory to obtain a stability criterion against the shell burning (Schwarzschild and Härm 1965). Soon thereafter, Weigert (1966) and Rose (1966) showed numerically that the instability results in recurrent thermal pulses.

About the same time Hayashi, Hōshi and Sugimoto (1965) constructed a model of less massive star in which a helium shell flash begins in an electron degenerate helium zone surrounding a carbon-oxygen core. By extending the model into a shell flash of *finite* amplitude, Sugimoto and Yamamoto (1966) showed the followings; (1) the shell burning is unstable even after the electron-degeneracy has been lifted, (2) effect of radiation pressure is essential in quenching the unstable nuclear shell-burning, and (3) a strong convection appears in the helium zone and it might reach the bottom of the hydrogen-rich envelope to trigger mixing between the helium zone and the envelope. However, they did not properly recognize that their shell flash was a different manifestation of the same instablity that Schwarzschild and Härm (1965) had met with.

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Now it is known that Sugimoto and Yamamoto's (1966) equilibrium model overestimated the degeneracy in the helium zone and that the helium shell flash would have actually begun at somewhat lower density. Nevertheless, general understanding of the shell flash of *finite* amplitude has already been obtained in their study. Later on such theory was somewhat more advanced by Hayakawa and Sugimoto (1968) and was applied for nova-like explosion in which the hydrogen shell-burning was saturated by beta decays in CNO cycles.

After Schwarzschild and Härm (1965) many numerical computations are done. Indeed the shell flash, which is sometimes called flicker, thermal pulse, relaxation oscillation etc., has now a wide variety of applications. Helium shell-flash takes place deep in the interior of red giant stars, which is discussed in relation with the origin of carbon stars and the *s*-process nucleosynthesis (Sackmann 1980 and papers referred therein). Hydrogen shell-flash takes place near the surface of accreting white dwarfs, which is related with nova explosion. Such theory of nova explosion was greatly advanced by constructing detailed models (Nariai, Nomoto and Sugimoto 1980 and papers referred therein), by computing detailed nuclear reaction networks, and by comparing them with observed novae in detail (Gallagher and Starrfield 1978 and papers referred therein). Relatively strong helium shell-flash takes place in accreting helium or carbon-oxygen white dwarf which is related also with explosive phenomena (Taam 1980 and papers referred therein). The shell-flash near the surface of neutron stars is discussed in relation with X-ray bursters (Joss 1980 and papers referred therein).

Despite such wide applications and many numerical computations the physics involved in the shell flash of *finite* amplitude has not come to wide understanding. (One can cite many wrong statements concerning the shell flash if one wishes.) Of course, it is not only for the shell flash, but it is a widespread tendency in the theory of stellar *structure* and evolution. We are not saying about physical processes taking place locally under the condition at given temperature and density, which are rather well understood. On the contrary the stellar structure is non-linear and non-local in the sense that physical situations in other shells of the star affect those in a specific shell. Nevertheless we can advance almost analytical theories relatively easily in the case of the shell flash, which have been published piece by piece in different papers. This is the reason why such theories have not been understood even among the specialists. Therefore, it seems useful to summarize them. When they are posed with concepts of thermodynamics of dissipative open system, they will be clearly understood as will be shown in the following sections.

2. RELATIONS BETWEEN SHELL FLASHES IN RED GIANT STARS AND THOSE IN ACCRETING STARS

The shell flash was first noticed to occur in the bottom of the helium zone, which surrounds the carbon-oxygen core and which is immersed deep in the hydrogen-rich envelope. However, the pressure $\underline{P}$ at the bottom of the hydrogen-rich envelope (subscript H) is much lower than the pressure at the bottom of the helium zone (subscript He); for example, P_H/P_{He} is typically as small as 3×10^{-3}. Therefore, the existence of the hydrogen-rich envelope has practically nothing to do with the stability and further development of the helium shell-flash, except for possible mixing between the envelope and the helium zone.

Therefore the situation can be discussed in more generalized context, in which helium is being accreted onto a carbon-oxygen white dwarf or hydrogen is being accreted onto a helium or carbon-oxygen white dwarf. Models of accreting white dwarfs are specified by three parameters, i.e., the mass of the white dwarf M_1, the accretion rate dM/dt, and a parameter specifying thermal status of the core. The last parameter is typically represented by the intrinsic luminosity $L^{(0)}$ of the white dwarf just before onset of the accretion, or by the time elapsed between the formation of the white dwarf and the onset of accretion (Sugimoto et al. 1979; Fujimoto and Sugimoto 1979a).

In the case of red giant stars, these parameters are reduced into a single parameter, the core mass. The mass of white dwarf should be equal to the core mass. The accretion rate corresponds to the growth rate of the helium zone which is determined by the energy generation rate due to the hydrogen shell-burning L_H by

$$dM_1/dt = L_H/X_e E_H \ . \tag{1}$$

Here X_e is the concentration of hydrogen in the envelope, E_H is the energy release from unit mass of hydrogen, and L_H is determined by the core-mass to luminosity relation (Paczyński 1970) which is shown in Figure 1. The third parameter is also specified by the evolutionary history of the star which is also parametrized by the core mass. Therefore, the shell flash in red giant stars can be treated as a special case of the generalized shell flash in accreting stars. In Figure 1 parameters, which were applied in existing model computations, are plotted both for hydrogen and helium shell flashes.

