Forum of Mathematics, Sigma (2022), **Vol. 10**:e32 1–36
doi:10.1017/fms.2022.25

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RESEARCH ARTICLE

The space $L_1(L_p)$ is primary for $1 < p < \infty$

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Received: 19 January 2022; **Revised:** 8 April 2022; **Accepted:** 19 April 2022

2020 Mathematics subject classification: *Primary* – 46B09, 46B25, 46B28, 47A68

Abstract

The classical Banach space $L_1(L_p)$ consists of measurable scalar functions f on the unit square for which

$$\|f\| = \int_0^1 \Big(\int_0^1 |f(x,y)|^p \, dy \Big)^{1/p} dx < \infty.$$

We show that $L_1(L_p)(1 < p < \infty)$ is primary, meaning that whenever $L_1(L_p) = E \oplus F$, where E and F are closed subspaces of $L_1(L_p)$, then either E or F is isomorphic to $L_1(L_p)$. More generally, we show that $L_1(X)$ is primary for a large class of rearrangement-invariant Banach function spaces.

Contents

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https://doi.org/10.1017/fms.2022.25 Published online by Cambridge University Press

1. Introduction

The decomposition of normed linear spaces into direct sums and the analysis of the associated projection operators is central to important chapters in the theory of modern and classical Banach spaces. In a seminal paper, Lindenstrauss [23] set forth an influential research program aiming at detailed investigations of complemented subspaces and operators on Banach spaces.

The main question addressed by Lindenstrauss was this: Which are the spaces X that cannot be further decomposed into two `essentially different, infinite-dimensional subspaces'? That is to say, which are the Banach spaces X that are not isomorphic to the direct sum of two infinite-dimensional spaces Y and Z, where neither Y nor Z is isomorphic to X? This condition would be satisfied if X were indecomposable: that is, for any decomposition of X into two spaces, one of them has to be finite-dimensional. Separately, such a space could be primary, meaning that for any decomposition of X into two spaces, one of them has to be isomorphic to X. The first example of an indecomposable Banach space was constructed by Gowers and Maurey [18], who also showed that their space X_{GM} is not primary – the infinite-dimensional component of $X_{\mathrm{GM}} \sim X \oplus Y$ is not isomorphic to the whole space.

While indecomposable spaces play a tremendous role [3, 18, 27] in the present-day study of nonclassical Banach spaces, a wide variety of Banach function spaces may usually be decomposed, for instance, by restriction to subsets, by taking conditional expectations, and so on. This provides the background for the program set forth by Lindenstrauss to determine the 'classical' spaces that are primary.

1.1. Background and history

The term 'classical Banach space' – while not formally defined – certainly applies to the space $C[0, 1]$ and to scalar and vector-valued Lebesgue spaces. The space of continuous functions was shown to be primary by Lindenstrauss and Pełczyński [24], who posed the corresponding problem for scalar-valued L_p spaces. Its elegant solution by Enflo via Maurey [26] introduced a ground-breaking method of proof that applies equally well to each of the L_p spaces $(1 \le p < \infty)$. Later alternative proofs were given by Alspach, Enflo and Odell [1] for L_p in the reflexive range $1 < p < \infty$ and by Enflo and Starbird [16] for L_1.

Exceptionally deep results on the decomposition of Bochner-Lebesgue spaces $L_p(X)$ are due to Capon [12, 11], who obtained that those spaces are primary in the following cases:

- X is a Banach space with a symmetric basis, and $1 \le p < \infty$.
- $X = L_q$, where $1 < q < \infty$ and $1 < p < \infty$.

This leaves the spaces $L_1(L_p)$ and $L_p(L_1)$ among the most prominent examples of classical Banach spaces for which primariness is open.

After Capon's paper [12], the focus was concentrated mostly on nonseparable Banach spaces, where Bourgain [7] developed a very flexible method based on localisation to a sequence of quantitative finite-dimensional factorisation problems. This method led to results like the primariness of $\mathcal{L}(\ell_2)$ [6] and the primariness of BMO and its predual H_1 by the third author [29].

The purpose of the present paper is to prove that $L_1(L_p)$ is primary. Our proof works equally well for real and complex-valued functions. Before we describe our work, we review in some detail the development of methods pertaining to the spaces L_p and, more broadly, to rearrangement-invariant spaces.

Projections on those spaces are studied effectively alongside the Haar system and the reproducing properties of its block bases. The methods developed for proving that a particular Lebesgue space L_p is primary may be divided into two basic classes, depending on whether the Haar system is an unconditional Schauder basis.

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In case of unconditionality, the most flexible method goes back to the work of Alspach, Enflo and Odell [1]. For a linear operator T on L_p, it yields a block basis of the Haar system $\widetilde{h}_I$ and a bounded sequence of scalars a_I forming an approximate eigensystem of T such that

$$T\widetilde{h}_I = a_I\widetilde{h}_I + \text{a small error} \tag{1}$$

and $\widetilde{h}_I$ spans a complemented copy of the space L_p. Thus, when restricted to $span\widetilde{h}_I$, the operator T acts as a bounded Haar multiplier. Since the Haar basis is unconditional, the Haar multiplier is invertible if $|a_I| > \delta$ for some $\delta > 0$.

Alspach, Enflo and Odell [1] arrive at equation (1) by ensuring that for $\varepsilon_{I,J} > 0$ sufficiently small, the following linearly ordered set of constraints holds true

$$|\langle T\widetilde{h}_I, \widetilde{h}_J\rangle| + |\langle\widetilde{h}_I, T^*\widetilde{h}_J\rangle| \le \varepsilon_{I,J} \quad \text{for} \quad I \prec J, \tag{2}$$

where the relation $\prec$ refers to the lexicographic order on the collection of dyadic intervals. Utilising that the independent $\{-1, +1\}$-valued Rademacher system $\{r_n\}$ is a weak null sequence in L_p, $(1 \le p < \infty)$, Alspach, Enflo and Odell [1] obtain, *by induction* along $\prec$, the block basis $\widetilde{h}_I$ satisfying equation (2).

The Alspach-Enflo-Odell method provides the basic model for the study of operators on function spaces in which the Haar system is unconditional; this applies in particular to rearrangement of invariant spaces in [20] and [15].

In L_1, the Haar system is a Schauder basis but fails to be unconditional. The basic methods for proving that L_1 is primary are due to Enflo via Maurey [26] on the one hand and Enflo and Starbird [16] on the other hand. For operators T on L_1, the Enflo-Maurey method yields a block basis of the Haar basis $\widetilde{h}_I$ and a bounded measurable function g, such that

$$(Tf)(t) = g(t)f(t) + \text{a small error}, \tag{3}$$

for $f \in \operatorname{span}\{\widetilde{h}_I\}$, and $\widetilde{h}_I$ spans a copy of L_1. Thus the restricted operator T acts as a bounded multiplication operator and is invertible if $|g| > \delta$ for some $\delta > 0$. The full strength of the proof by Enflo-Maurey is applied to show that the representation in equation (3) holds true.

Enflo-Maurey [26] exhibit in their proof of equation (3) a sequence of bounded scalars a_I such that

$$T\widetilde{h}_I = a_I\widetilde{h}_I + \text{a very small error}. \tag{4}$$

Since the Rademacher system $\{r_n\}$ is a weakly null sequence in L_1, equation (4) may be obtained directly by choosing a block basis for which the constraints in equation (2) and

$$|\langle T\widetilde{h}_I, \widetilde{h}_J\rangle| \le 2\varepsilon_{I,J} \quad \text{for} \quad I \ne J, \tag{5}$$

hold true. Remarkably, *until very recently* [22], eigensystem representations such as equation (4) were not exploited in the context of L^1, where the Haar system is not unconditional.

The powerful precision of L_1-constructions with dyadic martingales and block basis of the Haar system is in full display in [19] and [36]. Johnson, Maurey and Schechtman [19] determined a normalised weakly null sequence in L_1 such that each of its infinite subsequences contains in its span a block basis of the Haar system $\widetilde{h}_I$, spanning a copy of L_1. Thus L_1 fails to satisfy the unconditional subsequence property, a problem posed by Maurey and Rosenthal [28]. By contrast, Talagrand [36] constructed a dyadic martingale difference sequence $g_{n,k}$ such that neither $X = \overline{\operatorname{span}}^{L_1}\{g_{n,k}\}$ nor L_1/X contains a copy of L_1.

The investigation of complemented subspaces in Bochner Lebesgue spaces was initiated by Capon [12, 11] who pushed hard to further the development of the scalar methods and proved that $L_p(X)(1 \le p < \infty)$ is primary when X is a Banach space with a symmetric basis, say (x_k). Specifically, Capon [12] showed that for an operator T on $L_p(X)$, there exists a block basis of the Haar basis $\widetilde{h}_I$, a subsequence

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of the symmetric basis (x_{k_n}) and a bounded measurable g such that

$$(T(f \otimes x_{k_n}))(t) = g(t)f(t) \otimes x_{k_n} + \text{a small error},$$

for $f \in \operatorname{span}\{\widetilde{h}_I\}$. Thus on $\operatorname{span}\{\widetilde{h}_I\} \otimes \operatorname{span}\{x_{k_n}\}$ the operator T acts like $M_g \otimes Id$, where M_g is the multiplication operator induced by g. Simultaneously, Capon shows that the tensor products form an approximate eigensystem,

$$T(\widetilde{h}_I \otimes x_{k_n}) = a_I \widetilde{h}_I \otimes x_{k_n} + \text{a small error}$$

where a_I is a bounded sequence of scalars and $\widetilde{h}_I$ spans a copy of L_p.

In the mixed norm space $L_p(L_q)$ where $1 < q < \infty$ and $1 < p < \infty$, the biparameter Haar system forms an unconditional basis. Displaying extraordinary combinatorial strength, Capon [11] exhibited a so-called local product block basis $k_{I\times J}$ spanning a complemented copy of $L_p(L_q)$ such that

$$Tk_{I\times J} = a_{I\times J} k_{I\times J} + \text{a small error}.$$

1.2. The present paper

Now we describe the main ideas in the approach of the present paper.

Introducing a transitive relation between operators S, T on a Banach space X, we say that T is a projectional factor of S if there exist transfer operators $A, B \colon X \to X$ such that

$$S = ATB \quad \text{and} \quad BA = Id_X. \tag{6}$$

If merely $S = ATB$, without the additional constraint $BA = Id_X$, we say that T is a factor of S, or equivalently that S factors through T.

Clearly, if T is a projectional factor of S and S one of R, then T is a projectional factor of R: that is, being a projectional factor is a transitive relation. Given any operator $T : L_1(L_p) \to L_1(L_p)$, the goal is to show that either T or $Id - T$ is a factor of the identity $Id : L_1(L_p) \to L_1(L_p)$. In section 2.1, we expand on the quantitative aspects of the transitive relation in equation (6) and the role it plays in providing a step-by-step reduction of the problem, allowing for the replacement of a given operator with a simpler one that is easier to work with.

Let $T : L_1(L_p) \to L_1(L_p)$ be a bounded linear operator. It is represented by a matrix $T = (T^{I,J})$ of operators $T^{I,J} : L_1 \to L_1$, indexed by pairs of dyadic intervals (I, J): that is, on $f \in L_1(L_p)$ with Haar expansion

$$f = \sum x_J h_J / |J|^{1/p}, \qquad x_J \in L_1, \tag{7}$$

the operator T acts by

$$Tf = \sum_I \Big(\sum_J T^{I,J} x_J\Big) h_I / |I|^{1/p}. \tag{8}$$

Theorem 6.1, the main result of this paper, asserts that there exists a bounded operator $T^0 : L_1 \to L_1$ such that

$$T \text{ is a projectional factor of } T^0 \otimes Id_{L_p},$$

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meaning there exist bounded transfer operators $A, B : L_1(L_p) \to L_1(L_p)$ such that $BA = Id_{L_1(L_p)}$ and

$$\begin{array}{ccc} L_1(L_p) & \xleftarrow[A]{} & L_1(L_p) \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle T^0 \otimes Id} \\ L_1(L_p) & \xrightarrow[B]{} & L_1(L_p) \end{array} . \tag{9}$$

The ideas involved in the proof of Theorem 6.1 are based on the interplay of topological, geometric and probabilistic principles. Specifically, we build on compact families of L_1-operators, extracted from $span\{T^{I,J}\}$, and large deviation estimates for empirical processes:

(a) (Compactness.) We utilise the Semenov-Uksusov characterisation [34, 35] of Haar multipliers on L_1 and uncover compactness properties of the operators $T^{I,J} : L_1 \to L_1$. See Theorem 3.2 and Theorem 3.4.
(b) (Stabilisation.) Large deviation estimates for the empirical distribution method gave rise to a novel connection between factorisation problems on $L_1(L_p)$ and the concentration of measure phenomenon. See Lemma 5.3 and Lemma 5.4.

Step 1. We say that T is a diagonal operator if $T^{I,J} = 0$ for $I \neq J$, in which case we put $T^L = T^{L,L}$. The first step provides the reduction to diagonal operators. Specifically, Theorem 4.1 asserts that for any operator $T = (T^{I,J})$, there exists a diagonal operator $T_{\text{diag}} = (T^L)$ such that

$$T \text{ is a projectional factor of } T_{\text{diag}} = (T^L). \tag{10}$$

The reduction in equation (10) results from compactness properties for the family of L_1 operators $T^{I,J}$ established in Theorem 3.2 and Theorem 3.4. Specifically, if $f \in L_1$, then the set

$$\{T^{I,J} f : I, J \in \mathcal{D}\} \subset L_1 \quad \text{is weakly relatively compact;} \tag{11}$$

if, moreover, $T^{I,J}$ satisfies uniform off-diagonal estimates

$$\sup_{I,J} |\langle T^{I,J} h_L, h_M \rangle| < \varepsilon_{L,M}, \quad \text{for} \quad L \neq M, \tag{12}$$

then, for $\eta > 0$, there exists a stopping time collection of dyadic intervals $\mathcal{A}$ satisfying $|\limsup \mathcal{A}| > 1 - \eta$ such that the set of operators

$$\{T^{I,J} P_{\mathcal{A}} : I, J \in \mathcal{D}\} \subset L(L_1) \quad \text{is relatively norm-compact.} \tag{13}$$

Recall that $\mathcal{A} \subseteq \mathcal{D}$ is a stopping time collection if for $K, L \in \mathcal{A}$ and $J \in \mathcal{D}$, the assumption $K \subset J \subset L$ implies that $J \in \mathcal{A}$. By Theorem 2.6, the orthogonal projection

$$P_{\mathcal{A}}(f) = \sum_{I \in \mathcal{A}} \langle f, h_I \rangle h_I / |I|$$

is bounded on L_1 when $\mathcal{A}$ is a stopping time collection of dyadic intervals.

Step 2. Next we show that it suffices to prove the factorisation in equation (9) for diagonal operators satisfying uniform off-diagonal estimates. We say that $T = (R^L)$ is a reduced diagonal operator if the $R^L : L_1 \to L_1$ satisfy

$$\sup_{L} |\langle R^L h_I, h_J \rangle| < \varepsilon_{I,J}, \quad \text{for} \quad I \neq J. \tag{14}$$

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Proposition 5.6 asserts that there exists a reduced diagonal operator $T_{\text{diag}}^{\text{red}} = (R^L)$ satisfying equation (14), such that

$$T_{\text{diag}} = (T^L) \text{ is a projectional factor of } T_{\text{diag}}^{\text{red}} = (R^L). \tag{15}$$

To prove equation (15), we use the compactness properties of $T_{\text{diag}} = (T^L)$ together with measure concentration estimates [8, 33] associated to the empirical distribution method. See Lemma 5.3 and Lemma 5.4.

Step 3. Next we show that we may replace reduced diagonal operators by stable diagonal operators. We say that $T_{\text{diag}}^{\text{stbl}} = (S^L)$ is a stable diagonal operator if

$$\|S^L - S^M\| < \varepsilon_M, \tag{16}$$

for dyadic intervals M, L satisfying $L \subseteq M$, we obtain in Proposition 5.2 that for any reduced diagonal operator $T_{\text{diag}}^{\text{red}}$, there exists a stable diagonal operator $T_{\text{diag}}^{\text{stbl}}$ such that

$$T_{\text{diag}}^{\text{red}} = (R^L) \text{ is a projectional factor of } T_{\text{diag}}^{\text{stbl}} = (S^L). \tag{17}$$

We verify equation (17), exploiting again the compactness properties of $T_{\text{diag}}^{\text{red}} = (R^L)$ in tandem with the probabilistic estimates of Lemma 5.3 and Lemma 5.4.

