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FUNCTIONS INVARIANT UNDER THE BOCHNER–MARTINELLI INTEGRAL

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We give an elementary proof of the statement that a function f on the closed unit ball of $\mathbf{C}^n$, integrable on the unit sphere, is holomorphic if it is invariant under the Bochner–Martinelli integral transform.

The classical Bochner–Martinelli integral formula can be written explicitly in the unit ball B_n of $\mathbf{C}^n$. Indeed (see, for example, [**2**, 16.5.8]) if $f \in C^1(\overline{B_n})$ and $z \in B_n$, then

$$f(z) = \int_S \frac{1 - \langle \xi, z\rangle}{|\xi - z|^{2n}} f(\xi)\, d\sigma(\xi) - \frac{1}{n}\sum_{k=1}^{n}\int_{B_n} \frac{\overline{D}_k f(w)(\overline{w}_k - \overline{z}_k)}{|w - z|^{2n}} d\nu(w) \tag{1}$$

where S is the unit sphere, σ is the rotation invariant, positive Borel measure on S with $\sigma(S) = 1$ and $\langle \xi, z\rangle = \sum \xi_k \overline{z}_k$ denotes the inner product in $\mathbf{C}^n$ as we follow standard notations of [**2**]. From the formula (1), we see that if f is, in addition, holomorphic in B_n then for $z \in B_n$

$$f(z) = \int_S \frac{1 - \langle \xi, z\rangle}{|\xi - z|^{2n}} f(\xi)\, d\sigma(\xi).$$

Now, for $f \in L^1(S)$, we define an integral transform Bf on B_n by

$$(Bf)(z) = \int_S \frac{1 - \langle \xi, z\rangle}{|\xi - z|^{2n}} f(\xi)\, d\sigma(\xi) \qquad \text{for } z \in B_n \tag{2}$$

We call Bf the Bochner–Martinelli transform of f. For $n = 1$, Bf is the Cauchy integral (thus it is holomorphic), but in general not holomorphic when $n \geqslant 2$ although it is harmonic in B_n for every $f \in L^1(S)$.

In 1978 Romanov [**1**] showed, by using the iterates of the operator B, that a function $f \in C(B_n) \cap L^2(S)$ satisfies $Bf = f$ on B_n, then it is holomorphic. Here we prove, by an elementary method, a function on the closed unit ball, integrable on S, which is invariant under the Bochner-Martinelli transform is holomorphic.

THEOREM 1. *If a function f on $\overline{B_n}$ which is integrable on S satisfies*

$$f(z) = (Bf)(z) \qquad \textit{for all } z \in B_n \tag{3}$$

then f is holomorphic in B_n.

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PROOF: It is trivial for $n = 1$, thus we assume that $n \geqslant 2$. Since f is harmonic in B_n by (3), $f(z) = Pf(z)$ for all $z \in B_n$ where

$$(Pf)(z) = \int_S \frac{1 - |z|^2}{|\xi - z|^{2n}} f(\xi)\, d\sigma(\xi)$$

is the Poisson integral of f. Thus for each $z \in B_n$, we have

$$\begin{aligned} 0 &= Pf(z) - Bf(z) \\ &= \int_S \frac{\langle \xi, z\rangle - |z|^2}{|\xi - z|^{2n}} f(\xi)\, d\sigma(\xi) \\ &= \sum_k \overline{z}_k \int_S \frac{\xi_k - z_k}{|\xi - z|^{2n}} f(\xi)\, d\sigma(\xi) \qquad (4) \\ &= \sum_k \overline{z}_k \frac{\partial}{\partial \overline{z}_k} h(z) \end{aligned}$$

where

$$h(z) = \frac{1}{n-1} \int_S \frac{f(\xi)}{|\xi - z|^{2n-2}}\, d\sigma(\xi)$$

(it is the single layer potential with movement f).

Since h is real analytic (indeed it is harmonic) in B_n, the power series expression of h at the origin shows that (4) implies that h is holomorphic.

Hence the function

$$g(z) = \sum z_k \frac{\partial}{\partial z_k} h(z)$$

is also holomorphic.

Once again by the same calculation as (4), we can see that

$$g(z) = \int_S \frac{\langle z, \xi\rangle - |z|^2}{|\xi - z|^{2n}} f(\xi)\, d\sigma(\xi).$$

Therefore

$$\begin{aligned} Bf(z) &= \int_S \frac{1 - \langle \xi, z\rangle}{|\xi - z|^{2n}} f(\xi)\, d\sigma(\xi) \\ &= \int_S \left(\frac{\langle z, \xi\rangle - |z|^2}{|\xi - z|^{2n}} + \frac{|\xi - z|^2}{|\xi - z|^{2n}} \right) f(\xi)\, d\sigma(\xi) \\ &= g(z) + (n-1)h(z) \end{aligned}$$

The holomorphicity of g and h implies that Bf is holomorphic.

Since $f = Bf$, f is also holomorphic and this proves the theorem. $\square$

References

[1] A. Romanov, 'Spectral analysis of the Martinelli–Bochner operator for the ball in $\mathbf{C}^n$ and its application', *Funct. Anal. Appl.* **12** (1978), 232–234.

[2] W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$ (Springer-Verlag, Berlin, Heidelberg, New York, 1980).

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