Interesting cases of the hydrogen shell-flash are concerned only with accreting stars. They can also be discussed in the same framework as in the helium shell-flash, since the mechanism and essential features are common. Differences between them lie only in the local physics, i.e., nuclear reaction rate, equation of state and opacity.

3. THERMAL HISTORY OF ACCRETION AND ITS EFFECT ON THE DEVELOPMENT OF THE FLASH

When gas is accreting, the accretion rate is limited by the Eddington's critical accretion rate which is given by

$$(dM/dt)_{cr} = 4\pi c R_1/\kappa \ . \tag{2}$$

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Here κ is the opacity and R_1 is the radius of the (corresponding) white dwarf. The critical accretion rate is also shown in Figure 1.

If gas is accreting at a rate close or higher than the critical accretion, the gas will be just piled up on the stellar surface. Then, the stellar radius will increase and the value of the critical accretion rate will also increase as seen in equation (2) until the latter exceeds the given rate of accretion. Though not exactly for the supercritical accretion, such sequences of events were computed for accreting main-sequence stars (Kippenhahn and Meyer-Hofmeister 1977; Neo et al. 1977). [In many papers such increase in stellar radius is described as an expansion, but in actual a Lagrange shell is contracting and newly added shells are just being piled up (Neo et al. 1977).]

For accreting white dwarf Nomoto, Nariai and Sugimoto (1979) computed a case of rapid accretion at the rate that is just equal to the critical accretion rate for its initial radius. This accretion rate is higher than the mean rate at which the hydrogen is processed into helium according to equation (1). Therefore the envelope is accumulating and the star evolves into a red giant.

For lower accretion rates the hydrogen/helium zone grows in time. The bottom of this zone is gradually compressed which tends to increase the temperature thereof. On the other hand heat diffuses out of this zone which tends to lower the temperature. As the accumulated mass ΔM becomes larger, the former effect weighs over the latter (see e.g. Nomoto and Sugimoto 1977), and at last the nuclear burning ignites when a certain mass ΔM_{ig} has been accumulated.

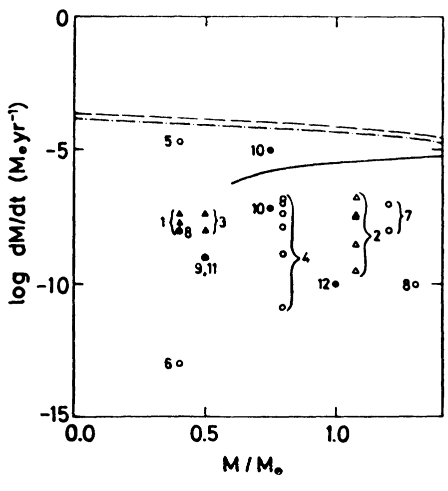

Fig. 1. Masses and accretion rates for some of existing computations. Triangles and circles are the helium- and hydrogen-shell flashes, respectively. Filled marks indicate that the computations were stopped at a stage preceding the peak of the flash. Also shown are the accretion rates corresponding to the core-mass to luminosity relation (solid curve), and the critical accretion rates for hydrogen-rich gas (dash-dotted) and for helium gas (dashed). References are as follows; 1. Nomoto and Sugimoto (1977), 2. Fujimoto and Sugimoto (1979b), 3. Taam (1980), 4. Paczyński and Żytkow (1978), 5. Nomoto et al. (1979), 6. Sugimoto et al. (1979), 7. Sion et al. (1979), 8. Nariai et al. (1980), 9. Giannone and Weigert (1967), 10. Rose (1968), 11. Redkoborodyi (1972), 12. Taam and Faulkner (1975).

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The timescale of the compression is inversely proportional to the accretion rate, while the timescale of the heat diffusion depends mainly on ΔM and is relatively insensitive to the accretion rate. Therefore, ΔM_{ig} is smaller when the accretion rate is higher (Sugimoto et al. 1979). Its actual value can be calculated by taking account of thermal history of accretion. The temperature rise by compression is sometimes called *heating* erroneously, but the entropy of the mass element is decreasing in actual. Anyhow, such competition between the compression and the heat diffusion (and also neutrino loss) are concerned not only with the accreting white dwarf but also with various models; the growth of the stellar cores leading to helium flash in the center, to carbon-deflagration supernova, and to electron capture supernova, and also the accreting helium or carbon-oxygen white dwarf leading to strong flash in a shell or to helium detonation (see Sugimoto and Nomoto 1980 for more details).

In the case of the accreting white dwarf such thermal history of accretion was computed in the earliest work (Giannone and Weigert 1967) and in some of the later works (Taam and Faulkner 1975, Taam 1978). However, in such works the succeeding phase of the shell flash was computed only in the relatively early stages preceding the peak of the energy generation. In most of the later works, in which nova explosion was extensively discussed, the shell flash was computed through the peak (see, e.g., Gallagher and Starrfield 1978 and papers referred therein). However, the preceding thermal history of accretion was not computed in obtaining their initial model. Instead the mass ΔM_{ig} was arbitrarily assumed and the initial temperature distribution in the hydrogen-rich envelope was assumed to be in thermal equilibrium which corresponds to an infinitely slow accretion (see, e.g., Truran et al. 1977).