Step 4. Proposition 6.2 provides the final step of the argument. It asserts that for any stable diagonal operator $T_{\text{diag}}^{\text{stbl}} = (S^L)$, there exists a bounded operator $T^0 : L_1 \to L_1$ such that

$$T_{\text{diag}}^{\text{stbl}} \text{ is a projectional factor of } T^0 \otimes Id_X. \tag{18}$$

To prove equation (18), we set up a telescoping chain of operators connecting any of the S^L to $S^{[0,1]}$ and invoke the stability estimates in equation (16) available for the operators S^I when $L \subset I \subset [0, 1]$. Thus we may finally take $T^0 = S^{[0,1]}$.

Step 5. Retracing our steps, taking into account that the notion of projectional factors forms a transitive relation, yields equation (9).

2. Preliminaries

2.1. Factors and projectional factors up to approximation

A common strategy in proving the primariness of spaces such as L_p is to study the behaviour of a bounded linear operator on a σ-subalgebra on a subset of $[0, 1)$ of positive measure. This process may have to be repeated several times. We introduce some language that will make this process notationally easier.

Definition 2.1. Let X be a Banach space, $T, S : X \to X$ be bounded linear operators and $C \geq 1$, $\varepsilon \geq 0$.

(a) We say that *T is a C-factor of S with error ε* if there exist $A, B : X \to X$ with $\|BTA - S\| \leq \varepsilon$ and $\|A\|\|B\| \leq C$. We may also say that *S C-factors through T with error ε*.
(b) We say that *T is a C-projectional factor of S with error ε* if there exists a complemented subspace Y of X that is isomorphic to X with associated projection and isomorphism $P, A : X \to Y$ (i.e., $A^{-1}PA$ is the identity on X), so that $\|A^{-1}PTA - S\| \leq \varepsilon$ and $\|A\|\|A^{-1}P\| \leq C$. We may also say that *S C-projectionally factors through T with error ε*.

When the error is $\varepsilon = 0$, we will simply say that T is a C-factor or C-projectional factor of S.

Remark 2.2. If T is a C-projectional factor of S with error ε, then $I - T$ is a C-projectional factor of $I - S$ with error ε. Indeed, if P and A are as in Definition (b), then $PA = A$ and therefore $A^{-1}P(I - T)A = I - A^{-1}PTA$: that is, $\|A^{-1}P(I - T)A - (I - S)\| \leq \varepsilon$.

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In a certain sense, being an approximate factor or projectional factor is a transitive property.

Proposition 2.3. *Let X be a Banach space and $R, S, T : X \to X$ be bounded linear operators.*

(a) *If T is a C-factor of S with error ε and S is a D-factor of R with error δ, then T is a CD-factor of R with error $D\varepsilon + \delta$.*
(b) *If T is a C-projectional factor of S with error ε and S is a D-projectional factor of R with error δ, then T is a CD-projectional factor or R with error $D\varepsilon + \delta$.*

Proof. The first statement is straightforward, and thus we only provide a proof of the second one. Let Y and Z be complemented subspaces of X, which are isomorphic to X. Let $P : X \to Y$ and $Q : X \to Z$ be the associated projections, and $A : X \to Y$ and $B : X \to Z$ the associated isomorphisms satisfying $\|A\|\|A^{-1}P\| \leq C$, $\|B\|\|B^{-1}Q\| \leq D$. $\|A^{-1}PT - S\| \leq \varepsilon$ and $\|B^{-1}QSB - R\| \leq \delta$.

We define $\tilde{P} = AQA^{-1}P$ and $\tilde{A} = AB$. Then $\tilde{P}$ is a projection onto $\tilde{A}[X]$ and $\|\tilde{P}\|\|\tilde{A}^{-1}P\| \leq CD$. We obtain

$$\|B^{-1}Q(A^{-1}PTA)B - B^{-1}QSB\| \leq \|B^{-1}Q\|\|B\|\|A^{-1}PTA - S\| \leq D\varepsilon$$

and thus $\|B^{-1}QA^{-1}PTAB - R\| \leq D\varepsilon + \delta$. Finally, observe that

$$\tilde{A}^{-1}\tilde{P} = B^{-1}A^{-1}AQA^{-1}P = B^{-1}QA^{-1}P$$

and thus $\|\tilde{A}^{-1}\tilde{P}T\tilde{A} - R\| \leq D\varepsilon + \delta$. ◻

The following explains the relation between primariness and approximate projectional factors.

Proposition 2.4. *Let X be a Banach space that satisfies Pełczyński's accordion property: that is, for some $1 \leq p \leq \infty$, we have that $X \simeq \ell_p(X)$. Assume that there exist $C \geq 1$ and $0 < \varepsilon < 1/2$ so that every bounded linear operator $T : X \to X$ is a C-projectional factor with error ε of a scalar operator: that is, a scalar multiple of the identity. Then for every bounded linear operator $T : X \to X$, the identity $2C/(1 - 2\varepsilon)$ factors through either T or $I - T$. In particular, X is primary.*

Proof. Let Y be a subspace of X that is isomorphic to X and complemented in X, with associated projection and isomorphism $P, A : X \to Y$, so that $\|A^{-1}P\|\|A\| \leq C$ and so that there exists a scalar λ with $\|(A^{-1}P)TA - \lambda I\| \leq \varepsilon$. If $|\lambda| \geq 1/2$, then

$$\Big\|\underbrace{\lambda^{-1}A^{-1}PTA}_{=:B} - I\Big\| \leq 2\varepsilon < 1$$

and thus B^{-1} exists with $\|B^{-1}\| \leq 1/(1 - 2\varepsilon)$. We obtain that if $S = B^{-1}\lambda^{-1}A^{-1}P$, then $STA = I$ and $\|S\|\|A\| \leq 2C/(1-2\varepsilon)$. If, on the other hand $|\lambda| < 1/2$, then, because $\|A^{-1}P(I-T)A - (1-\lambda)I\| \leq \varepsilon$, we achieve the same conclusion for $I - T$ instead of T.

If $X = Y \oplus Z$ and $Q : X \to Y$ is a projection, then we deduce that either Y or Z contains a complemented subspace isomorphic to X. To see that we can assume that for some scalar λ, with $|\lambda| \geq 1/2$, Q is a C-projectional factor with error $\varepsilon \in (0, 1/2)$ of λI. Otherwise, we replace Q by $I - Q$. From what we have proved so far, we deduce that there are operators $S, A : X \to X$ so that $SQA = I$. Then $W = QA(X)$ is a subspace of Y that is isomorphic to X. It is also complemented via the projection $R = (S|_W)^{-1}S : X \to W$. So we obtain that Y is a complemented subspace of X and X is isomorphic to complemented subspace of Y. Since in addition X satisfies the accordion property, it follows from Pełczyński's famous classical argument from [30] that $X \simeq Y$. Similarly, if $(I - Q)$ is a factor of the identity, we deduce $X \simeq Z$. ◻

At this point, it is appropriate to point out that the above proposition applies to the space $L_1(X)$ for any Banach space X. Indeed, $L_1(X)$ is isomorphic to an ℓ_1 sum of infinitely many copies of itself (see, e.g., [41, Example 22(a), page 44]).

2.2. The Haar system in L_1

We denote by L_1 the space of all (equivalence classes of) integrable scalar functions f with domain $[0, 1)$ endowed with the norm $\|f\|_1 = \int_0^1 |f(s)|ds$. We will denote the Lebesgue measure of a measurable subset A of $[0, 1)$ by $|A|$.

We denote by $\mathcal{D}$ the collection of all dyadic intervals in $[0, 1)$, namely

$$\mathcal{D} = \left\{\left[\frac{i-1}{2^j}, \frac{i}{2^j}\right) : j \in \mathbb{N} \cup \{0\}, 1 \le i \le 2^j\right\}.$$

We define the bijective function $\iota : \mathcal{D} \to \{2, 3, \ldots\}$ by

$$\left[\frac{i-1}{2^j}, \frac{i}{2^j}\right) \overset{\iota}{\mapsto} 2^j + i.$$

The function ι defines a linear order on $\mathcal{D}$. We recall the definition of the Haar system $(h_I)_{I\in\mathcal{D}}$. For $I = [(i-1)/2^j, i/2^j) \in \mathcal{D}$, we define $I^+, I^- \in \mathcal{D}$ as follows: $I^+ = [(i-1)/2^j, (2i-1)/2^{j+1})$, $I^- = [(2i-1)/2^{j+1}, i/2^j)$, and

$$h_I = \chi_{I^+} - \chi_{I^-}.$$

We additionally define $h_\emptyset = \chi_{[0,1)}$ and $\mathcal{D}^+ = \mathcal{D} \cup \{\emptyset\}$. We also define $\iota(\emptyset) = 1$. Then $(h_I)_{I\in\mathcal{D}^+}$ is a monotone Schauder basis of L_1, with the linear order induced by ι. Henceforth, whenever we write $\sum_{I\in\mathcal{D}^+}$, we will always mean the sum is taken with this linear order ι.

For each $n \in \mathbb{N} \cup \{0\}$, we define

$$\mathcal{D}_n = \{I \in \mathcal{D} : |I| = 2^{-n}\} \text{ and } \mathcal{D}^n = \{\emptyset\} \cup (\cup_{k=0}^n \mathcal{D}_k).$$

An important realisation that will be used multiple times in the sequel is the following. Let $I \in \mathcal{D}$. Then there exist a unique $k_0 \in \mathbb{N}$ and a unique decreasing sequence of intervals $(I_k)_{k=0}^{k_0}$ in $(\mathcal{D}^+)$ so that $I_0 = \emptyset$, $I_1 = [0, 1)$ and $I_{k_0} = I$; and for $k = 1, 2, \ldots, k_{0-1}$, $I_{k+1} = I_k^+$ or $I_{k+1} = I_k^-$. In other words, $(I_k)_{k=1}^{k_0}$ consists of all elements of $\mathcal{D}^+$ that contain I, decreasingly ordered. For $k = 1, 2, \ldots, k_0 - 1$, put $\theta_k = 1$, if $I_{k+1} = I_k^+$ and $\theta_k = -1$ if $I_{k+1} = I_k^-$. We then have the following formula, already discovered by Haar:

$$|I|^{-1}\chi_I = |I_{k_0}|^{-1}\chi_{I_{k_0}} = h_{I_0} + \sum_{k=1}^{k_0-1} \theta_k |I_k|^{-1} h_{I_k}. \tag{19}$$

Note that in the above representation, if we define $I_{k_0} = I$, then $I_k = I_{k-1}^-$ or $I_k = I_{k-1}^+$ for $k = 2, \ldots, k_0$. To simplify notation, we will henceforth make the convention $\theta_0 = 1$ and $|I_0|^{-1} = |\emptyset|^{-1} = 1$ to be able to write

$$|I_{k_0}|^{-1}\chi_{I_{k_0}} = \sum_{k=0}^{k_0-1} \theta_k |I_k|^{-1} h_{I_k}. \tag{20}$$

This representation will be used multiple times in this paper.

A relevant definition is that of $[\mathcal{D}^+]$, the collection of all sequences $(I_k)_{k=0}^\infty$ in $\mathcal{D}^+$ so that $I_0 = \emptyset$, $I_1 = [0, 1)$, and for each $k \in \mathbb{N}$, $I_{k+1} = I_k^+$ or $I_{k+1} = I_k^-$. Note that for $(I_k)_{k=0}^\infty \in [\mathcal{D}^+]$ and $k \in \mathbb{N}$, $I_k \in \mathcal{D}_{k-1}$. Each $(I_k)_{k=0}^\infty$ defines a sequence $(\theta_k)_{k=1}^\infty$ as described in the paragraph above. This yields a bijection between $[\mathcal{D}^+]$ and $\{-1, 1\}^{\mathbb{N}}$. This fact will be used more than once. On $\{-1, 1\}^{\mathbb{N}}$, we will consider the product of the uniform distribution on $\{-1, 1\}$, which via this bijection generates a probability on $[\mathcal{D}^+]$, which we will also denote by $|\cdot|$. Also, we consider on $[\mathcal{D}^+]$ the image topology of the product of the discrete topology on $\{-1, 1\}$ via that bijection.

2.3. *Haar multipliers on* L_1

A Haar multiplier is a linear map D, defined on the linear span of the Haar system, for which every Haar vector h_I is an eigenvector with eigenvalue a_I. We denote the space of bounded Haar multipliers $D : L_1 \to L_1$ by $\mathcal{L}_{HM}(L_1)$. In this subsection, we recall a formula for the norm of a Haar multiplier that was observed by Semenov and Uksusov in [34, 35]. We then use Haar multipliers to sketch a proof of the fact that every bounded linear operator on L_1 is an approximate 1-projectional factor of a scalar operator.

This recent formula of Semenov and Uksusov is a very elegant characterisation of boundedness on Haar multipliers on L_1. In that spirit, Girardi studied related operators on L_p and $L_p(X)$ of a multiplier type [17]. Wark has since simplified the proof of Semenov-Uksusov [38] as well as extended the formula to the vector-valued case [39, 40].

Proposition 2.5. *Let* $(I_k)_{k=0}^\infty \in [\mathcal{D}^+]$ *be associated to* $(\theta_k)_{k=1}^\infty \in \{-1, 1\}^{\mathbb{N}}$. *For* $k \in \mathbb{N}$ *define* $B_k = I_k \setminus I_{k+1}$, *and let* $(a_k)_{k=0}^n$ *be a sequence of scalars.*

Then we have

$$\frac{1}{3}\Big(\sum_{k=1}^n |a_k - a_{k-1}| + |a_n|\Big) \le \Big\|\sum_{k=0}^n a_k\theta_k |I_k|^{-1} h_{I_k}\Big\|_{L_1} \le \sum_{k=1}^n |a_k - a_{k-1}| + |a_n|, \tag{21}$$

and for any $1 \le m < n$,

$$\Big\|\Big(\sum_{k=0}^n a_k\theta_k |I_k|^{-1} h_{I_k}\Big)\big|_{\bigcup_{j=m}^n B_j}\Big\|_{L_1} \ge \frac{1}{3}\Big(\sum_{k=m+1}^n |a_k - a_{k-1}| + |a_n|\Big). \tag{22}$$

Proof. Note that the sequence $(B_k)_{k=1}^\infty$ is a partition of $[0, 1)$, and for $k \in \mathbb{N}$, B_k is the set in $[0, 1]$ of measure 2^{-k} on which $\theta_k h_{I_k}$ takes the value -1. Let $f = a_0 h_\emptyset + \sum_{k=1}^n \theta_k a_k |I_k|^{-1} h_{I_k}$. For $k \in \mathbb{N}$, put $b_k = a_k$ if $k \le n$ and $b_k = 0$ otherwise. For each $k \in \mathbb{N}$, the function f is constant on B_k and in fact for $s \in B_k$, we have

$$f(s) = b_0 + \sum_{j=1}^{k-1} |I_j|^{-1} b_j - |I_k|^{-1} b_k = b_0 + \sum_{j=1}^{k-1} 2^{j-1} b_j - 2^{k-1} b_k =: c_k.$$

Therefore, for any $m = 1, 2, \dots n$,

$$\Big\|f\chi_{\bigcup_{j=m}^n B_j}\Big\|_{L_1} = \sum_{k=m}^\infty |X_k|, \tag{23}$$

where for each $k \in \mathbb{N}$,

$$X_k = \frac{c_k}{2^k} = \frac{1}{2^k} b_0 + \sum_{j=1}^{k-1} \frac{2^{j-1}}{2^k} b_j - \frac{1}{2} b_k.$$