Recently the whole process of the shell flash has been computed starting from the onset of accretion through the peak of the flash upto the stage of envelope expansion including the transition phases from quasistatic accretion through dynamical stages (Nariai et al. 1980). Such computations are important in two respects. First, ΔM_{ig} can be determined only by means of such computation of accretion (see, e.g., Sugimoto et al. 1979). Second, the recurrence of the flash can be discussed only when the whole process of accretion through flash is computed (Paczyński and Żytkow 1978, Sion et al. 1979).

We have discussed that the value of ΔM_{ig} is determined by the three parameters of the problem, i.e., M_1, dM/dt and $L^{(0)}$, of which the latter two are particularly important. However, further development of the shell flash and its strength, in particular, are almost exactly determined only by the values of M_1 and ΔM_{ig} as discussed by Sugimoto et al. (1979). In other words, once ΔM_{ig} is properly chosen, the preceding thermal history of accretion is unimportant any more in determining the further development of the shell flash in finite amplitude. In this sense computations with even an arbitrarily assumed value of ΔM_{ig} will be a good approximation except for initial stages of the flash and for the recurrence of the flash, if and only if appropriate value of ΔM_{ig} is chosen.

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4. GRAVOTHERMAL SPECIFIC HEAT AND GENERALIZED STABILITY CRITERIA

Schwarzschild and Härm (1965) were the first to develop the linearized stability theory for nuclear shell burning. It compares the change in the nuclear reaction rate with the corresponding change in the diffusion, both of which are induced by perturbation in specific entropy δs, for example, upon model in thermal equilibrium. Here the thermal equilibrium implies that the nuclear energy generation rate ε_n just balances the rate of the heat diffusion dL_r/dM_r. In order to compute the change in ε_n, i.e. $\delta\varepsilon_n$, the temperature change $\delta \ln T$ or the pressure change $\delta \ln P$ in the burning shell should be computed. This is accomplished by solving linearized equations of stellar structure in terms of Green's function, in principle.

Schwarzschild and Härm (1965) discussed also approximate nature of the Green's function and its relation with the notion of *thin* shell. However, such discussions were not fine enough in the sense that the notion of the thin shell was not formulated quantitatively. In the bottom of the thin helium zone, helium shell-burning is unstable. In the bottom of the hydrogen-rich envelope which was accreted on a white dwarf, the hydrogen shell-burning is unstable also. On the contrary, in the deep interior of the red giant stars, the hydrogen shell-burning is stable, though it is much thinner in mass than in the case of the accreted envelope. Here, the thinness of the shell was formulated as $\Delta M(H_p)/M_r << 1$, i.e., mass contained in unit scale hight of pressure $\Delta M(H_p)$ is much smaller than mass contained interior to its shell M_r. However, the important parameter for the instability is the thinness in radius, i.e., $H_p/r << 1$ as will be shown below in this section.

After Schwarzschild and Härm (1965), some works were done improving the approximations (Hōshi 1968; Unno 1970; Dennis 1971) but within essentially the same framework. On the other hand, Sackmann (1977) analyzed detailed numerical computations, and tried to extract some *empirical* relations for the shell flash of finite amplitude. Sugimoto and Fujimoto (1978) elaborated the non-linear theory for the shell flash of finite amplitude to well analytical and accurate one. Here we will discuss the shell flash mainly along their theory but in more generalized fashion by adding some new developments.

The hydrostatic equilibrium of stellar structure can be computed when the relation between the pressure $\underline{P}$ and the density ρ is given as in the case of polytropes for instance. Alternatively, when the distribution of the specific entropy $s(M_r)$ is given, we can compute the hydrostatic equilibrium of the star irrespectively of its thermal state. Therefore, the theory of stability criterion is easily extended to include the case *out of thermal equilibrium*, i.e., the case of finite amplitude.

The energy equations are given by

$$dL_r/dM_r = \varepsilon_n - \varepsilon_\nu + \varepsilon_g \ , \qquad (3)$$

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$$\varepsilon_g \equiv -(T\, ds/dt + \sum_k \tilde{\mu}_k\, dN_k/dt). \tag{4}$$

Here, ε_ν is the neutrino loss rate, ε_g is the so-called gravitational energy release or, more exactly, the rate of heat energy flowing out of an element of unit mass, and $\tilde{\mu}_k$ and N_k are the chemical potential of a particle and the number of particles of the k-th kind which are contained in unit mass of matter. In what follows we will take account of neither ε_ν nor $\tilde{\mu}_k dN_k$ for brevity; they can be easily included when necessary. We call the case to be in thermal equilibrium when ε_g vanishes.

When a perturbation in specific entropy $\delta s(M_r)$ is applied, the temperature changes as much as

$$\delta \ln T = (1/c_g^*)\,\delta s\ , \tag{5}$$

$$\frac{1}{c_g^*} \equiv \frac{1}{c_p} + \left(\frac{\partial \ln T}{\partial \ln P}\right)_s \frac{d \ln P}{ds} \tag{6}$$

Here c_p is the usual thermodynamic specific heat, and $d \ln P/ds$ is the change of pressure $\delta \ln P(M_r)$ divided by the change in the specific entropy $\delta s(M_r)$ of the shell at M_r. The change $\delta \ln P(M_r)$ is called hydrostatic readjustment and is affected not only by δs in the same shell but also by those in other shells. Mathematically, such effect is described by means of Green's function as

$$\delta \ln P(M_r) = \int_0^M G(M_r, M_r')\,\delta s(M_r')\,dM_r'\ . \tag{7}$$

Schwarzschild and Härm's (1965) discussion was essentially the same as obtaining the Green's function. Later on it was explicitly formulated by Henyey and Ulrich (1972) in relation with the shell flash and by Hachisu and Sugimoto (1978) in relation with gravothermal catastrophe of selfgravitating system. Off-diagonal part of $G(M_r, M_r')$ represents the effect of the hydrostatic readjustment throughout the star. Therefore c_g^* defined in equation (6) will be called *gravothermal specific heat*.

In practical problems the change $T\delta s$ in the time interval δt comes from the nuclear energy generation and the heat diffusion, i.e.,

$$T\delta s \equiv (T\delta s)_0 + (T\delta s)_1\ , \tag{8}$$

$$(T\delta s)_0 = [\varepsilon_n^{(0)} - dL_r^{(0)}/dM_r]\,\delta t, \tag{9}$$

$$(T\delta s)_1 = [\delta\varepsilon_n - d\delta L_r/dM_r]\,\delta t\ , \tag{10}$$

where the zero-th order term $(T\delta s)_0$ does or does not vanish according to the thermal equilibrium or non-equilibrium, respectively. The first order term $(T\delta s)_1$ can be written down in terms of $\delta \ln P$, $\delta \ln r$, $\delta \ln T$, $d(\delta \ln T)/dr$ and derivatives of ε_n and κ.

Table 1. GENERALIZED STABILITY CRITERIA AND SEQUENCE OF EVENTS ALONG THE SHELL FLASH.

Phase/ Stage	stability	c_g^{*-1}	$(T\delta s)_0$	$(T\delta s)_1$	Region in Figure 2	Y		
1	unstable	+	0	+	I	~1	onset	main pulse
2	unstable	+	+		I	~1	growing	
3	neutral	0	+		I/II	~1	peak	
4	stable	−	+		II	~1	decaying	
5	stable	−	0	+	II		depletion of fuel	
6	neutral	−	0	0	II/III	~0	stagnation	
7	unstable	−	0	−	III	~0	onset	sub-pulse
8	unstable	−	−		III	~0	growing	
9	neutral	0	−		III/IV	~0	peak	
10	stable	+	−		IV	~0	decaying	
11	stable	+	0	−	IV		steady accretion	
12	neutral	+	0	0	IV/I	~1	stagnation	

When c_g^{*-1} and $T\delta s$ have the same sign, the shell burning is thermally unstable, i.e., the temperature of the burning shell is increasing as a result of perturbation [when $(T\delta s)_0 = 0$] or as a result of the *existing* deviation from thermal equilibrium [when $(T\delta s)_0 \neq 0$]. There are twelve cases for the signs of c_g^{*-1} and $T\delta s$. They are summarized in Table 1 along the sequence of events which will be discussed in detail in the next section. Usually only the cases with $(T\delta s)_0 = 0$ are referred to in the stability criterion of the linearized theory.

As seen in equations (8)-(10) the sign of $T\delta s$ is determined as a result of competition between the nuclear reaction and the heat diffusion. It was discussed by many authors and is easily understood. More important is the notion of c_g^{*-1}. We may ask what characteristics of the stellar structure determines the sign of c_g^{*-1}. It is determined by the sign and the magnitude of $d \ln P/ds$, i.e., by the effect of the hydrostatic readjustment. Thus it depends on the geometry in the hydrostatic equilibrium and on the equation of state.

For the plane-parallel configuration the pressure at a shell is determined only by the weight of the overlying layers. In other words it does not depend on the value of the entropy. Therefore, $d \ln P/ds$ should vanish, and c_g^{*-1} is always positive as seen in equation (6). On the contrary $d \ln P/ds$ is negative at the center of the spherical star. However, its magnitude depends on the equation of state. For ideal gas plus radiation pressure its absolute value is large enough to make c_g^{*-1}

negative. When electrons are degenerate, its absolute value is so small that c_g^{*-1} is positive [see, e.g., Sugimoto and Nomoto (1980) for more detail].

In the case of the burning shell which we are discussing, the configuration is intermediate between the plane-parallel and the spherical symmetry. Its degree will be parametrized by means of the ratio of the radius of the spherical shell to the pressure scale height, i.e.,

$$V \equiv r/H_p = -d\ln P/d\ln r = GM_r\rho/rP \; . \tag{11}$$

The plane-parallel configuration is obtained in the limit of $V \to \infty$ and the spherical one corresponds to $V \to 4$ [see Sugimoto and Nomoto (1980) for reasoning].