Putting $X_0 = 0$, a calculation yields that for all $k \in \mathbb{N}$,

$$X_k = \frac{1}{2} X_{k-1} + \frac{1}{2}(b_{k-1} - b_k). \tag{24}$$

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Applying the triangle inequality to equations (23) and (24), we conclude

$$\begin{aligned}\|f\|_{L_1} &= \sum_{k=m}^{\infty}|X_k| = \sum_{k=1}^{\infty}2|X_k| - |X_{k-1}|\\ &\leq \sum_{k=1}^{\infty}|2X_k - X_{k-1}| = \sum_{k=1}^{\infty}|b_k - b_{k-1}|,\end{aligned}$$

which yields the upper bound of equation (21). The lower bound is proved with a similar computation. To obtain equation (22), we deduce from equation (24)

$$\sum_{k=m+1}^{\infty}|X_k| \geq \frac{1}{2}\sum_{k=m+1}^{\infty}|b_k - b_{k-1}| - \frac{1}{2}\sum_{k=m}|X_k|$$

and therefore

$$\frac{3}{2}\sum_{k=m+1}^{\infty}|X_k| + \frac{1}{2}|X_m| \geq \frac{1}{2}\sum_{k=m+1}^{\infty}|b_k - b_{k-1}|,$$

which yields

$$\|f\chi_{\bigcup_{j=m}^{n}B_j}\|_{L_1} = \sum_{k=m}^{\infty}|X_k| \geq \sum_{k=m+1}^{\infty}|X_k| + \frac{1}{3}|X_m| \geq \frac{1}{3}\sum_{k=m+1}^{\infty}|b_k - b_{k-1}|$$

and proves equation (22). □

Theorem 2.6 (Semenov-Uksusov, [34, 35]). *Let $(a_I)_{I\in\mathcal{D}^+}$ be a collection of scalars and D be the associated Haar multiplier. Define*

$$|||D||| = \sup\Big(\sum_{k=1}^{\infty}\big|a_{I_k} - a_{I_{k-1}}\big| + \lim_k\big|a_{I_k}\big|\Big), \tag{25}$$

where the supremum is taken over all $(I_k)_{k=0}^{\infty} \in [\mathcal{D}^+]$. Then D is bounded (and thus extends to a bounded linear operator on $L_1(X)$) if and only if $|||D||| < \infty$. More precisely,

$$\|D\| \leq |||D||| \leq 3\|D\|. \tag{26}$$

Proof. By equation (19), D is always well defined on the linear span of the set $\mathcal{X} = \{|I|^{-1}\chi_I : I \in \mathcal{D}\}$. In fact, the closed convex symmetric hull of $\mathcal{X}$ is the unit ball of L_1. We deduce that $\|D\| = \sup\{\|Df\| : f \in \mathcal{X}\}$, under the convention that $\|D\| = \infty$ if and only if D is unbounded. Fix $f = |I|^{-1}\chi_I \in \mathcal{X}$. Use equation (19) to write

$$f = |I_{k_0}|^{-1}\chi_{I_{k_0}} = \sum_{k=0}^{k_0-1}\theta_k|I_k|^{-1}h_{I_k},\ that is,\ Df = \sum_{k=0}^{k_0-1}a_k\theta_k|I_k|^{-1}h_{I_k}.$$

Extend $(I_k)_{k=0}^{k_0}$ to a branch $(I_k)_{k=0}^{\infty}$. By equation (21), we have

$$\frac{1}{3}\Big(\sum_{k=1}^{k_0-1}|a_{I_k} - a_{I_{k-1}}| + |a_{I_{k_0-1}}|\Big) \leq \|Df\|_{L_1} \leq \sum_{k=1}^{k_0-1}|a_{I_k} - a_{I_{k-1}}| + |a_{I_{k_0-1}}|. \tag{27}$$

By the triangle inequality, $\|Df\|_{L_1} \leq \sum_{k=1}^{\infty}|a_{I_k} - a_{I_{k-1}}| + \lim_k|a_{I_k}| \leq |||D|||$. The lower bound is achieved by taking in equation (27) all $f \in \mathcal{X}$. □

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The following special type of Haar multiplier will appear in the sequel.

Example 2.7. Let $\mathscr{A} \subset [\mathcal{D}^+]$ be a nonempty set, and define the set $\mathcal{A} = \cup_{k_0=0}^{\infty}\{I_{k_0} : (I_k)_{k=0}^{\infty} \in \mathscr{A}\} \subset \mathcal{D}^+$. Let $P_{\mathscr{A}}$ denote the Haar multiplier that has entries $a_I = 1$ for $I \in \mathcal{A}$ and $a_I = 0$ otherwise. Then by Theorem 2.6, $\|P_{\mathscr{A}}\| \leq |||P_{\mathscr{A}}||| = 1$, and therefore $P_{\mathscr{A}}$ defines a norm-one projection onto $Y_{\mathscr{A}} = \overline{\langle\{h_I : I \in \mathcal{A}\}\rangle}$.

The following elementary remark will be useful eventually.

Remark 2.8. Let $\mathscr{A}$ be a nonempty closed subset of $[\mathcal{D}^+]$ and $\mathcal{A} = \cup_{k_0=0}^{\infty}\{I_{k_0} : (I_k)_{k=0}^{\infty} \in \mathscr{A}\}$. Let D be a Haar multiplier with entries that are zero outside $\mathcal{A}$. Then

$$|||D||| = \sup_{(I_k)_{k=0}^{\infty} \in \mathscr{A}} \Big(\sum_{k=1}^{\infty} |a_{I_k} - a_{k-1}| + \lim_k |a_{I_k}|\Big).$$

Haar multipliers provide a short path to a proof of the fact that every operator on L_1 is an approximate 1-projectional factor of a scalar operator, which in turn yields Enflo's theorem [26] that L_1 is primary.

Theorem 2.9. *The following are true in the space* L_1.

(i) *Let $D : L_1 \to L_1$ be a bounded Haar multiplier. For every $\varepsilon > 0$, D is a 1-projectional factor with error ε of a scalar operator.*
(ii) *Let $T : L_1 \to L_1$ be a bounded linear operator. For every $\varepsilon > 0$, T is a 1-projectional factor with error ε of a bounded Haar multiplier $D : L_1 \to L_1$.*

In particular, for every $\varepsilon > 0$, every bounded linear operator $T : L_1 \to L_1$ is a 1-projectional factor with error ε of a scalar operator.

We wish to provide a sketch of the proof of the above. First, we will use it at the end of the paper; and second, it provides an introduction to the basis of the methods used in the paper. Now, and numerous times in the sequel, we require the following notation and definition.

Notation. For every disjoint collection Δ of $\mathcal{D}^+$ and $\theta \in \{-1,1\}^{\Delta}$, we denote $h_{\Delta}^{\theta} = \sum_{J\in\Delta}\theta_J h_J$. If $\theta_J = 1$ for all $J \in \Delta$, we write $h_{\Delta} = \sum_{j\in\Delta} h_J$. For a finite disjoint collection Δ of $\mathcal{D}$, we denote $\Delta^* = \cup\{I : I \in \Delta\}$.

Definition 2.10. A *faithful Haar system* is a collection $(\tilde{h}_I)_{I\in\mathcal{D}^+}$ so that for each $I \in \mathcal{D}^+$, the function $\tilde{h}_I$ is of the form $\tilde{h}_I = h_{\Delta_I}^{\theta_I}$, for some finite disjoint collection Δ_I of $\mathcal{D}$, and so that

(i) $\Delta_{\emptyset}^* = \Delta_{[0,1)}^* = [0,1)$, and for each $I \in \mathcal{D}$, we have $|\Delta_I| = |I|$,
(ii) for every $I \in \mathcal{D}$, we have that $\Delta_{I^+}^* = [\tilde{h}_{\emptyset}\tilde{h}_I = 1]$ and $\Delta_{I^-}^* = [\tilde{h}_{\emptyset}\tilde{h}_I = -1]$.

Remark 2.11. It is immediate that $(\tilde{h}_{\emptyset}\tilde{h}_I)_{I\in\mathcal{D}^+}$ is distributionally equivalent to $(h_I)_{I\in\mathcal{D}^+}$. Therefore, $(\tilde{h}_I)_{I\in\mathcal{D}^+}$ is isometrically equivalent to $(h_I)_{I\in\mathcal{D}^+}$, both in L_1 and in L_∞. In particular,

$$Pf = \sum_{I\in\mathcal{D}^+}\langle\tilde{h}_I, f\rangle|I|^{-1}\tilde{h}_I$$

defines a norm-one projection onto a subspace Z of L_1 that is isometrically isomorphic to L_1. Note that unless $h_{\emptyset} = 1$, P is not a conditional expectation as $P\chi_{[0,1)} = 0$. Instead, it is of the form $Pf = \tilde{h}_{\emptyset}E(\tilde{h}_{\emptyset}f|\Sigma)$, where $\Sigma = \sigma(\tilde{h}_{\emptyset}\tilde{h}_I)_{I\in\mathcal{D}^+}$. Since $\tilde{h}_{\emptyset}$ is not Σ-measurable, it cannot be eliminated. The advantage of the notion of a faithful Haar system is that one can be constructed in every tail of the Haar system. The drawback is that it causes a slight notational burden when having to adjust for the initial function $\tilde{h}_{\emptyset}$ in several situations.

We will several times recursively construct faithful Haar systems $(\tilde{h}_I)_{I\in\mathcal{D}^+}$, which means we first choose $\tilde{h}_{\emptyset}$, second $\tilde{h}_{[0,1)}$ and then $\tilde{h}_I$, $I \in \mathcal{D}$, assuming that $\tilde{h}_J$ was chosen for all $J \in \mathcal{D}^+$ with $\iota(J) < \iota(I)$.

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Proof of Theorem 2.9. Let us sketch the proof of the first statement. Let $(a_I)_{I\in\mathcal{D}^+}$ be the entries of D. For every $I \in \mathcal{D}$, denote by Q_I the Haar multiplier that has entries 1 for all $J \subset I$ and zero for all others. Then $|||Q_I||| = 1$. First, note that for every $\varepsilon > 0$, there exists $I_0 \in \mathcal{D}^+$ so that $\left|\left|\left|DQ_{I_0} - a_{I_0}Q_{I_0}\right|\right|\right| \le \varepsilon$. Otherwise, we could easily deduce $|||D||| = \infty$. Construct a dilated and renormalised faithful Haar system $(\tilde h_I)_{I\in\mathcal{D}^+}$ with closed linear span Z in the range of Q_{I_0}, and let $P : L_1 \to Z$ be the corresponding norm-one projection and $A : L_1 \to Z$ be an onto isometry. Then $\|A^{-1}PDA - a_{I_0}I\| \le \varepsilon$.

For the second part, we will use that the Rademacher sequence $(r_n)_n$ (i.e., $r_n = \sum_{L\in\mathcal{D}_n} h_L$, for $n \in \mathbb{N}$)) is weakly null in L_1 and w^*-null in $(L_1)^* \equiv L_\infty$. Using this fact, we inductively construct a faithful Haar system $(\tilde h_I)_{I\in\mathcal{D}^+}$ so that for each $I \neq J$, we have

$$\left|\left\langle \tilde h_I, T\left(|J|^{-1}\tilde h_J\right)\right\rangle\right| \le \varepsilon_{(I,J)},$$

where $(\varepsilon_{(I,J)})_{(I,J)\in\mathcal{D}^+}$ is a prechosen collection of positive real numbers with $\sum \varepsilon_{(I,J)} \le \varepsilon$. This is done as follows. If we have chosen $\tilde h_I$ for $\iota(I) = 1, \ldots, k-1$. Let $I \in \mathcal{D}^+$ with $\iota(I) = k$, and let I_0 be the predecessor of I: that is, either $I = I_0^+$ or $I = I_0^-$. Let us assume $I = I_0^+$. We then choose the next function $\tilde h_I$ among the terms of a Rademacher sequence with support $[\tilde h_\emptyset \tilde h_{I_0} = 1]$. Denote by Z the closed linear span of $(\tilde h_I)_{I\in\mathcal{D}^+}$, and take the canonical projection $P : L_1 \to Z$ as well as the onto isometry $A : L_1 \to Z$ given by $Ah_I = \tilde h_I$. Consider the operator $S = A^{-1}PTA : L_1 \to L_1$, and note that for all $I \neq J$, we have $|\langle h_I, S(|J|^{-1}h_J)\rangle| = |\langle \tilde h_I, T(|J|^{-1}\tilde h_J)\rangle| \le \varepsilon_{(I,J)}$. It follows that the entries $a_I = \langle h_I, S(|I|^{-1}h_I)\rangle$ define a bounded Haar multiplier D and $\|S - D\| \le \varepsilon$: that is, T is a 1-projectional factor with error ε of D. □

2.4. Haar system spaces

We define Haar system spaces. These are Banach spaces of scalar function generated by the Haar system in which two functions with the same distribution have the same norm. This abstraction does not impose any notational burden on the proof of the main result. The only difference to the case $X = L_p$ is the normalisation of the Haar basis. Properties such as unconditionality of the Haar system and reflexivity of L_p are never deployed.

Definition 2.12. A *Haar system space* X is the completion of $Z = \langle\{h_L : L \in \mathcal{D}^+\}\rangle = \langle\{\chi_I : I \in \mathcal{D}\}\rangle$ under a norm $\|\cdot\|$ that satisfies the following properties:

(i) If f, g are in Z and $|f|$, $|g|$ have the same distribution, then $\|f\| = \|g\|$.
(ii) $\|\chi_{[0,1)}\| = 1$.

We denote the class of Haar system spaces by $\mathcal{H}$.

Obviously, property (ii) may be achieved by scaling the norm of a space that satisfies (i). We include it anyway for notational convenience.

An important class of spaces that satisfy Definition 2.12, according to [25, Proposition 2.c.1], are separable rearrangement-invariant function spaces on $[0, 1]$. Recall that a (nonzero) Banach space Y of measurable scalar functions on $[0, 1)$ is called *rearrangement invariant* (or, as in [31], *symmetric*) if the following conditions hold true. First, whenever $f \in Y$ and g is a measurable function with $|g| \le |f|$ a.e., then $g \in Y$ and $\|g\|_Y \le \|f\|_Y$. Second, if u, v are in Y and they have the same distribution, then $\|u\|_Y = \|v\|_Y$.

The following properties of a Haar system space X follow from elementary arguments. For completeness, we provide the proofs.

Proposition 2.13. *Let X be a Haar system space.*

(a) *For every $f \in Z = \langle\{\chi_I : I \in \mathcal{D}\}\rangle$, we have $\|f\|_{L_1} \le \|f\| \le \|f\|_{L_\infty}$. Therefore, X can be naturally identified with a space of measurable scalar functions on $[0, 1)$ and $\overline{Z}^{\|\cdot\|_{L_\infty}} \subset X \subset L_1$.*
(b) *$Z = \langle\{\chi_I : I \in \mathcal{D}\}\rangle$ naturally coincides with a subspace of X^*, and its closure $\overline{Z}$ in X^* is also a Haar system space.*

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(c) *The Haar system, in the usual linear order, is a monotone Schauder basis of X.*
(d) *For a finite union A of elements of $\mathcal{D}$, we put $\mu_A = \|\chi_A\|_X^{-1}$ and $\nu_A = \|\chi_A\|_{X^*}^{-1}$. Then $\mu_A\nu_A = |A|^{-1}$. In particular, $(\nu_L h_L, \mu_L h_L)_{L\in\mathcal{D}^+}$ is a biorthogonal system in $X^* \times X$.*
(e) *A faithful Haar system $(\widehat{h}_L)_{L\in\mathcal{D}^+}$ is isometrically equivalent to $(h_L)_{L\in\mathcal{D}^+}$. In particular, $Pf = \sum_{L\in\mathcal{D}^+}\langle \nu_L\widehat{h}_L, f\rangle \mu_L\widehat{h}_L$ defines a norm-one projection onto a subspace of X that is isometrically isomorphic to X.*

Proof. By the first condition in Definition 2.12, we have

$$\Big\|\sum_{I\in\mathcal{D}_n} a_I\chi_{\pi(I)}\Big\| = \Big\|\sum_{I\in\mathcal{D}_n} a_I\chi_I\Big\|$$

for all $n\in\mathbb{N}$, all permutations π on $\mathcal{D}_n$ and all scalar families $(a_I : I\in\mathcal{D}_n)$.

To show the first inequality in (a), let $n\in\mathbb{N}$, $f = \sum_{I\in\mathcal{D}_n} a_I\chi_I \in Z$, and let $\pi : \mathcal{D}_n \to \mathcal{D}_n$, be cyclic (i.e., $\{\pi^r(I) : r = 1, 2\dots, 2^n\} = \mathcal{D}_n$ for $I\in\mathcal{D}_n$). Then

$$\|f\| = \big\||f|\big\| \ge \frac{1}{2^n}\Big\|\sum_{r=1}^{2^n}\sum_{I\in\mathcal{D}_n}|a_I|\chi_{\pi^r(I)}\Big\| = \frac{1}{2^n}\Big|\sum_{I\in\mathcal{D}_n}|a_I|\Big| = \|f\|_{L_1}.$$

The second inequality in (a) follows from the observation that for each $n\in\mathbb{N}$, the family $(\chi_I : I\in\mathcal{D}_n)$ is 1-unconditional.

We identify each $g\in Z$ with the bounded functional x_g^*, defined by $x_g^*(f) = \int_0^1 fg$, and we denote the dual norm by $\|\cdot\|_*$. From this representation it is clear that $\|\cdot\|_*$ also satisfies the first condition in Definition 2.12. Since $\|1_{[0,1)]}\| = 1$, and since for all $f\in Z$, $\int f \le \|f\|_1 \le \|f\|$, we deduce that the second condition in Definition 2.12 holds true for the norm $\|\cdot\|_*$.

Let (h_n) be the Haar basis linearly ordered in the usual way, meaning that if $m < n$, then either $\mathrm{supp}(h_n)\subset\mathrm{supp}(h_m)$ or $\mathrm{supp}(h_n)\cap\mathrm{supp}(h_m) = \emptyset$. The claim of condition (c) follows from the fact that if $f = \sum_{j=1}^n a_jh_j \in Z$, then for any scalar a_{n+1}, the absolute values of the functions $f + a_{n+1}h_{n+1}$ and $f - a_{n+1}h_{n+1}$ have the same distribution and their average is f.