Sugimoto and Fujimoto (1978) have shown that c_g^{*-1} is expressed analytically as a function of V_1 and the polytropic index N_1 of the burning shell (denoted by the subscript 1) when the mass contained in the overlying layers ΔM is small, i.e., when $\Delta M/M_r \ll 1$. It is to be noticed that the results depend only slightly upon the polytropic index in the overlying layers. In doing so they solved the stellar structure equation analytically for the envelope of $\Delta M/M_r \ll 1$. Its solution gave also a relation among ΔM_1, V_1, r_1 and P_1. The value of r_1, i.e., the radius of the core is well approximated by that of zero-temperature white-dwarf with mass equal to the core mass $M_r = M_1$. Therefore, if the density of the shell ρ_1 *or* the specific entropy of the shell s_1 is given, all physical quantities are known including the value of c_g^{*-1}. Along the evolutionary change s_1 is increasing when the nuclear energy generation is dominant over the heat diffusion. Then the layers overlying the burning shell expand to make H_p larger. Therefore the evolutionary sequence can be followed by decreasing the value of V_1.

5. LIMIT CYCLE OF NON-LINEAR OSCILLATION IN DISSIPATIVE SYSTEM

Sequence of events and recurrence of shell flashes are clearly understood in a plane such as Figure 2a, where the change of the temperature in the burning shell is schematically shown against its value of V_1. Numbers attached to the curve indicate specific phases or stages summarized in Table 1. According to the signs of c_g^{*-1} and $T\delta s$ the plane is divided into four regions, I - IV. Since the sign of $T\delta s$ depends on the concentration of the nuclear fuel $\underline{Y}$, the boundary between the regions shifts as a result of depletion of the nuclear fuel. Two cases of different values of $\underline{Y}$ are shown in Figure 2a. Directions of evolution are also shown by arrows.

Several remarks should be given. In Phase 5 the nuclear shell-burning is in stable equilibrium. Therefore, the burning shell should stay at the corresponding point unless there would be any change in physical parameters. In actual, helium is being depleted and the curve of $T\delta s = 0$ shifts toward the upper-right. From Stage 6 on, the diffusion is dominant over the nuclear energy generation, and the point moves *unstably* through

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Phase 8. After Stage 9 the shell is cooled down, and the accretion brings the point back to the initial stage of the histeresis.

In Figure 2b a histeresis curve is taken from a model, which is repeating thermal pulses of steady-state amplitude in the deep interior of a red giant star (Fujimoto and Sugimoto 1979a). In such a model a surface convection zone in the hydrogen-rich envelope penetrates into the helium zone during Phases 7-9. It reduces the mass of the helium zone and shifts the histeresis curve towards larger value of V_1 so that it even crosses the curve in Phase 2. During Phase 11 the mass of the helium zone turns to increase again by the hydrogen shell-burning, which makes V_1 smaller.

For clarity of discussion it is better to turn back to the idealized case of Figure 2a which would be obtained if the recurrence of the