Let $n\in\mathbb{N}$ and $I\in\mathcal{D}_n$ using for $k > n$ cyclic permutations on $\{J\in\mathcal{D}_k, J\subset I\}$. We deduce that $\sup_{f\in Z,\|f\|\le 1}\int_I f$ is attained for $f = \chi_I/\|\chi_I\|$ and thus $\|\chi_I\|\cdot\|\chi_I\|_* = 2^{-n}$. Since, second, for each n, $(\chi_I : I\in\mathcal{D}_n)$ is an orthogonal family, we deduce (d).

Since faithful Haar systems have the same joint distribution, we deduce the first part of (e). Since by (b), this is also true with respect to the dual norm, we deduce the second part of (e). □

In different parts of the proof, we will require additional properties of Haar system spaces. The following class of Haar system spaces is the one for which we prove our main theorem.

Definition 2.14. We define $\mathcal{H}^*$ as the class of all Banach spaces X in $\mathcal{H}$ satisfying

(I) the Rademacher sequence $(r_n)_n$ is not equivalent to the ℓ_1-unit vector basis.

We define $\mathcal{H}^{**}$ as the class of all Banach spaces X in $\mathcal{H}$ satisfying

(II) no subsequence of the X-normalised Haar system $(\mu_Lh_L)_{L\in\mathcal{D}^+}$ is equivalent to the ℓ_1-unit vector basis.

Remark 2.15. Examples of Haar system spaces that satisfy (⋆) and (⋆⋆) are separable reflexive r.i. spaces.

We note and will use several times that (⋆) for Haar system spaces is equivalent, with the condition that the Rademacher sequence (r_n) is weakly null. To see this, first note that for any $(a_n)\in c_{00}$, any $\sigma = (\sigma_n)\subset\{\pm 1\}$ and permutation π on $\mathbb{N}$, the distribution of $\sum_{n\in\mathbb{N}} a_n\sigma_nr_{\pi(n)}$, does not depend on σ on π. It follows that (r_n) is a symmetric basic sequence in X. This implies that either r_n is equivalent to the ℓ_1 unit vector basis or it is weakly null in X. Indeed, if it is not equivalent to the unit vector basis of ℓ_1, and by symmetry no subsequence is equivalent to the ℓ_1 unit vector basis, it must by Rosenthal's

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ℓ_1 Theorem have weakly Cauchy subsequence; and thus, for some subsequence $(n_k) \subset \mathbb{N}$, the sequence $(r_{n_{2k}} - r_{n_{2k-1}} : k \in \mathbb{N})$ is weakly null. But then the sequence $(r_{n_{2k}} + r_{n_{2k-1}} : k \in \mathbb{N})$ is also weakly null, and thus $r_{n_{2k}}$ is weakly null and by symmetry (r_n) is weakly null.

2.5. *Complemented subspaces of* $L_1(X)$ *isomorphic to* $L_1(X)$

Let E, F be Banach spaces. The projective tensor product of E and F is the completion of the algebraic tensor product $E \otimes F$ under the norm

$$\|u\| = \inf\Big\{ \sum_{n=1}^{N} \|e_n\|\|f_n\| : u = \sum_{n=1}^{N} e_n \otimes f_n \Big\}. \tag{28}$$

It is well known and follows from the definition of Bochner-Lebesque spaces that for any Banach space X, $L_1 \otimes_\pi X \equiv L_1(X)$ via the identification $(f \otimes x)(s) = f(s)x$. Then $L_\infty(X^*)$ canonically embeds into $(L_1(X))^*$ via the identification $\langle u, v\rangle = \int_0^1 \langle u(s), v(s)\rangle ds$. Recall that by the definition of tensor norms, the projective tensor norm satisfies the following property we will use.

(○) For any pair of bounded linear operators $T : E \to E$ and $S : F \to F$, there exists a unique bounded linear operator $T \otimes S : E \otimes_\pi F \to E \otimes_\pi F$ with $(T \otimes S)(e \otimes f) = (Te) \otimes (Sf)$ and $\|T \otimes S\| = \|T\|\|S\|$.

The next standard statement explains one of the main features of the projective tensor product. For the sake of completeness, and because it is essential in this paper, we include the proof.

Proposition 2.16. *Let Z be a subspace of L_1 that is isometrically isomorphic to L_1 via $A : L_1 \to Z$ and 1-complemented in L_1 via $P : L_1 \to Z$. Let X be a Banach space, and let W be a subspace of X that is isometrically isomorphic to X via $B : X \to W$ and 1-complemented in X via $Q : X \to W$.*

Then the space $Z(W) = \overline{Z \otimes W}^{L_1(X)}$ coincides with $Z \otimes_\pi W$ and is isometrically isomorphic to $L_1(X)$ via $A \otimes B : L_1(X) \to Z(W)$ and 1-complemented in $L_1(X)$ via $P \otimes Q : L_1(X) \to Z(W)$.

Proof. It is immediate that $P \otimes Q$ is a norm-one projection onto $Z(W)$ and that $A \otimes B$ is a norm-one map with dense image. It also follows that $A \otimes B$ is 1-1 on $L_1 \otimes X$. One way to see this is to identify $L_1 \otimes X$ and $Z \otimes W$ with spaces of bilinear forms on $(L_1)^* \times X^*$ and $Z^* \times W^*$, respectively. To conclude that $A \otimes B$ is an isometry and that $Z(W) = Z \otimes_\pi W$, take u in $L_1 \otimes X$. Note that $v := (A \otimes B)(u)$ is in $Z \otimes W \subset L_1 \otimes X$, and write $v = \sum_{i=1}^n f_i \otimes x_i$, where $f_1, \ldots, f_n \in L_1$ and $x_1, \ldots, x_n \in X$. We will see that $\sum_{i=1}^n \|f_i\|\|x_i\| \geq \|u\|$, which will imply the conclusion by the definition of $\|v\|$. Indeed, $v = (P \otimes Q)(v) = \sum_{i=1}^n (Pf_i) \otimes (Qx_i)$ and

$$\begin{aligned} \|v\| &\geq \sum_{i=1}^n \|Pf_i\|\|Qx_i\| = \sum_{i=1}^n \|A^{-1}Pf_i\|\|B^{-1}Qx_i\| \\ &\geq \Big\| \underbrace{\sum_{i=1}^n \big(A^{-1}Pf_i\big) \otimes \big(B^{-1}Qx_i\big)}_{=:y} \Big\|. \end{aligned}$$

It is immediate that $(A \otimes B)(y) = v$ and thus $y = u$. □

The following standard example will be used often to define projectional factors of an operator $T : L_1(X) \to L_1(X)$.

Example 2.17. Let $(\widetilde{h}_I)_{I\in\mathcal{D}^+}$, $(\widehat{h}_L)_{L\in\mathcal{D}^+}$ be a faithful Haar systems, and let X be a Haar system space. Take

$$Z = \overline{\langle \widetilde{h}_I : I \in \mathcal{D}^+\rangle} \subset L_1 \text{ and } W = \overline{\langle \widehat{h}_I : I \in \mathcal{D}^+\rangle} \subset X.$$

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Then the map $P : L_1(X) \to L_1(X)$ given by

$$Pu = \sum_{I\in\mathcal{D}^+}\sum_{L\in\mathcal{D}^+} \langle \widetilde{h}_I \otimes \nu_L \widehat{h}_L, u\rangle |I|^{-1}\widetilde{h}_I \otimes \mu_L \widehat{h}_L$$

(recall that $\mu_I = \|\chi_I\|_X^{-1}$ and $\nu_L = \|\chi_L\|_{X^*}^{-1}$) is a norm-one projection onto $Z(X) = \overline{\langle \widetilde{h}_I \otimes \widehat{h}_L : I, L \in \mathcal{D}^+\rangle}$, and the map

$$A : L_1(X) \to L_1(X) \text{ given by } A(h_I \otimes h_L) = \widetilde{h}_I \otimes \widehat{h}_L$$

is a linear isometry onto $Z(X)$. Then any bounded linear operator $T : L_1(X) \to L_1(X)$ is a 1-projectional factor of $S = A^{-1}PTA : L_1(X) \to L_1(X)$, so that for all $I, J, L, M \in \mathcal{D}^+$, we have

$$\langle h_I \otimes h_L, S(h_J \otimes h_M)\rangle = \langle \widetilde{h}_I \otimes \widehat{h}_L, T(\widetilde{h}_J \otimes \widehat{h}_M)\rangle.$$

Proposition 2.18. *Let $\mathscr{A} \subset [\mathcal{D}^+]$ be a subset that has positive measure. Denote by $\mathcal{A} = \cup_{k_0=0}^{\infty}\{I_{k_0} : (I_k)_{k=0}^{\infty} \in \mathscr{A}\}$ and $Y_{\mathscr{A}} = \overline{\langle\{h_I : I \in \mathcal{A}\}\rangle}$. Then there exists a subspace Z of $Y_{\mathscr{A}}$, which is isometrically isomorphic to L_1 and 1-complemented in L_1.*

Proof. By approximating $\mathscr{A}$ in measure by closed sets from the inside, we can assume that $\mathscr{A}$ is closed. For $k \in \mathbb{N}$, let $A_k = \cup\{I : I \in \mathcal{A} \cap \mathcal{D}_k\}$ and

$$\mathscr{A}_k = \{(I_n)_{n=0}^{\infty} \in [\mathcal{D}^+] : I_k \in \mathcal{D}_k, I_k \subset A_k\},$$

that is, $\mathscr{A}_k$ is the set of all sequences $(I_n)_{n=0}^{\infty}$ in $[\mathcal{D}^+]$ such that the k'th entry I_k is a subset of A_k. Then it follows that $\mathscr{A} = \bigcap_k \mathscr{A}_k$, and letting $A = \cap_k A_k$, we deduce that

$$|A| = \lim_{k\to\infty} |A_k| = \lim_{k\to\infty} |\mathscr{A}_k| = |\mathscr{A}|.$$

But also, for any $J \notin \mathcal{A}$, we have $J \cap A = \emptyset$. It follows that for any $f \in L_1$ with $f|_{A^c} = 0$ and $J \notin \mathcal{A}$, we have $\langle h_J, f\rangle = 0$ and thus $f \in Y_{\mathscr{A}}$. In particular, the restriction operator $R_A : L_1 \to L_1$ is a 1-projection onto a subspace that is isometrically isomorphic to L_1. □

The above proposition leads to the following example, which will be useful in the sequel.

Example 2.19. Let $\mathscr{A} \subset [\mathcal{D}^+]$ be a subset that has positive measure, and let X be a Banach space. Then there exists a subspace Z of $Y_{\mathscr{A}}$ that is isometrically isomorphic to L_1 via $A : L_1 \to Z$ and 1-complemented in L_1 via $P : L_1 \to Z$. In particular, for any Banach space X, the space

$$Z(X) = \overline{Z \otimes X} \subset L_1(X)$$

is isometrically isomorphic to $L_1(X)$ via $A \otimes I : L_1(X) \to Z(X)$ and 1-complemented in $L_1(X)$ via $P \otimes I$.

2.6. Decompositions of operators on $L_1(X)$

We begin by listing further standard facts about projective tensor products. We then use these facts to associate to each bounded linear operator $T : L_1(X) \to L_1(X)$ a family of bounded linear operators on L_1. In the next section, we will study the compactness properties of this family. In later sections, we use these properties to extract information about projectional factors of the operator T.

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Let E, F be Banach spaces.

(a) For every $e_0^* \in E^*$ and $f_0^* \in F^*$, we may define the bounded linear maps $q_{(e_0^*)} : E \otimes_\pi F \to F$ and $q^{(f_0^*)} : E \otimes_\pi F \to E$ given by $q_{(e_0^*)}(e \otimes f) = e_0^*(e) f$ and $q^{(f_0^*)}(e \otimes f) = f_0^*(f) e$. Then $\|q_{(e_0^*)}\| = \|e_0^*\|$ and $\|q^{(f_0^*)}\| = \|f_0^*\|$.
(b) For every $e_0 \in E$ and $f_0 \in F$, we may define the maps $j_{(e_0)} : F \to E \otimes_\pi F$ and $j^{(f_0)} : E \to E \otimes_\pi F$ given by $j_{(e_0)} f = e_0 \otimes f$ and $j^{(f_0)} e = e \otimes f_0$. Then $\|j_{(e_0)}\| = \|e_0\|$ and $\|j^{(f_0)}\| = \|f_0\|$.
(c) For every bounded linear operator $T : E \otimes_\pi F \to E \otimes_\pi F$, $f_0^* \in F^*$, and $f_0 \in F$, the map $T^{(f_0^*, f_0)} := q^{(f_0^*)} T j^{(f_0)} : E \to E$ is the unique bounded linear map so that for all $e^* \in E^*$ and $e \in E$, we have $\langle e^*, T^{(f_0^*, f_0)} e \rangle = \langle e^* \otimes f_0^*, T(e \otimes f_0) \rangle$.
(d) For every bounded linear operator $T : E \otimes_\pi F \to E \otimes_\pi F$, $e_0^* \in E^*$ and $e_0 \in E$, the map $T_{(e_0^*, e_0)} := q_{(e_0^*)} T j_{(e_0)} : F \to F$ is the unique bounded linear map so that for all $f^* \in F^*$ and $f \in F$, we have $\langle f^*, T_{(e_0^*, e_0)} f \rangle = \langle e_0^* \otimes f^*, T(e_0 \otimes f) \rangle$.

Notation. Let X be a Haar system space. For $L \in \mathcal{D}^+$, we denote

(i) $q^L = q^{(\nu_L h_L)} : L_1(X) \to L_1$,
(ii) $j^L = j^{(\mu_L h_L)} : L_1 \to L_1(X)$, and
(iii) $P^L = j^L q^L : L_1(X) \to L_1(X)$.

Note that for any $k \in \mathbb{N}$, $\| \sum_{\{L : \iota(L) \le k\}} P^L \| = 1$. This is because this operator coincides with $I \otimes P^{[\iota \le k]}$, where $P^{[\iota \le k]} : X \to X$ is the basis projection onto $(\mu_L h_L)_{\iota(L) \le k}$ (this is easy to verify on vectors of the form $u = h_I \otimes h_L$ whose linear span is dense in $L_1(X)$). We may therefore state the following.

Remark 2.20. Let X be a Haar system space.

(i) For each $L \in \mathcal{D}^+$, P^L is a projection with image

$$Y^L = \{ f \otimes (\mu_L h_L) : f \in L_1 \}$$

that is isometrically isomorphic to L_1.
(ii) $(Y^L)_{L \in \mathcal{D}^+}$ forms a monotone Schauder decomposition of $L_1(X)$. In particular, for every $u \in L_1(X)$,

$$u = \sum_{L \in \mathcal{D}^+} P^L u = \sum_{L \in \mathcal{D}^+} (q^L u) \otimes (\mu_L h_L).$$

Thus, u admits a unique representation $u = \sum_{L \in \mathcal{D}^+} f_L \otimes (\mu_L h_L)$.

2.7. *Operators on L_1 associated to an operator on $L_1(X)$*

For a Haar system space X, we represent every bounded linear operator $T : L_1(X) \to L_1(X)$ as a matrix of operators $(T^{(L,M)})_{(L,M) \in \mathcal{D}^+}$, each of which is defined on L_1.

Notation. Let X be a Haar system space, and let $T : L_1(X) \to L_1(X)$ be a bounded linear operator. For $L, M \in \mathcal{D}^+$, we denote $T^{(L,M)} = T^{(\nu_L h_L, \mu_M h_M)}$ (recall from Proposition 2.13 that the scalars μ_M and ν_L are positive and chosen so that $\mu_M h_M$ is normalised in X^* and $\nu_L h_L$ is normalised in X), so that for every $u \in L_1(X)$, we have

$$\begin{aligned} Tu &= \sum_{L \in \mathcal{D}^+} P^L T \Big(\sum_{M \in \mathcal{D}^+} P^M u \Big) = \sum_{L \in \mathcal{D}^+} \sum_{M \in \mathcal{D}^+} j^L T^{(L,M)} q^M u \\ &= \sum_{L \in \mathcal{D}^+} \sum_{M \in \mathcal{D}^+} \Big(T^{(L,M)} (q^M u) \Big) \otimes (\mu_L h_L). \end{aligned} \tag{29}$$

For $L \in \mathcal{D}^+$, we denote $T^L = T^{(L,L)}$.

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The following type of operator is essential as it is easier to work with. A significant part of the paper shows that within the constraints of the problem under consideration, every operator $T : L_1(X) \to L_1(X)$ is a 1-projectional factor with error ε of an X-diagonal operator.