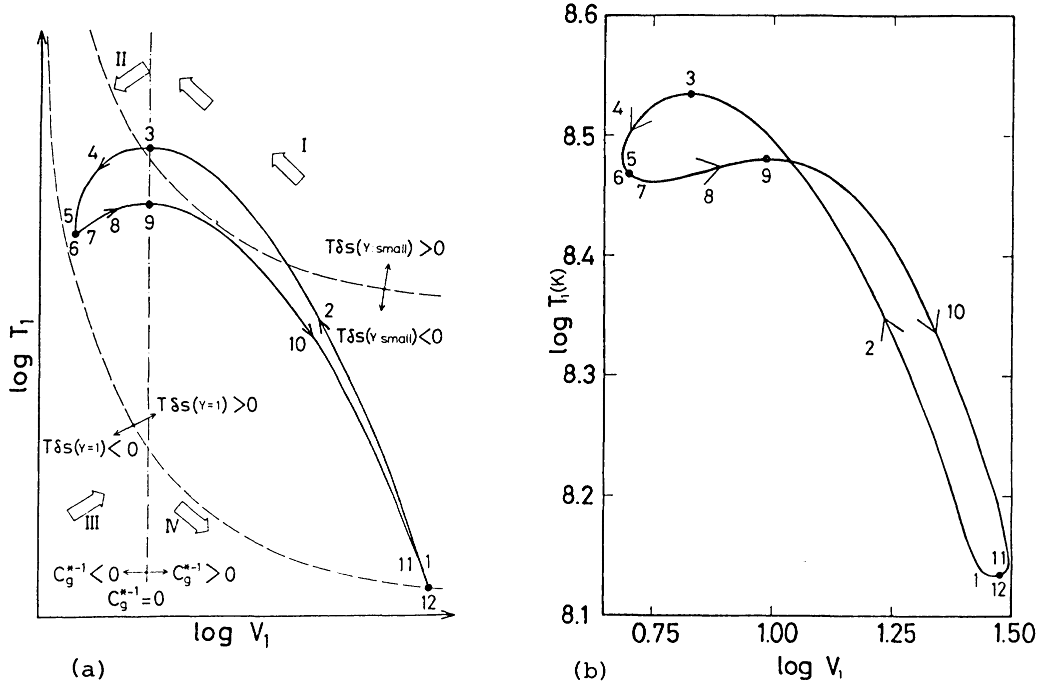

Fig. 2. Evolutionary change of the temperature in the burning shell is drawn by solid curve against V_1 thereof, which will be called histeresis curve. See the text for more details.
(a) Schematic diagram: Dash-dotted line corresponds to $c_g^{*-1} = 0$. Dashed curves correspond to $T\delta s = 0$ for $Y = 1.0$ and for small $\underline{Y}$ as indicated in the figure. These line and curve divide the plane into four regions I-IV. The boundary between the regions I/II and III/IV shifts up as nuclear fuel $\underline{Y}$ is depleted. Note that the histeresis curve encloses the point where both c_g^{*-1} and $T\delta s$ for small $\underline{Y}$ vanish simultaneously. Numbers attached along the histeresis curve are the phase/stage number in Table 1.
(b) Histeresis curve obtained for a model of helium shell-flash in red giant stars.

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hydrogen shell-flash be computed for accreting white dwarf for instance. For relatively small $\underline{Y}$, the singular point, where both c_g^{*-1} and $T\delta s$ vanish, lies inside the histeresis curve. This is an unstable singular point. Therefore the recurrent shell flash meets with the necessary condition for a limit cycle of non-linear oscillation in dissipative system. Actually Fujimoto and Sugimoto (1979a) obtained two types of solutions. When the initial model is taken inside (outside) of the histeresis curve of the steady state, the peak of the shell flash becomes stronger (weaker), i.e., the histeresis curve is diverging (converging), as it recurs. Then it tends to the limit cycle that they called *steady-state solution*.

6. TWO BRANCHES IN THE SOLUTIONS OF STELLAR STRUCTURE

As seen in Table 1 and Figure 2a, there are two typical stages which are in thermal equilibrium. They are Stage/Phase 1 and 5 which are in unstable and stable equilibrium, respectively. This implies that there are two solutions for given values of the envelope mass ΔM_1 and the core mass M_1. We can consider a linear series of thermal equilibrium solutions with the total mass $M = M_1 + \Delta M_1$ fixed as illustrated in Figure 3. The parameter of the series is either M_1 or ΔM_1. The two equilibrium solutions cited above imply that there are two branches in the linear series, which we shall call white-dwarf branch corresponding to Stage/Phase 1 and red-giant branch corresponding to 5. (Such situation has been known long since in relation with evolution off the red-giant branch of the globular cluster into a hot white dwarf, which takes place when almost all of hydrogen in the envelope is exhausted.) In this context the shell flash is understood to be a phenomenon in which the star makes a transition from the white-dwarf branch to the red-giant branch by removing the condition of thermal equilibrium (see Figure 3). Then, the effects of cooling and of accretion bring about a complete transition back to the white-dwarf branch through the interpulse phase (Phases 6-12).

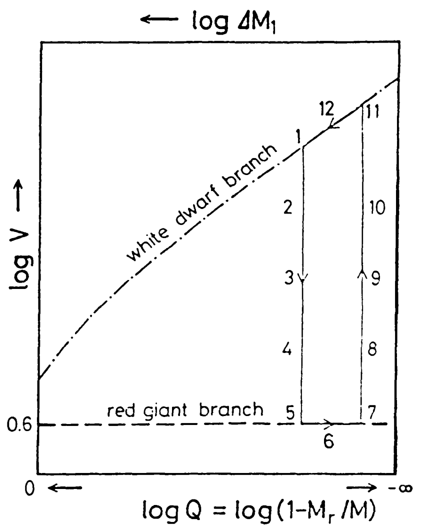

Fig. 3. Linear series of stellar structure in which an electron-degenerate core is surrounded by a relatively thin envelope. There are two branches of solutions as indicated in the figure. A cycle of thermal pulse is superposed by solid curve. See the text for more details.

Even for practical purpose such understanding is important as will be discussed below. In the white-dwarf branch the configuration in the envelope is almost plane-parallel with large values of $\underline{V}$. In other words the pressure at the bottom of the envelope is close to the weight per unit area of the overlying layers, i.e., $f \simeq 1$, in the expression

$$P = f(GM_1/r_1^2)(\Delta M/4\pi r_1^2). \quad (12)$$

Therefore c_g^* is positive and the shell burning is unstable.

On the contrary the red-giant branch is characterized by $f \ll 1$. It can be stated in another expression: In the bottom of the envelope, i.e., in the burning shell, mass contained in unit scale height of pressure $\Delta M(H_p)$ is much smaller than the mass contained in the whole envelope, since equation (12) is rewritten as

$$\Delta M(H_p) = -(dM_r/d\ln P)_1 = f\Delta M. \quad (13)$$

Therefore, the pressure at the bottom of the envelope is much lower than the weight of the overlying cone. The configuration is in spherical symmetry rather than plane-parallel and thus c_g^* is negative and the shell burning is stable.