Definition 2.21. Let X be a Haar system space. A bounded linear operator $T : L_1(X) \to L_1(X)$ is called *X-diagonal* if for all $L \neq M \in \mathcal{D}^+$, $T^{(L,M)} = 0$. We then call $(T^L)_{L\in\mathcal{D}^+}$ the entries of T.

Note that T is X-diagonal if and only if for all $f \in L_1$ and $L \in \mathcal{D}^+$, we have $T(f \otimes (\mu_L h_L)) = (T^L f) \otimes (\mu_L h_L)$ if and only if for all $L \in \mathcal{D}^+$, the space Y^L is T-invariant.

Remark 2.22. If X is a Haar system space and $T : L_1(X) \to L_1(X)$ is a bounded linear operator so that $\sum_{L\neq M} \|T^{(L,M)}\| = \varepsilon < \infty$, then equation (29) yields that there exists an X-diagonal operator $\bar{T} : L_1(X) \to L_1(X)$ with entries $(T^L)_{L\in\mathcal{D}^+}$ so that $\|T - \bar{T}\| \leq \varepsilon$.

3. Compactness properties of families of operators

In this section, we extract compactness properties of families of operators associated to a $T : L_1(X) \to L_1(X)$. These results will eventually be applied to families that resemble ones of the form $(T^{(L,M)})_{(L,M)\in\mathcal{D}^+}$. The achieved compactness will be used later in a regularisation process that will allow us to extract 'nicer' operators that projectionally factor through T. We have chosen to present this section in a more abstract setting that permits more elegant statements and proofs.

3.1. *WOT-sequentially compact families*

Taking WOT-limits of certain sequences of operators of the form $T^{(x^*,x)}$ is an important component of the proof. This element was already present in the approach of Capon [11, 12].

This essential Lemma due to Rosenthal is necessary in this subsection as well as the next one. A proof can be given, for example, by induction on j for $\varepsilon = 2^{-j} \sup_n \|\xi_n\|_1$.

Lemma 3.1 ([32, Lemma 1.1]). *Let $(\xi_n)_n$ be a bounded sequence of elements of ℓ_1 and $\varepsilon > 0$. Then there exists an infinite set $N = \{n_j : j \in \mathbb{N}\} \in [\mathbb{N}]^\infty$ so that for every $j_0 \in \mathbb{N}$, we have $\sum_{j\neq j_0} |\xi_{n_{j_0}}(n_j)| \leq \varepsilon$.*

Here, WOT stands for the weak operator topology in $L_1(X)$.

Theorem 3.2. *Let X be a Banach space, $T : L_1(X) \to L_1(X)$ be a bounded linear operator and A, B be bounded subsets of X^* and X, respectively. Assume that B contains no sequence that is equivalent to the unit vector basis of ℓ_1. Then for every $f \in L_1$, the set*

$$\left\{T^{(x^*,x)} f : (x^*, x) \in A \times B\right\}$$

is a uniformly integrable (and thus weakly relatively compact) subset of L_1. In particular, every sequence in $\{T^{(x^*,x)} : (x^*, x) \in A \times B\}$ has a WOT-convergent subsequence.

Proof. The 'in particular' part follows from the separability of L_1 and the fact that the set in question if bounded by $\|T\| \sup_{(x^*,x)\in A\times B} \|x^*\|\|x\|$.

Fix a sequence $(x_n^*, x_n) \in A \times B$. Assume that $(T^{(x_n^*,x_n)} f)_n$ is not uniformly integrable. Then after passing to a subsequence, there exist $\delta > 0$ and a sequence of disjoint measurable subsets $(A_n)_n$ of $[0, 1)$ so that for all $n \in \mathbb{N}$, we have

$$\delta \leq \left| \int_{A_n} (T^{(x_n^*,x_n)} f)(s)ds \right| = \left|\langle \chi_{A_n}, T^{(x_n^*,x_n)} f\rangle\right| = \left|\langle \chi_{A_n} \otimes x_n^*, T(f \otimes x_n)\rangle\right|.$$

For every $n \in \mathbb{N}$, define the scalar sequence $\xi_n = (\xi_n(m))_m$ given by $\xi_n(m) = \langle \chi_{A_m} \otimes x_m^*, T(f \otimes x_n)\rangle$.

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Then for every $m_0 \in \mathbb{N}$, we have that for appropriate scalars $(\zeta_m)_{m=1}^N$ of modulus one,

$$\begin{aligned}\sum_{m=1}^{m_0} |\xi_n(m)| &= \Big|\Big\langle \sum_{m=1}^{m_0} \chi_{A_m} \otimes \zeta_m x_m^*, T(f \otimes x_n)\Big\rangle\Big| \\ &\leq \underbrace{\Big\| \sum_{m=1}^{m_0} \chi_{A_m} \otimes (\zeta_m x_m^*)\Big\|}_{=\max_{1\leq m\leq m_0} \|x_m^*\|} \|T\|\|f\|\|x_n\| \\ &\leq \|T\|\|f\| \sup_{(x^*,x)\in A\times B} \|x^*\|\|x\|.\end{aligned} \tag{30}$$

By Rosenthal's Lemma 3.1, there exists an infinite subset $N = \{n_j : j \in \mathbb{N}\}$ of $\mathbb{N}$ so that for all $i_0 \in \mathbb{N}$, we have $\sum_{j\neq i_0} |\xi_{n_{i_0}}(n_j)| \leq \delta/2$. After relabelling, for all $n_0 \in \mathbb{N}$, we have

$$\sum_{m\neq n_0} |\xi_{n_0}(m)| \leq \delta/2.$$

We now show that $(x_n)_n$ is equivalent to the unit vector basis of ℓ_1. Fix scalars $a_1, \ldots, a_N$. For appropriate scalars $\theta_1, \ldots, \theta_N$ of modulus 1, we have

$$\sum_{n=1}^N a_n\theta_n\langle \chi_{A_n} \otimes x_n^*, T(f \otimes x_n)\rangle \geq \delta \sum_{n=1}^N |a_n|. \tag{31}$$

Put

$$\begin{aligned}\Lambda &= \Big|\Big\langle \sum_{m=1}^{N} \chi_{A_m} \otimes (\theta_m x_m^*), T\Big(\sum_{n=1}^N f \otimes a_n x_n\Big)\Big\rangle\Big| \\ &\leq \underbrace{\Big\| \sum_{m=1}^{N} \chi_{A_m} \otimes (\theta_m x_m^*)\Big\|}_{=\max_{1\leq m\leq N} \|x_m^*\|} \|T\|\|f\|\Big\|\sum_{n=1}^N a_n x_n\Big\| \\ &\leq \|T\|\|f\| \sup_{x^*\in A} \|x^*\|\Big\|\sum_{n=1}^N a_n x_n\Big\|.\end{aligned}$$

Also,

$$\begin{aligned}\Lambda &= \Big|\sum_{n=1}^N a_n\theta_n\langle \chi_{A_n} \otimes x_n^*, T(f \otimes x_n)\rangle + \sum_{n=1}^N a_n \sum_{m\neq n} \theta_m\langle \chi_{A_m} \otimes x_m^*, T(f \otimes x_n)\rangle\Big| \\ &\geq \delta \sum_{n=1}^N |a_n| - \sum_{n=1}^N |a_n| \sum_{m\neq n} |\xi_n(m)| \geq \delta/2 \sum_{n=1}^N |a_n|.\end{aligned}$$

Thus, $\|\sum_{n=1}^N a_n x_n\| \geq c \sum_{n=1}^N |a_n|$, where $c = \delta/(2\|T\|\|f\| \sup_{x^*\in A} \|x^*\|)$. □

3.2. *Compactness in operator norm*

We discuss families that are uniformly eventually close to multipliers and how to obtain compact sets from them. This is particularly important in the sequel because compactness will be essential in achieving strong stabilisation properties of operators $T : L_1(X) \to L_1(X)$.

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Notation. For $n \in \mathbb{N}$, we denote by $P_{(\leq n)} : L_1 \to L_1$ the norm-one canonical basis projection onto $\langle\{h_I : I \in \mathcal{D}^n\}\rangle$. We also denote $P_{(>n)} = I - P_{(\leq n)}$.

Definition 3.3. A set $\mathscr{T}$ of bounded linear operators on L_1 is called *uniformly eventually close to Haar multipliers* if there exists a collection $(D_T)_{T\in\mathscr{T}}$ in $\mathcal{L}_{HM}(L_1)$ so that

$$\lim_n \sup_{T\in\mathscr{T}} \Big(\|(T - D_T)P_{(>n)}\| + \|P_{(>n)}(T - D_T)\| \Big) = 0.$$

The main result of this subsection is the first one in the paper that requires a certain amount of legwork.

Theorem 3.4 (Fundamental Lemma). *Let X be a Banach space, A, B be bounded subsets of X^* and X, respectively, and $C \subset A \times B$. Let $T : L_1(X) \to L_1(X)$ be a bounded linear operator, and assume the following:*

(i) *The set B contains no sequence that is equivalent to the unit vector basis of ℓ_1.*
(ii) *The set $\{T^{(x^*,x)} : (x^*, x) \in C\}$ is uniformly eventually close to Haar multipliers.*

Then for every $\eta > 0$, there exists a closed subset $\mathscr{A}$ of $[\mathcal{D}^+]$ with $|\mathscr{A}| > 1 - \eta$ so that the set $\{T^{(x^*,x)} P_{\mathscr{A}} : (x^*, x) \in C\}$ is relatively compact in the operator norm topology.

Remark 3.5. It is not hard to see that the unit ball of $\mathcal{L}_{HM}(L_1)$ is a compact set in the strong operator topology of L_1. In fact, this is the w^*-topology inherited by a predual of $\mathcal{L}_{HM}(L_1)$, namely Rosenthal's Stopping Time space studied by Bang and Odell in [4, 5], by Dew in [14] and by Apatsidis in [2]. The Fundamental Lemma (Theorem 3.4) states that under the right conditions, strong operator convergence yields convergence in operator norm on a big subspace of L_1. Therefore, this is a type of Egorov Theorem. We point out that some restriction on the family of operators is necessary for the conclusion to hold. If one takes, for example, $D_n = P_{(\leq n)}$, then this converges to I in the strong operator topology. But for no nonempty set of branches $\mathscr{A}$, the set $\{D_n P_{\mathscr{A}} : n \in \mathbb{N}\}$ is relatively compact in the operator norm topology.

Lemma 3.6. *Let $r > 0$, $(I_k)_{k=0}^\infty \in [\mathcal{D}^+]$ associated to $(\theta_k)_{k=1}^\infty \in \{-1, 1\}^{\mathbb{N}}$ and $(a_k^n)_{(k,n)\in(\{0\}\cup\mathbb{N})\times\mathbb{N}}$ be a collection of scalars. Assume that there exist $k_1 < \ell_1 < k_2 < \ell_2 < \cdots$ so that for each $n \in \mathbb{N}$, we have*

$$\sum_{k=k_n+1}^{\ell_n} |a_k^n - a_{k-1}^n| \geq r.$$

For every $\ell, n \in \mathbb{N}$, define $f_n^\ell = \sum_{k=0}^{\ell} a_k^n \theta_k |I_k|^{-1} h_{I_k}$. Then there exists a strictly increasing sequence of disjoint measurable subsets $(A_n)_n$ of $[0, 1)$ so that for all $n \in \mathbb{N}$ and $\ell \geq \ell_n$, we have

$$f_n^\ell(s) = f_n^{\ell_n}(s) \text{ on } A_n \text{ and } \int_{A_n} \big|f_n^{\ell_n}(s)\big| ds \geq r/3.$$

Proof. Let (B_k) be the partition of $[0, 1)$ defined by $B_k = I_k \setminus I_{k+1}$, $k \in \mathbb{N}$. We conclude from the inequality in equation (22) in Proposition 2.5 that:

(i) for every $k \leq \ell_n \leq \ell \in \mathbb{N}$ and $s \in B_k$, we have $f_n^\ell(s) = f_n^{\ell_n}(s)$ and
(ii) for every $m \leq \ell_n \in \mathbb{N}$, we have

$$\int_{\cup_{k=m}^{\ell_n} B_k} |f_n^{\ell_n}(s)| ds \geq \frac{1}{3} \sum_{k=m+1}^{\ell_n} |a_k - a_{k-1}|.$$

Put $A_n = \cup_{i=k_n}^{\ell_n} B_i$. The conclusion follows directly from (i) and (ii). □

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Proof of Theorem 3.4. Put $\mathscr{T} = \{T^{(x^*,x)} : (x^*,x) \in A \times B\}$. Take a family $(D_T)_T$ as in Definition 3.3. For each $T \in \mathscr{T}$, we have

$$\|(T - D_T)P_{(>k)}\| \leq \sup_{S \in \mathscr{T}} \left(\left\|(S - D_S)P_{(>k)}\right\|\right) = \varepsilon_k. \tag{32}$$

$$\begin{aligned}\|P_{(>k)}TP_{(\leq k)}\| &\leq \underbrace{\left\|P_{(>k)}D_T P_{(\leq k)}\right\|}_{=0} + \left\|P_{(>k)}(T - D_T)P_{(\leq k)}\right\| \\ &\leq \left\|P_{(>k)}(T - D_T)\right\| \\ &\leq \sup_{S \in \mathscr{T}} \left(\left\|P_{(>k)}(S - D_S)\right\|\right) = \delta_k. \end{aligned} \tag{33}$$

Both $(\varepsilon_k)_k$ and $(\delta_k)_k$ tend to zero. For each $T \in \mathscr{T}$, denote by $(a_I^T)_{I \in \mathcal{D}^+}$ the entries of D_T.

Claim: Fix $\sigma = (I_k)_{k=1}^{\infty} \in [\mathcal{D}^+]$ and $r > 0$. Then there exists $k_0 \in \mathbb{N}$ so that for all $T \in \mathscr{T}$, we have $\sum_{k=k_0}^{\infty} |a_{I_k}^T - a_{I_{k-1}}^T| \leq r$.

We will assume that the claim is true and proceed with the rest of the proof. For every $N, k_0, \in \mathbb{N}$, let

$$\mathcal{A}_{N,k_0} = \left\{\sigma = (I_k)_{k=1}^{\infty} \in [\mathcal{D}^+] : \sup_{T \in \mathscr{T}} \sum_{k=k_0}^{\infty} |a_{I_k}^T - a_{I_{k-1}}^T| \leq 2^{-N}\right\},$$

which is a closed subset of $[\mathcal{D}^+]$, and by the claim, we have $\cup_{k_0} \mathscr{A}_{N,k_0} = [\mathcal{D}^+]$ for all $N \in \mathbb{N}$. We may therefore pick a strictly increasing sequence of natural numbers (k_N) so that for each N, we have $|\mathscr{A}_{N,k_N}| \geq 1 - \eta/2^N$. We put $\mathscr{A} = \cap_N \mathscr{A}_{N,k_N}$, and we demonstrate that this is the desired set.

To show that $\{TP_{\mathscr{A}} : T \in \mathscr{T}\}$ is relatively compact with respect to the operator norm, we fix $\varepsilon > 0$ and $(T_n)_n$ in $\mathscr{T}$. For each $n \in \mathbb{N}$, denote $D_n = D_{T_n}$. We will find $M \in [\mathbb{N}]^{\infty}$ so that for all $n, m \in M$, we have $\|T_n P_{\mathscr{A}} - T_m P_{\mathscr{A}}\| \leq 11\varepsilon$. Fix $N \in \mathbb{N}$ so that $2^{-N} \leq \varepsilon$, $\varepsilon_{k_N} \leq \varepsilon$, and $\delta_{k_N} \leq \varepsilon$. For each $n \in \mathbb{N}$, write

$$T_n = D_n P_{(>k_N)} + \underbrace{(T_n - D_n)P_{(>k_N)}}_{=:A_n} + \underbrace{P_{(>k_N)}T_n P_{(\leq k_N)}}_{=:B_n} + \underbrace{P_{(\leq k_N)}T_n P_{(\leq k_N)}}_{=:C_n}.$$

Then we have $\|A_n\| \leq \varepsilon_{k_N} \leq \varepsilon$ and $\|B_n\| \leq \delta_{k_N} \leq \varepsilon$. By passing to a subsequence of (T_n), we may assume that for all $n, m \in \mathbb{N}$, we have (letting $a_I^n = a_I^{T_n}$)

$$\sum_{\left\{\substack{I \in \mathcal{D}^+ \\ |I| \geq 1/2^{k_N+1}}\right\}} |a_I^n - a_I^m| \leq \varepsilon. \tag{34}$$

Since the C_n are bounded elements of a finite-dimensional space, we can also assume that $\|C_n - C_m\| \leq \varepsilon$, for $m, n \in \mathbb{N}$. Therefore, for $n, m \in \mathbb{N}$, we have

$$\|T_n P_{\mathscr{A}} - T_m P_{\mathscr{A}}\| \leq \underbrace{\|D_n P_{(>k_N)} P_{\mathscr{A}} - D_m P_{(>k_N)} P_{\mathscr{A}}\|}_{=:\Lambda} + 5\varepsilon.$$

Luckily, the remaining quantity Λ is the norm of a Haar multiplier on L_1, and we know how to compute this. If for $\sigma = (I_k)_{k=0}^{\infty} \in \mathscr{A}$, we put

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$$\Lambda_\sigma = \sum_{k=k_N+1}^{\infty} |(a^n_{I_k} - a^m_{I_k}) - (a^n_{I_{k-1}} - a^m_{I_{k-1}})| + |a^n_{I_{k_N}} - a^m_{I_{k_N}}| + \lim_k |a^n_{I_k} - a^m_{I_k}|$$
$$\leq 2 \underbrace{\sum_{k=k_N+1}^{\infty} |(a^n_{I_k} - a^m_{I_k}) - (a^n_{I_{k-1}} - a^m_{I_{k-1}})|}_{\leq 2/2^N \leq 2\varepsilon} + 2 \underbrace{|a^n_{I_{k_N}} - a^m_{I_{k_N}}|}_{\leq \varepsilon} \leq 6\varepsilon.$$

Then by Remark 2.8, $\Lambda = \sup_{\sigma\in\mathscr{A}} \Lambda_\sigma$ and thus $\|T_n P_{\mathscr{A}} - T_m P_{\mathscr{A}}\| \leq 11\varepsilon$.

We now provide the required proof of the claim. We fix $\sigma = (I_k)_{k=0}^\infty$, with associated signs $(\theta_k)_{k=0}^\infty$. Let us assume that the conclusion fails. Then we may find $(T_n)_n = (T^{(x_n^*, x_n)})_n$ in $\mathscr{T}$, each T_n is associated with a D_n (each D_n has entries $(a^n_I)_{I\in\mathcal{D}^+}$) and $k_1 < \ell_1 < k_2 < \ell_2 < \cdots$ so that for all $n \in \mathbb{N}$

$$\sum_{k=k_n+1}^{\ell_n} |a^n_{I_k} - a^n_{I_{k-1}}| \geq r.$$

Pick $k_0 \in \mathbb{N}$ so that $\varepsilon_{k_0} \leq r/12$. For $k, n \in \mathbb{N}$ define $b^n_k = 0$ if $k \leq k_0$ and $b^n_k = a^n_{I_k}$ if $k > k_0$. If we additionally assume that $k_1 > k_0$, then for all $n \in \mathbb{N}$, we have

$$\sum_{k=k_n+1}^{\ell_n} |b^n_k - b^n_{k-1}| \geq r. \tag{35}$$

For each $n, \ell \in \mathbb{N}$, put

$$f_n^\ell = \sum_{k=0}^{\ell} b^n_k \theta_k |I_k|^{-1} h_{I_k} = D_n P_{(>k_0)}(\underbrace{|I_{\ell+1}|^{-1}\chi_{I_{\ell+1}}}_{=:\psi_\ell}).$$

By Lemma 3.6, we may find a sequence of $(A_n)_n$ of disjoint measurable sets so that for each $n \in \mathbb{N}$, the sequence $(f_n^\ell(s))_{\ell\geq\ell_n}$ is constant for all $s \in A_n$ and $\|f_n^{\ell_n}|_{A_n}\|_{L_1} \geq r/3$. For each $n \in \mathbb{N}$ fix g_n in the unit sphere of L_∞ with support in A_n so that for all $\ell \geq \ell_n$

$$r/3 \leq \|f_n^{\ell_n}|_{A_n}\|_{L_1} = \left|\langle g_n, f_n^\ell\rangle\right| = \left|\langle g_n, D_n P_{(>k_0)}(\psi_\ell)\rangle\right|$$
$$\leq \left|\langle g_n, T_n P_{(>k_0)}(\psi_\ell)\rangle\right| + r/12 = \left|\langle g_n \otimes x_n^*, T\big(\underbrace{(P_{(>k_0)}\psi_\ell)}_{\phi_\ell} \otimes x_n\big)\rangle\right| + r/12.$$

Note that for all $\ell \in \mathbb{N}$, $\|\phi_\ell\|_{L_1} \leq 2$. Then for all $n \in \mathbb{N}$ and $\ell \geq \ell_n$,

$$\left|\langle g_n \otimes x_n^*, T\big(\phi_\ell \otimes x_n\big)\rangle\right| \geq r/4.$$

Pick an $L \in [\mathbb{N}]^\infty$ so that for each $m, n \in \mathbb{N}$, the limit

$$\xi_n(m) := \lim_{\ell\in L} \langle g_m \otimes x_m^*, T(\psi_\ell \otimes x_n)\rangle \text{ exists.}$$

Because the sequence $(g_m)_m$ is disjointly supported, an identical calculation as in equation (30) yields that for all $n \in \mathbb{N}$, we have

$$\sum_m |\xi_n(m)| \leq 2\|T\| \sup_{(x^*,x)\in C} \|x^*\|\|x\|.$$

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Thus, by Rosenthal's Lemma 3.1, we may pass to a subsequence and relabel so that for all $n_0 \in \mathbb{N}$, we have $\sum_{m\neq n_0} |\xi_{n_0}(m)| \leq r/8$.

We will show that $(x_n)_n$ must be equivalent to the unit vector basis of ℓ_1, which would contradict our assumption and thus finish the proof. Fix scalars $a_1, \dots, a_N$, and for $\ell \in L$ with $\ell \geq \ell_N$ pick appropriate scalars $\zeta_1^\ell, \dots, \zeta_N^\ell$ of modulus one so that we have

$$\frac{r}{4}\sum_{n=1}^N |a_n| \leq \sum_{n=1}^N \langle g_n \otimes (\zeta_n^\ell x_n^*), T(\phi_\ell \otimes (a_n x_n))\rangle$$

and put

$$\begin{aligned}\Lambda_\ell &= \Big|\Big\langle \sum_{m=1}^N g_n \otimes (\zeta_m^\ell x_m^*), \sum_{n=1}^N T\big(\phi_\ell \otimes (\sum_{n=1}^N a_n x_n)\big)\Big\rangle\Big| \\ &\leq \Big(2\|T\| \sup_{x^*\in A} \|x^*\|\Big)\Big\|\sum_{n=1}^N a_n x_n\Big\|.\end{aligned}$$

But also,

$$\begin{aligned}\lim_{\ell\in L} \Lambda_\ell &= \lim_{\ell\in L}\Big|\sum_{n=1}^N \langle g_n \otimes (\zeta_n^\ell x_n^*), T(\phi_\ell \otimes (a_n x_n))\rangle \\ &\quad + \sum_{n=1}^N a_n \sum_{m\neq n} \zeta_m^\ell \langle g_m \otimes x_m^*, T(\psi_\ell \otimes x_n)\rangle\Big| \\ &\geq \frac{r}{4}\sum_{n=1}^N |a_n| - \sum_{n=1}^N |a_n| \sum_{m\neq n} |\xi_n(m)| \geq \frac{r}{8}\sum_{n=1}^N |a_n|.\end{aligned}$$

Therefore, $\|\sum_{n=1}^N a_n x_n\| \geq r/(16\|T\| \sup_{x^*\in A}\|x^*\|)\sum_{n=1}^N |a_n|$. □

4. Projectional factors of X-diagonal operators

The main purpose of the section is to prove the following first step towards the final result. The Fundamental Lemma (Theorem 3.4) is a necessary part of the proof.

Theorem 4.1. *Let X be in $\mathcal{H}^*$, and let $T : L_1(X) \to L_1(X)$ be a bounded linear operator. Then for every $\varepsilon > 0$, T is a 1-projectional factor with error ε of an X-diagonal operator $S : L_1(X) \to L_1(X)$.*

The strategy is to first pass to an operator S with the family $(S^{(L,M)})_{L\neq M}$ uniformly eventually close to Haar multipliers (in reality, S satisfies something slightly stronger). We will then use the Fundamental Lemma to eliminate these entries. The following result states how uniform eventual proximity to Haar multipliers is achieved in practice.

Lemma 4.2. *Let $\mathscr{T}$ be a subset of $\mathcal{L}(L_1)$ and $(\varepsilon_{(I,J)})_{(I,J)\in\mathcal{D}^+\times\mathcal{D}^+}$ be a summable collection of positive real numbers. If for every $I \neq J \in \mathcal{D}^+$ and $T \in \mathscr{T}$, we have $\big|\langle h_I, T(|J|^{-1}h_J)\rangle\big| \leq \varepsilon_{(I,J)}$, then $\mathscr{T}$ is uniformly eventually close to Haar multipliers.*

Proof. For fixed $T \in \mathscr{T}$, put $a_I = \langle h_I, T(|I^{-1}|h_I)\rangle$. This collection defines a bounded Haar multiplier D_T because for all f in the unit ball of L_1, $\|(T - D_T)f\| \leq \sum_{I\in\mathcal{D}^+}\sum_{\{J\in\mathcal{D}^+ : J\neq I\}} |\langle h_I, T(|J|^{-1}h_J)\rangle| < \infty$.

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Also, for all $n \in \mathbb{N}$,

$$\left\|TP_{(>n)} - D_T P_{(>n)}\right\| \leq \sum_{I\in\mathcal{D}^+}\sum_{J\in\mathcal{D}^+\setminus\mathcal{D}^n} \varepsilon_{(I,J)} =: \varepsilon_n \text{ and}$$
$$\left\|P_{(>n)}T - P_{(>n)}D_T\right\| \leq \sum_{I\in\mathcal{D}^+\setminus\mathcal{D}^n}\sum_{J\in\mathcal{D}^+} \varepsilon_{(I,J)} =: \delta_n.$$

Both $(\varepsilon_n)_n$ and $(\delta_n)_n$ tend to zero. □

The next lemma is the basic tool used to achieve the first step.

Lemma 4.3. *Let X be in $\mathcal{H}^*$ and $\mathcal{T} \subset \mathcal{L}(X)$, $G \subset X^*$ and $F \subset X$ be finite sets. Then for any $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ so that for any disjoint collection Δ of $\mathcal{D}^+$ with $\min \iota(\Delta) \geq i_0$ and any $\theta \in \{-1,1\}^\Delta$, we have*

$$\max_{g\in G, T\in\mathcal{T}} \left|\left\langle g, T(h_\Delta^\theta)\right\rangle\right| \leq \varepsilon \textit{ and } \max_{f\in F, T\in\mathcal{T}} \left|\left\langle h_\Delta^\theta, T(f)\right\rangle\right| \leq \varepsilon$$

(recall that h_Δ^θ was introduced before Definition 2.10).

Proof. The result is an immediate consequence of the following fact: let (Δ_k) be a sequence of finite disjoint collections of $\mathcal{D}^+$ with $\lim_k \min \iota(\Delta_k) = \infty$, and for every $k \in \mathbb{N}$ let $\theta_k \in \{-1,1\}^{\Delta_k}$.

(a) The sequence $(h_{\Delta_k}^{\theta_k})_k$ is weakly null.
(b) The sequence $(h_{\Delta_k}^{\theta_k})_k$ is a bounded block sequence in X^* and thus is w^*-null.

There is nothing further to say about statement (b). We now explain how statement (a) is achieved. Note that any sequence of independent $\{-1,1\}$-valued random variables of mean 0 is distributionally equivalent to $(r_n)_n$ and thus weakly null. Any sequence as in statement (a) has a subsequence that is of the form $(\frac{r_n + r_n'}{2})$, where (r_n) and (r_n') are both sequences of independent $\{-1,1\}$-valued random variables of mean 0. Thus, it is weakly null as well. □

We carry out the first step towards the proof of Theorem 4.1

Proposition 4.4. *Let X be in $\mathcal{H}^*$, and denote by C the set of all pairs (g,f) in $B_{X^*} \times B_X$ so that g and f have finite and disjoint supports with respect to the Haar system. Then every bounded linear operator $T : L_1(X) \to L_1(X)$ is a 1-projectional factor of a bounded linear operator $S : L_1(X) \to L_1(X)$ so that the family $\{S^{(f,g)} : (f,g) \in C\}$ is uniformly eventually close to Haar multipliers.*

Proof. We will inductively construct faithful Haar systems $(\widetilde{h}_I)_{I\in\mathcal{D}^+}$ and $(\widehat{h}_L)_{L\in\mathcal{D}^+}$. In each step k of the induction, we will define $\widetilde{h}_I$ and then $\widehat{h}_L$ with $k = \iota(I) = \iota(L)$ (i.e., $I = L$, but we separate the notation for clarity). These vectors are of the form $\widetilde{h}_I = \sum_{J\in\Delta_I} h_J$ and $\widehat{h}_L = \sum_{M\in\Gamma_L} h_M$. The inductive assumption is the following.

For every $J, J', M, M' \in \mathcal{D}^+$ with $\iota(J') \neq \iota(J) \leq k$ and $\iota(M') \neq \iota(M) \leq k$, we have

$$\left|\left\langle \widetilde{h}_J \otimes \nu_M \widehat{h}_M, T\left(|J'|^{-1}\widetilde{h}_{J'} \otimes \mu_{M'}\widehat{h}_{M'}\right)\right\rangle\right| \leq 2^{-(\iota(J)+\iota(J')+\iota(M)+\iota(M'))}. \tag{36}$$

We may start by picking $\widetilde{h}_\emptyset = \widehat{h}_\emptyset = h_\emptyset$. We now carry out the kth inductive step. Let $I = L \in \mathcal{D}^+$ with $\iota(I) = \iota(L) = k$. We will apply Lemma 4.3 twice, once for L_1 and once for X. First, define the following finite sets:

$$\mathcal{T}_1 = \{T^{(\nu_M\widehat{h}_M, \mu_{M'}\widehat{h}_{M'})} : \iota(M), \iota(M') < k\} \subset \mathcal{L}(L_1)$$
$$G_1 = \{\widetilde{h}_J : \iota(J) < k\} \subset (L_1)^* \text{ and } F_1 = \{|J|\widetilde{h}_J : \iota(J) < k\} \subset L_1.$$

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Use Lemma 4.3 to pick $\widetilde{h}_I$ so that

$$\max_{g\in G_1, T\in\mathscr{T}_1} \left|\left\langle g, T(\widetilde{h}_I)\right\rangle\right| \le 2^{-4k} \text{ and } \max_{f\in F_1, T\in\mathscr{T}_1} \left|\left\langle \widetilde{h}_I, T(f)\right\rangle\right| \le |I|2^{-4k}.$$

Next, we take the finite sets

$$\mathscr{T}_2 = \{T_{(\widetilde{h}_J, |J'|^{-1}\widetilde{h}_{J'})} : \iota(J), \iota(J') \le k\} \subset \mathcal{L}(X)$$
$$G_2 = \{\nu_M \widehat{h}_M : \iota(M) < k\} \subset (L_1)^* \text{ and } F_2 = \{\mu_M \widehat{h}_M : \iota(M) < k\} \subset X.$$

Use Lemma 4.3 to pick $\widehat{h}_M$ so that

$$\max_{g\in G_2, T\in\mathscr{T}_2} \left|\left\langle g, T(\widehat{h}_L)\right\rangle\right| \le \mu_L^{-1}2^{-4k} \text{ and } \max_{f\in F_2, T\in\mathscr{T}_2} \left|\left\langle \widehat{h}_L, T(f)\right\rangle\right| \le |I|\nu_L^{-1}2^{-4k}.$$

The inductive step is complete, and it is straightforward to check that the inductive hypothesis is preserved.

Take the operator S given in Example 2.17. We will show that it has the desired property. Fix $g \in B_{X^*}$, $f \in B_X$ with $g = \sum_{M\in E} b_M \nu_M h_M$ and $f = \sum_{M\in F} a_M \mu_M h_M$ so that F, G are finite and disjoint. Then

$$S^{(g,f)} = \sum_{M\in E} \sum_{M'\in F} b_M a_{M'} S^{(M,M')}$$

and for $I \ne J \in \mathcal{D}^+$, we have

$$\begin{aligned}\left|\left\langle h_I, S^{(g,f)}(|J|^{-1}h_J)\right\rangle\right| &\le \sum_{M\in E} \sum_{M'\in F} \left|\left\langle h_I, S^{(M,M')}(|J|^{-1}h_J)\right\rangle\right| \\ &= \sum_{M\in E} \sum_{M'\in F} \left|\left\langle h_I \otimes (\nu_M h_M), S\big((|J|^{-1}h_J) \otimes (\mu_{M'} h_{M'})\big)\right\rangle\right| \\ &= \sum_{M\in E} \sum_{M'\in F} \left|\left\langle \widetilde{h}_I \otimes (\nu_M \widehat{h}_M), T\big((|J|^{-1}\widetilde{h}_J) \otimes (\mu_{M'} \widehat{h}_{M'})\big)\right\rangle\right| \\ &\le \sum_{M\in E} \sum_{M'\in F} 2^{-(\iota(J)+\iota(J')+\iota(M)+\iota(M'))} \le 2^{-(\iota(I)+\iota(J))} =: \varepsilon_{(I,J)}.\end{aligned}$$

By Lemma 4.2, the family under consideration is uniformly eventually close to Haar multipliers. □

Remark 4.5. Proposition 4.4 can be achieved if we merely assume that X is a Haar system space, as condition (⋆) of Definition 2.14 can be replaced with a probabilistic argument. We presented the slightly simpler proof that assumes (⋆).