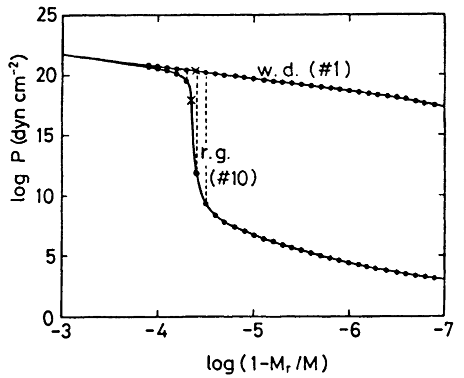

Fig. 4. Pressure distributions in solutions of white-dwarf branch (#1) and of red-giant branch (#10). Note that there is a sharp drop in pressure in the red-giant solution. See the text for more details.

How small is the value of f in Phases 5-7? It depends upon the height of the peak in the shell flash. In a typical helium shell-flash depicted in Figure 2b (Fujimoto and Sugimoto 1979a) the peak temperature reaches $T = 3.3 \times 10^8$ K (Stage 3) and f decreases down to 0.30 (Stage 5). In a nova model of accreting white dwarf of 1.3 $M_\odot$ with normal abundance of the CNO elements (Nariai et al. 1980), the peak temperature becomes as high as $T = 3.5 \times 10^8$ K even for the *hydrogen* shell-burning and f becomes as small as 0.0055. In order to compute such stellar structure we need mesh points 180 times finer than for a white-dwarf model.

It is instructive to illustrate solutions of the nova model discussed above. Their pressure distributions are shown in Figure 4 for solutions in the white-dwarf branch (#1) and in the red-giant branch (#10). As for the latter we see that the pressure drops very sharply in the region just outside the hydrogen-burning shell. This was computed by using mesh pints with the separation even as fine as $\Delta\ln(1-M_r/M) = 0.001$ near the hydrogen-burning shell. In another typical nova model, Sparks et al. (1978) used mesh points with $\Delta\ln(1-M_r/M) = 0.223$. (In other works mesh points are not described.) Their mesh points are superposed on Figure 4. As seen in this figure they are fine enough for models near the white-dwarf branch, while they would be too coarse for models in the red-giant branch if such a solution were the case.

Here we should give two warnings to those computing models near the red-giant branch. First, the fine mesh points are necessary, in particular, near the bottom of the envelope. Second, if one has used a relatively coarse mesh points and obtained a relatively slow decrease in pressure, it does not always mean that the accurate solution is actually of the type having such a slow decrease in pressure. Such nature of the red-giant envelope was relatively well known before the computer age, but it has been almost forgotten and now proper attention is not being paid for.

7. STEADY STATE SOLUTION AND RECURRENCE OF THE FLASH

In numerical computations of thermal pulses, the pulse height grows or decays pulse by pulse. However, the discussion in section 5 suggests that the pulse height corresponding to the limit cycle will have a limiting amplitude. The change in the pulse height results from a change in the thermal structure (distribution of temperature) near the outer edge of the core. Therefore the limiting amplitude will result from the model which are thermally well relaxed before the onset of the flash.

It is convenient to introduce a variable of mass fraction

$$q \equiv M_r/M(t) \ , \tag{14}$$

where the total mass M(t) increases in time at the accretion rate. Then every quantity can be rewritten as a function of (q, t) instead of (M_r, t). In particular, equation (4) is divided into homologous term $\varepsilon_g^{(h)}$ and non-homologous term $\varepsilon_g^{(nh)}$, i.e.,

$$\varepsilon_g = \varepsilon_g^{(h)} + \varepsilon_g^{(nh)} \ , \tag{15}$$

$$\varepsilon_g^{(h)} \equiv \frac{d \ln M(t)}{dt} \, T\left(\frac{\partial s}{\partial \ln q}\right)_t \ , \quad \varepsilon_g^{(nh)} \equiv -T\left(\frac{\partial s}{\partial t}\right)_q \ . \tag{16}$$

Before the onset of the flash there is a long quiescent phase of accretion. In this phase the heat loss by diffusion dL_r/dM_r in equation (3) is balanced by $\varepsilon_g^{(h)}$, i.e., by the addition of entropy due to the inward propagation of a Lagrangian shell in q-coordinate. [Though $\varepsilon_g^{(nh)}$ is essential during the flash, the energy released by the flash diffuses out in a relatively early stages of the accretion phase.]

If we neglect $\varepsilon_g^{(nh)}$, the differential equations of stellar structure do not contain the time coordinate explicitly any more. Therefore they are reduced to ordinary differential equations. When the accretion rate and the envelope mass ΔM_1 are specified, they are solved as a boundary value problem. We obtain a higher temperature at the bottom of the envelope for a larger ΔM_1. For $\Delta M_1 = \Delta M_{ig}$ the temperature is just high enough to ignite unstable nuclear shell burning. As a result the value of ΔM_{ig} is obtained for each set of the accretion rate and the core mass.