We now eliminate the off-diagonal entries to obtain an X-diagonal operator that projectionally factors through T.

Proof of Theorem 4.1. By Proposition 4.4, T is a 1-projectional factor of an $S : L_1(X) \to L_1(X)$ that satisfies condition (ii) of Theorem 3.4.

For a finite pairwise disjoint collection $\Gamma \subset \mathcal{D}^+$, we define $\Gamma(n) = \{D \in \mathcal{D}_n : D \subset \Gamma^*\}$. Note that $\Gamma(n)$ is a partition of Γ^* for large enough n. Also note that from our condition $(*)$ it follows that for two finite subsets Γ, Γ' of $\mathcal{D}$, with $\Gamma^* \cap (\Gamma')^* = \emptyset$, the set

$$C^{(\Gamma,\Gamma')} = \{(\nu_{(\Gamma')^*} h_{\Gamma'(n)}, \mu_{\Gamma^*} h_\Gamma) : n \in \mathbb{N}\} \cup \{(\nu_{\Gamma^*} h_\Gamma, \mu_{(\Gamma')^*} h_{\Gamma'(n)}) : n \in \mathbb{N}\}$$

satisfies condition (i) of Theorem 3.4. The following claim will be the main step towards recursively defining an appropriate faithful Haar system $(\tilde{h}_L)$.

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Claim. There are $\mathscr{A} \subset [\mathcal{D}]$, with $|\mathscr{A}| > 1 - \eta$, and $\mathscr{U} \in [\mathbb{N}]^\infty$, so that

$$\lim_{n\in\mathscr{U}} S^{(\nu_{(\Gamma')^*} h_{\Gamma'(n)}, \mu_{\Gamma^*} h_\Gamma)} P_{\mathscr{A}} = 0 \text{ and } \lim_{n\in\mathscr{U}} S^{(\nu_{\Gamma^*} h_\Gamma, \mu_{(\Gamma')^*} h_{\Gamma'(n)})} P_{\mathscr{A}} = 0$$

with respect to the operator norm, for all $\Gamma, \Gamma' \subset \mathcal{D}$, with $\Gamma^* \cap (\Gamma')^* = \emptyset$.

To show the claim, we choose for each pair (Γ, Γ'), with $\Gamma, \Gamma' \subset \mathcal{D}$ being finite, $\eta_{(\Gamma,\Gamma')} > 0$. with $\sum \eta_{(\Gamma,\Gamma')} < \eta$. Then, using Theorem 3.4, we choose a closed set $\mathscr{A}_{(\Gamma,\Gamma')}$ in $[\mathcal{D}^+]$ with $|\mathscr{A}_{(\Gamma,\Gamma')}| > 1 - \eta_{(\Gamma,\Gamma')}$, so that $\{S^{(g,f)} P_{\mathscr{A}_{(\Gamma,\Gamma')}} : (g, f) \in C^{(\Gamma,\Gamma')}\}$ is relatively compact in the operator norm topology. Put $\mathscr{A} = \cap \mathscr{A}_{(\Gamma,\Gamma')}$, and note that $|\mathscr{A}| > 1 - \eta$ and that for each (Γ, Γ'), we still have that $\{S^{(g,f)} P_{\mathscr{A}} : (g, f) \in C^{(\Gamma,\Gamma')}\}$ is relatively compact.

Via a Cantor diagonalisation, find $\mathscr{U} \in [\mathbb{N}]^\infty$ so that for every pair (Γ, Γ'), both limits

$$S_1^{(\Gamma,\Gamma')} := \lim_{n\in\mathscr{U}} S^{(\nu_{(\Gamma')^*} h_{\Gamma'(n)}, \mu_{\Gamma^*} h_\Gamma)} P_{\mathscr{A}} \text{ and } S_2^{(\Gamma,\Gamma')} := \lim_{n\in\mathscr{U}} S^{(\nu_{\Gamma^*} h_\Gamma, \mu_{(\Gamma')^*} h_{\Gamma'(n)})} P_{\mathscr{A}}$$

exist with respect to the operator norm. As we see right away, $S_1^{(\Gamma,\Gamma')} = S_2^{(\Gamma,\Gamma')} = 0$. Indeed, for any $g \in L_\infty$ and $f \in L_1$, we have

$$\begin{aligned}\langle g, S_2^{(\Gamma,\Gamma')} f\rangle &= \lim_{n\in\mathscr{U}} \big\langle g, S^{(\nu_{\Gamma^*} h_\Gamma, \mu_{(\Gamma')^*} h_{\Gamma'(n)})} (P_{\mathscr{A}} f)\big)\big\rangle \\ &= \lim_{n\in\mathscr{U}} \big\langle g \otimes (\nu_{\Gamma^*} h_\Gamma), S\big((P_{\mathscr{A}} f) \otimes (\mu_{(\Gamma')^*} h_{\Gamma'(n)})\big)\big\rangle = 0\end{aligned}$$

because $(h_{\Gamma'(n)})_n$ is weakly null in X, by Lemma 4.3. With the same computation, $S_2^{(\Gamma,\Gamma')} = 0$ because $(h_{\Gamma'(n)})_n$ is w^*-null in X^*. This finishes the proof of the claim.

We now choose inductively a faithful Haar system $(\widetilde{h}_L)_{L\in\mathcal{D}^+}$ so that for every $L \neq M \in \mathcal{D}^+$, we have

$$\|S^{(\nu_L \widetilde{h}_L, \mu_M \widetilde{h}_M)} P_{\mathscr{A}}\| \le \varepsilon 2^{-(\iota(L)+\iota(M))}. \tag{37}$$

Assume $M \in \mathcal{D}$, and $\widetilde{h}_L = h_{\Gamma_L}$ has been chosen for all $L \in \mathcal{D}^+$ with $\iota(L) < \iota(M)$, ($\widetilde{h}_\emptyset = h_\emptyset$ and $\widetilde{h}_{(0,1)} = h_{[0,1)}$ by definition). Without loss of generality, we can assume that $M = K^+$ for some $K \in \mathcal{D}$ with $\iota(K) < \iota(M)$. Thus we will choose Γ_M so that $\Gamma_M^* = [\widetilde{h}_K = 1]$. For large enough $n_0 \in \mathbb{N}$, it follows that $[\widetilde{h}_K = 1] = (\Gamma')^*$ for some $\Gamma' \subset \mathcal{D}_{n_0}$. Then we can use our claim that for large enough $n > 0$, we let $\Gamma_M = \Gamma'(n)$. We deduce equation (37) for all $L \in \mathcal{D}$, with $\iota(L) < \iota(M)$

Apply Proposition 2.18 to find a subspace Z of $Y_{\mathscr{A}}$ (i.e., in the image of $P_{\mathscr{A}}$) that is 1-complemented in L_1 via $P : L_1 \to Z$ and isometrically isomorphic to L_1 via $A : L_1 \to Z$. Let also W be the closed linear span of $(\widetilde{h}_L)_{L\in\mathcal{D}^+}$ in X, $Q : X \to W$ be the canonical 1-projection and $B : X \to W$ be the canonical onto isometry.

By Proposition 2.16, the operator $R = ((A^{-1}P) \otimes (B^{-1}Q))S(A \otimes B)$ is a 1-projectional factor of S and thus also of T. It remains to see that R is ε-close to an X-diagonal operator. Fix $L \neq M$. To compute the norm of $R^{(L,M)}$, we also fix $g \in B_{L_\infty}$ and $f \in B_{L_1}$.

$$\begin{aligned}\big|\big\langle g, R^{(L,M)} f\big\rangle\big| &= \big|\big\langle g \otimes (\nu_L h_L), R(f \otimes \mu_M h_M)\big\rangle\big| \\ &= \Big|\Big\langle \underbrace{P^* A^{-1*} g}_{:=v\in B_{L_\infty}} \otimes \underbrace{(Q^* B^{-1*} \nu_L h_L)}_{=\nu_L \widetilde{h}_L}, S(\underbrace{Af}_{=:u\in B_{Y_{\mathscr{A}}}} \otimes \underbrace{B\mu_M h_M}_{=\mu_M \widetilde{h}_M})\Big\rangle\Big| \\ &= \big|\big\langle v, S^{(\nu_L \widetilde{h}_L, \mu_M \widetilde{h}_M)} (P_{\mathscr{A}} u)\big\rangle\big| \le \varepsilon 2^{-(\iota(L)+\iota(M))}.\end{aligned}$$

By Remark 2.22, R is ε-close to an X-diagonal operator. □

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5. Stabilising entries of X-diagonal operators

Once we have an X-diagonal operator at hand, we can pass to another X-diagonal operator whose entries are stable in an extremely strong sense.

Theorem 5.1. *Let X be in $\mathcal{H}^{**}$, and let $T : L_1(X) \to L_1(X)$ be a bounded X-diagonal operator. Then for any collection of positive real numbers $(\varepsilon_L)_{L\in\mathcal{D}^+}$, T is a 1-projectional factor of an operator $S : L_1(X) \to L_1(X)$ with the following properties:*

(a) *S is X-diagonal with entries $(S^L)_{L\in\mathcal{D}^+}$ and*
(b) *for every $L, M \in \mathcal{D}^+$ with $L \subset M$, we have $\|S^L - S^M\| \le \varepsilon_M$.*

The above theorem is proved in two steps. The first one is to pass from an arbitrary X-diagonal operator to another one whose entries are uniformly eventually close to Haar multipliers. This is perhaps the most challenging part of the entire process. For presentation purposes, we momentarily skip this. Instead, we describe the step that follows it, which is the strong stabilisation of the entries, given the uniform eventual proximity to Haar multipliers. This is based on the Fundamental Lemma (Theorem 3.4) and a simple concentration inequality. This proof also serves as an icebreaker for the proof of the first step, which is presented later in this section.

Proposition 5.2. *Let X be in $\mathcal{H}^{**}$, and let $T : L_1(X) \to L_1(X)$ be a bounded X-diagonal operator. Assume that the set of entries $\{T^L : L \in \mathcal{D}^+\}$ of T is uniformly eventually close to Haar multipliers. Then for any collection of positive real numbers $(\varepsilon_L)_{L\in\mathcal{D}^+}$, T is a 1-projectional factor of an X-diagonal operator $S : L_1(X) \to L_1(X)$ so that for every $L, M \in \mathcal{D}^+$ with $L \subset M$, we have $\|S^L - S^M\| \le \varepsilon_M$.*

We start with the probabilistic component required in the proof.

Lemma 5.3. *Let $N \in \mathbb{N}$, $M \ge 0$, Ω be a uniform probability space with $2N$ elements, and let $\Omega = \sqcup_{n=1}^N \{\omega_n^{-1}, \omega_n^1\}$ be a partition of Ω into doubletons. For a function $G : \Omega \to [-M, M]$, define $\Phi : \{-1,1\}^N \to [-M, M]$ given by*

$$\Phi(\varepsilon) = \frac{1}{N}\sum_{n=1}^N G(\omega_n^{\varepsilon_n}). \tag{38}$$

Then $\mathbb{E}(\Phi) = \mathbb{E}(G)$ and $\mathrm{Var}(\Phi) \le M^2/N$, where on $\{-1,1\}^N$, we also consider the uniform probability measure. In particular, for any $\eta > 0$,

$$\mathbb{P}\Big(\Big|\Phi - \mathbb{E}(G)\Big| \ge \eta\Big) \le \frac{M^2}{N\eta^2}. \tag{39}$$

Proof. For $1 \le n \le N$, let $\Phi_n : \{-1,1\}^N \to [-M, M]$ given by $\Phi_n(\varepsilon) = G(\omega_n^{\varepsilon_n})$. This is an independent sequence of random variables, and for each $n \in \mathbb{N}$, we have

$$\mathbb{E}(\Phi_n) = \frac{1}{2}\big(G(\omega_n^{-1}) + G(\omega_n^1)\big) \text{ and } \mathrm{Var}(\Phi_n) = \frac{1}{4}\big(G(\omega_n^{-1}) - G(\omega_n^1)\big)^2.$$

Then $\mathbb{E}(\Phi) = (1/N)\sum_{n=1}^N \mathbb{E}(\Phi_n) = \mathbb{E}(G)$. By independence, we obtain

$$\mathrm{Var}(\Phi) = \frac{1}{N^2}\sum_{n=1}^N \mathrm{Var}(\Phi_n) \le \frac{1}{N^2}N\frac{4M^2}{4} = \frac{M^2}{N\eta^2}.$$

□

Lemma 5.4. *Let K be a relatively compact subset of a Banach space. Then for every $\varepsilon > 0$ and $\eta > 0$, there exists $N(K, \varepsilon, \eta) \in \mathbb{N}$ so that for every $N \ge N(K, \varepsilon, \eta)$, the following holds. For every uniform* *probability space* Ω *with* $2N$ *elements and partition* $\Omega = \sqcup_{n=1}^N \{\omega_n^{-1}, \omega_n^1\}$ *into doubletons, for any function* $G : \Omega \to K$ *if we define* $\Phi : \{-1,1\}^N \to \mathrm{conv}(K)$ *given by*

$$\Phi(\varepsilon) = \frac{1}{N}\sum_{n=1}^{N} G(\omega_n^{\varepsilon_n})$$

then $\mathbb{E}(\Phi) = \mathbb{E}(G)$ *and*

$$\mathbb{P}\Big(\Big\|\Phi - \mathbb{E}(G)\Big\| \geq \eta\Big) \leq \varepsilon. \tag{40}$$

Proof. The statement $\mathbb{E}(\Phi) = \mathbb{E}(G)$ is proved exactly as in the scalar valued scenario, and it is in fact independent of the choice of $N(K,\varepsilon,\eta)$. For the second part fix $\varepsilon, \eta > 0$, and take a finite $\eta/3$-net $(k_i)_{i=1}^{d(K,\eta)}$ of the set $\mathrm{conv}(K \cup (-K))$. Fix norm-one functionals $(f_i)_{i=1}^{d(k,\eta)}$ so that for each $1 \leq i \leq d(K,\eta)$, we have $f_i(k_i) = \|k_i\|$. In particular, for any $k_1, k_2 \in \mathrm{co}(K)$ with $\|k_1 - k_2\| \geq \eta$, there exists $1 \leq i \leq d(K,\eta)$ so that $|f_i(k_1) - f_i(k_2)| \geq \eta/3$. Also set $M = \sup_{k\in K}\|k\|$.