Fujimoto and Sugimoto (1979a) computed such steady-state solution for the helium shell-flash in the deep interior of the red giant star and showed also that such steady-state solution yields the limit cycle of thermal pulses as discussed in section 5. Nariai and Nomoto (1979) computed such steady-state solution extensively for accreting helium white dwarfs. The recurrence period of the flash is obtained by

$$\tau_{rec} = \Delta M_{ig}/(dM/dt) \ . \qquad (17)$$

As seen in Figure 1 of their paper ΔM_{ig} is smaller and thus τ_{rec} is still shorter for higher accretion rate. The smaller ΔM_{ig} yields the weaker flash (Sugimoto et al. 1979). Thus such model explains the tendency existing between ordinary novae (with very long τ_{rec}) and recurrent novae.

It is appropriate to note here about the role of mass flow in the (q, t)-coordinate for the limit cycle. If we describe equations (9) and (10) at constant $\underline{q}$ instead of constant M_r, then $\varepsilon_g^{(h)}$ and $\delta\varepsilon_g^{(h)}$ are to be included, respectively, in the right hand sides of these equations. Then $(T\,\delta\, s)_{q,0} = 0$ becomes the steady state with $\varepsilon_g^{(nh)} = 0$ instead of the thermal equilibrium state with $\varepsilon_g = 0$. Though it will be somewhat more complicated, such a formulation will lead to a more precise description of the limit cycle, especially for stages ii through 1. However, it is out of the scope of this brief review.

8. GENERAL FEATURE OF SHELL FLASH OF ACCRETING WHITE DWARFS

From the discussions in the preceding sections we can make the following statements concerning the shell flashes. We shall summarize here only those which have well analytical basis. In some papers do appear results of numerical computations and statements which are in contradiction with our statements. In such cases numerical computations and/or their interpretations should be examined more closely.
1) Mass contained in the envelope ΔM_{ig} is determined by the thermal history of accretion, i.e., by the mass of the white dwarf M_1, the accretion rate dM/dt, and the thermal structure of the white dwarf at the onset of the accretion.
2) ΔM_{ig} is larger for smaller M_1, for smaller dM/dt and for more cooled white dwarf in advance of the onset of the accretion.
3) After many cycles of recurrent shell flashes, it reaches a limit cycle of non-linear oscillation in dissipative system. In such a limit ΔM_{ig} is determined only by M_1 and dM/dt.
4) When the flash grows up to a finite amplitude, heat diffusion becomes negligible. Then further development of the flash is described only by the amount of nuclear energy thus far released. Therefore, the peak temperature, for instance, is determined only by the values of ΔM_{ig} and M_1. (Detailed expression of the nuclear reaction and its absolute value determine only the timescale to reach the peak temperature.) The peak temperature is higher for larger M_1 and for larger ΔM_{ig} because the pressure at the bottom of the envelope is higher.

In the statements above any dynamical effects have not been taken into account though expected in some cases of strongest flashes.

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DISCUSSION

Kippenhahn: If I remember the history correctly, Härm & Schwarzschild first found the instability numerically and explained it as a thermal runaway. It was Weigert who first found that it is a recurrent phenomenon. In these thermal pulses, convective regions come and go and change the chemical composition by mixing. From the theory which you just described, one gets a very good insight into the mechanism which drives the pulses but the theory does take into account the convective mixing. For practical application, one would like to know how the thermal pulses influence the evolution. One can do that by computing through thousands of thermal pulses. Is there some hope that analytical work could help us to predict the cumulative effects of many pulses (including the accompanying mixing effects) without following the evolution using a computer pulse by pulse?

Sugimoto: Your description of the history is right. Concerning the effects of convection, there are two points. During the stages where the convection zone is confined within the helium layer, for example, there is no additional problem other than just specifying the polytropic index to be the adiabatic one. You are talking rather of the effect of mixing between the helium layer and the hydrogen-rich envelope. It may take place in two different modes, depending on the conditions. Firstly, convection in the helium zone penetrates up into the hydrogen-rich envelope just after the peak of the flash. Secondly, the surface convection zone penetrates down into the helium layer in the early part of the interpulse accretion phase. When each takes place, the mass of the helium layer changes and the histeresis curve changes somewhat, as illustrated in the text. When we compute such things numerically for the limit cycle, however, we obtain the same amount of mixing for each consecutive pulse. Therefore, we can easily compute the behavior which should result after thousands of pulses. We can estimate the influence of the pulse without following the evolution pulse by pulse using a computer.

Wheeler: Have you considered the effect on the shell-flash problem of the difference between the Ledoux and the Schwarzschild criteria for convective instability? I believe Barkat's group in Jerusalem has found that if the Ledoux criterion is applied, the hydrogen shell ignites in the convective envelope, not in a radiative zone, and there are no shell flashes at all.

Sugimoto: As I have discussed, hydrogen shell-burning is always stable for the red-giant structure because of the spherical geometry and the negative gravothermal specific heat. The stability of helium shell-burning should have almost nothing to do with the structure of the hydrogen-rich envelope and thus almost nothing to do with the stability criteria for convective instability. The solutions obtained by Barkat's group deserve further investigation. If their numerical results and, in particular, their U-V curves are shown, we can diagnose at once the reason for the behavior they have found.