We now fix N, X, G and Φ as in the statement. For $1 \leq i \leq d(K,\eta)$, put $G_i = f_i \circ G$ and $\Phi_i = f_i \circ \Phi$; then

$$\big\{\omega : \|\Phi(\omega) - \mathbb{E}(G)\| > \eta\big\} \subset \bigcup_{1\leq i\leq d(K,\eta)} \big\{\omega : |\Phi_i(\omega) - \mathbb{E}(G_i)| \geq \eta/3\big\}$$

and thus, by Lemma 5.3, we have

$$\mathbb{P}\big(\|\Phi - \mathbb{E}(G)\| > \eta\big) \leq d(K,\eta)\frac{9M^2}{N\eta^2}.$$

Picking any $N(K,\varepsilon,\eta) \geq 9d(K,\eta)M^2/(\varepsilon\eta^2)$ completes the proof. □

Remark 5.5. Let X be an Haar system space, $T : L_1(X) \to L_1(X)$ be an X-diagonal operator and Γ be a disjoint collection of $\mathcal{D}^+$. Then for every $g \in L_\infty$, $f \in L_1$, and θ in $\{-1,1\}^\Gamma$, we have

$$\begin{aligned}
\langle g, T^{(\nu_{\Gamma^*}h_\Gamma^\theta, \mu_{\Gamma^*}h_\Gamma^\theta)} f\rangle &= \langle g \otimes \nu_{\Gamma^*}h_\Gamma^\theta, T(f \otimes \mu_{\Gamma^*}h_\Gamma^\theta)\rangle \\
&= |\Gamma^*|^{-1}\sum_{M\in\Gamma}\sum_{L\in\Gamma}\theta_M\theta_L\langle g\otimes h_M, T(f\otimes h_L)\rangle \\
&= |\Gamma^*|^{-1}\sum_{M\in\Gamma}\sum_{L\in\Gamma}\theta_M\theta_L\langle g\otimes h_M, (T^L f)\otimes h_L\rangle \\
&= |\Gamma^*|^{-1}\sum_{M\in\Gamma}\sum_{L\in\Gamma}\theta_M\theta_L\langle g, T^L f\rangle\langle h_M, h_L\rangle \\
&= \sum_{L\in\Gamma}\big(|L|/|\Gamma^*|\big)\langle g, T^L f\rangle.
\end{aligned}$$

In particular, the above expression does not depend on the choice of signs θ: that is, we may write

$$T^\Gamma := T^{(\nu_{\Gamma^*}h_\Gamma^\theta, \mu_{\Gamma^*}h_\Gamma^\theta)} = \sum_{L\in\Gamma}\big(|L|/|\Gamma^*|\big)T^L.$$

Proof of Proposition 5.2. Since X is in $\mathcal{H}^{**}$, the conditions of the Fundamental Lemma (Theorem 3.4) are satisfied for $B = \{\mu_L h_L : L \in \mathcal{D}^+\}$. Fix some $\eta \in (0,1)$. We apply the Fundamental Lemma to find a closed subset $\mathscr{A}$ of $[\mathcal{D}^+]$ with $|\mathscr{A}| > 1 - \eta$ and so that $\{T^L P_{\mathscr{A}} : L \in \mathcal{D}^+\}$ is relatively compact. By Proposition 2.18, there exists a subspace Z of $P_{\mathscr{A}}(L_1)$ that is isometrically isomorphic to L_1 via $A : L_1 \to Z$ and 1-complemented in L_1 via $P : L_1 \to Z$. The operator T is a 1-projectional factor of $S = ((A^{-1}P)\otimes I)T(A\otimes I)$, and in fact for every $L \in \mathcal{D}^+$, we have $S^L = A^{-1}PT^LA = A^{-1}PT^LP_{\mathscr{A}}A$.

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In particular, for every set $\{S^L : L \in \mathcal{D}^+\}$ is relatively compact. Let K be the closed convex hull of $\{S^L : L \in \mathcal{D}^+\}$, with respect to the operator norm.

As in the proof of Theorem 4.1, for every finite disjoint collection Γ of $\mathcal{D}^+$ and $n \in \mathbb{N}$, define $\Gamma(n) = \{L \in \mathcal{D}_n : L \subset \Gamma^*\}$. For finitely many $n \in \mathbb{N}$, $\Gamma(n)$ may be empty but eventually $\Gamma^* = \Gamma(n)^*$. Note that for n sufficiently large so that $\Gamma^* = \Gamma(n)^*$, we have

$$S^\Gamma_n := S^{\Gamma(n)} = \frac{1}{\#\Gamma(n)} \sum_{L \in \Gamma(n)} S^L \in K. \tag{41}$$

By the relative compactness of K, pass to an infinite subset $\mathcal{U}$ of $\mathbb{N}$ so that for each disjoint collection Γ, the limit $S^\Gamma_\infty = \lim_{n \in \mathcal{U}} S^\Gamma_n$ exists. We point out for later that for any partition $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_k$, we have

$$S^\Gamma_\infty = \big(|\Gamma_1^*|/|\Gamma^*|\big) S^{\Gamma_1}_\infty + \cdots + \big(|\Gamma_k^*|/|\Gamma^*|\big) S^{\Gamma_k}_\infty. \tag{42}$$

Pick $(\delta_L)_{L \in \mathcal{D}^+}$ so that for all $M \in \mathcal{D}^+$, we have $\sum_{L \subset M} \delta_L \leq \varepsilon_M/3$. We will recursively define a faithful Haar system $(\hat{h}_L)_{L \in \mathcal{D}^+}$ so that each $\hat{h}_L = \sum_{M \in \Gamma_L} \zeta_M h_M$ with $\Gamma_L \subset \mathcal{D}_{n_L}$, with $n_L \in \mathcal{U}$. We will require that additional conditions are satisfied.

For each L, put $\Gamma_L^+ = \{M \in \mathcal{D}_{n_L+1} : M \subset [\hat{h}_\emptyset \hat{h}_L = 1]\}$ and $\Gamma_I^- = \{M \in \mathcal{D}_{n_L+1} : M \subset [\hat{h}_\emptyset \hat{h}_L = -1]\}$. In the case $L = \emptyset$, the set $\Gamma_\emptyset^-$ is empty, and we don't consider it, which is consistent with the fact that there is only one immediate successor of $\emptyset$ in $\mathcal{D}^+$. For each L, we define a disjoint collection E_L of $\mathcal{D}^+$ with $E_L^* = \Gamma_L^*$. This auxiliary collection E_L will be chosen in the inductive step before Γ_L^*, and in fact it will be used to choose the latter. If $L = \emptyset$, put $E_L = \{[0, 1)\}$, if $L = [0, 1)$, put $E_L = \Gamma_\emptyset$, if $L = L_0^+$, put $E_L = \Gamma_{L_0}^+$, and if $L = L_0^-$, put $E_L = \Gamma_{L_0}^-$. Below are the additional requirements for each $L \in \mathcal{D}^+$.

(i) The set Γ_L is of the form $E_L(n_L)$ – that is, E_L is a disjoint collection of $\mathcal{D}^+$ and n_L is a sufficiently large positive integer to be chosen during the induction – and

$$E_L(n_L) = \Big\{M \in \mathcal{D}_{n_L} : M \subset E_L^*\Big\}.$$

(ii) $\|S^{E_L}_{n_L} - S^{E_L}_\infty\| \leq \delta_L$.

(iii) $\|S^{E_L}_\infty - S^{\Gamma_L^+}_\infty\| \leq \delta_L$ and $\|S^{E_L}_\infty - S^{\Gamma_L^-}_\infty\| \leq \delta_L$.

If we have achieved this construction, we define

$$Q : L_1(X) \to L_1(X), \text{ by } Q(f) = \sum_{J, M \in \mathcal{D}^+} \langle h_J \otimes \nu_M \hat{h}_M, f\rangle |J|^{-1} h_J \otimes \mu_M \hat{h}_M,$$
$$B : L_1(X) \to L_1(X), \text{ by } B(h_I \otimes h_L) = h_I \otimes \hat{h}_L.$$

Put $R = B^{-1}QSB$. It follows that R is X-diagonal and, by Remark 5.5, for each $L \in \mathcal{D}^+$, we have $R^L = S^{E_L}_{n_L}$. Then for each L, we have

$$\begin{aligned}
\|R^L - R^{L^+}\| &= \|S^{E_L}_{n_L} - S^{E_{L^+}}_{n_{L^+}}\| \leq \|S^{\Gamma_L^+}_\infty - S^{E_{L^+}}_{n_{L^+}}\| + \|S^{E_L}_{n_L} - S^{\Gamma_L^+}_\infty\| \\
&\leq \|S^{\Gamma_L^+}_\infty - S^{E_{L^+}}_{n_{L^+}}\| + 2\delta_L \text{ (by (ii) \& (iii))} \\
&\leq \|S^{\Gamma_L^+}_\infty - S^{E_{L^+}}_\infty\| + \|S^{E_{L^+}}_\infty - S^{E_{L^+}}_{n_{L^+}}\| + 2\delta_L \\
&\overset{\text{(ii)}}{\leq} \|S^{\Gamma_L^+}_\infty - S^{E_{L^+}}_\infty\| + \delta_{L^+} + 2\delta_L = 2\delta_L + \delta_{L^+},
\end{aligned} \tag{43}$$

because, by definition, $\Gamma_L^+ = E_{L^+}$. Similarly, we deduce $\|R^L - R^{L^-}\| \leq 2\delta_L + \delta_{L^-}$. Also, using $S^{\Gamma_\emptyset} = S^{\Gamma_{[0,1)}}$, we deduce $\|R^\emptyset - R^{[0,1)}\| \leq 2\delta_\emptyset$. By iterating this process, we may deduce that for every $L \subset M$, we have that $\|R^L - R^M\| \leq 3\sum_{N \subset M} \delta_N \leq \varepsilon_M$.

It remains to explain how we ensure that conditions (i), (ii), and (iii) are upheld. We start by putting $E_\emptyset = \{[0,1)\}$, by picking $n_\emptyset$ sufficiently large so that $\|S^{E_\emptyset}_{n_\emptyset} - S^{E_\emptyset}_\infty\| \le \delta_\emptyset$, and by taking $\zeta_M = 1$ for $M \in E_\emptyset(n_\emptyset) = \Gamma_\emptyset$. Assume that we have carried out the construction up to a certain point and the time has come to pick $\hat{h}_L$. Let L_0 be the immediate predecessor of L. We will assume $L = L_0^+$. Similar arguments work if $L = L_0^-$ or if $L = [0,1)$. Put $E_L = \Gamma^+_{L_0}$ and pick $n_L \in \mathscr{U}$ so that

$$\|S^{E_L}_{n_L} - S^{E_L}_\infty\| \le \delta_L \text{ and } \#E_L(n_L) \ge N(K, 1/2, \delta_L), \tag{44}$$

where $N(K,1/2,\delta_L)$ is given by Lemma 5.4 to the compact set K, defined in beginning of this proof. We now apply that Lemma to $G : E_L(n_L+1) \to K$ with $G(M) = S^{\{M\}}_\infty$. If we endow $E_L(n_L+1)$ with the uniform probability measure, by equation (42), $\mathbb{E}(G) = S^{E_L(n_L+1)}_\infty$. Because $E_L(n_L+1)^* = E_L^*$, we may instead write $\mathbb{E}(G) = S^{E_L}_\infty$. We partition $E_L(n_L+1)$ into doubletons by writing $E_L(n_L+1) = \sqcup_{M \in E_L(n_L)}\{M^+, M^-\}$. For $M \in E_L(n_L) = \Gamma_L$ define M^1 and M^{-1} as follows.

$$M^1 = \begin{cases} M^+ & \text{if } M \subset [\hat{h}_\emptyset = 1] \\ M^- & \text{if } M \subset [\hat{h}_\emptyset = -1] \end{cases} \text{ and } M^{-1} = \begin{cases} M^- & \text{if } M \subset [\hat{h}_\emptyset = 1] \\ M^+ & \text{if } M \subset [\hat{h}_\emptyset = -1] \end{cases}.$$

Take $\Phi : \{-1,1\}^{\Gamma_L} \to \mathrm{conv}(K)$ given by

$$\Phi(\zeta) = \frac{1}{\#\Gamma_L}\sum_{M \in \Gamma_L} G(M^{\zeta(M)}) = \frac{1}{\#\Gamma_L}\sum_{M \in \Gamma_L} S^{\{M^{\zeta(M)}\}}_\infty.$$

By the choice of n_L so that $\#E_L(n_L) \ge N(K,1/2,\delta_\cdot)$, there exists a choice $\zeta \in \{-1,1\}^{\Gamma_L}$ so that

$$\|\Phi(\zeta) - \mathbb{E}(G)\| = \|\Phi(\zeta) - S^{E_L}_\infty\| \le \delta_L.$$

By equation (42) and the definition of Φ, we deduce that $(1/2)(\Phi(\zeta) + \Phi(-\zeta)) = S^{E_L}_\infty$, and therefore we also have that

$$\|G(\zeta) - S^{E_L}_\infty\| \le \delta_L.$$

To finish the proof, it remains to observe that if we take $\hat{h}_L = \sum_{M \in \Gamma_L}\zeta(M)h_M$, we have that $S^{\Gamma_L^+} = \Phi(\zeta)$ and $S^{\Gamma_L^-} = \Phi(-\zeta)$. Indeed, taking a long and hard look at the definition of M^1 and M^{-1}, we eventually observe that for each $M \in \Gamma_L$, we have $(h_\emptyset\zeta(M)h_M)|_{M^{\zeta(M)}} = 1$ and $(h_\emptyset\zeta(M)h_M)|_{M^{-\zeta(M)}} = -1$. This can be seen, for example, by examining all four possible combinations of values of $h_\emptyset|_M$ and $\zeta(M)$. Therefore, it is now evident that

$$\Gamma_L^+ = \{M \in E_L(n_L+1) : M \subset [h_\emptyset h_L = 1]\} = \cup\{M^{\zeta(M)} : M \in \Gamma_L\}$$

and therefore $S^{\Gamma_L^+}_\infty = (\#\Gamma_L)^{-1}\sum_{M \in \Gamma_L} S^{\{M^{\zeta(M)}\}}_\infty = \Phi(\zeta)$. Finally, by using

$$\frac{1}{2}\Big(S^{\Gamma_L^+}_\infty + S^{\Gamma_L^-}_\infty\Big) = S^{\Gamma_L}_\infty = S^{E_L}_\infty = \frac{1}{2}\Big(\Phi(\zeta) + \Phi(-\zeta)\Big),$$

we see that $S^{\Gamma_L^-}_\infty = \Phi(-\zeta)$. □

Now that we are warmed up by the proof of Proposition 5.2, we are ready to proceed to the slightly more challenging proof of the following. We point out that Theorem 5.1 is an immediate consequence of Proposition 5.2 and the following Proposition 5.6.

Proposition 5.6. *Let X be in $\mathcal{H}^{**}$ and $T : L_1(X) \to L_1(X)$ be an X-diagonal operator, and let $(\varepsilon_{(I,J)})_{(I,J)\in(\mathcal{D}^+)^2}$ be a collection of positive real numbers. Then T is a 1-projectional factor of an*

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X-diagonal operator S with entries $(S_L)_{L\in\mathcal{D}^+}$ and the property that for every $L \in \mathcal{D}^+$ and $I \neq J \in \mathcal{D}^+$, we have

$$\left|\left\langle h_I, S^L\left(|J|^{-1}h_J\right)\right\rangle\right| \leq \varepsilon_{(I,J)}. \tag{45}$$

In particular, the entries of S are uniformly eventually close to Haar multipliers.

Proof. The ‘in particular’ part follows from Lemma 4.2; we therefore focus on achieving equation (45).

For each finite disjoint collection Γ of Δ^+, we define $\Gamma(n)$, T^Γ and T_n^Γ as in the proof of Proposition 5.2. Because $X \in \mathcal{H}^{**}$, by Theorem 3.2, applied to the set $B = \{\mu_L h_L : L \in \mathcal{D}^+\}$, for every $f \in L_1$, the set $\{T^L f : L \in \mathcal{D}^+\}$ is relatively compact and thus so is its convex hull. In particular, for every finite disjoint collection Γ, $\{T_n^\Gamma f : n \in \mathbb{N}\}$ is relatively compact by equation (41). By a Cantor diagonalisation, we may find $\mathcal{U} \in [\mathbb{N}]^\infty$ so that WOT-$\lim_{n\in\mathcal{U}} T_n^\Gamma = T_\infty^\Gamma$ exists for every finite disjoint collection Γ.

We will define inductively two faithful Haar systems $(\widetilde{h}_I)_{I\in\mathcal{D}^+}$, $(\widehat{h}_L)_{L\in\mathcal{D}^+}$. In each step of the induction, we will build a single vector $\widetilde{h}_I$, but we will build an entire level of vectors $\widehat{h}_L$. For example, in each of the first four steps of the inductive process, we will define, respectively, the collections of vectors

$$\{\widetilde{h}_\emptyset;\widehat{h}_\emptyset\},\ \{\widetilde{h}_{[0,1)};\widehat{h}_{[0,1)}\},\ \{\widetilde{h}_{[0,1/2)};\widehat{h}_{[0,1/2)},\widehat{h}_{[1/2,1)}\} \text{ and}$$
$$\{\widetilde{h}_{[1/2,1)};\widehat{h}_{[0,1/4)},\widehat{h}_{[1/4,1/2)},\widehat{h}_{[1/2,3/4)},\widehat{h}_{[3/4,1)}\}.$$

This asymmetric choice is necessary because whenever we pick a new vector $\widetilde{h}_I$, we have to stabilise its interaction with all $\widehat{h}_L$ that will be defined in the future. For each $I, L \in \mathcal{D}^+$, we will have $\widetilde{h}_I = \sum_{J\in\Delta_I} h_J$ and $\widehat{h}_L = \sum_{M\in\Gamma_L} \zeta_M h_M$, for some family $(\zeta_M : M \in \Gamma_L) \subset \{\pm 1\}$.

Let us set up the stage that will allow us to state the somewhat lengthy inductive hypothesis. For each $I \in \mathcal{D}^+$, let

$$\varepsilon'_I = \min_{\left\{\substack{J,J'\in\mathcal{D}^+:\\ \iota(J),\iota(J')\leq\iota(I)}\right\}} \varepsilon_{(J,J')}$$

and fix $(\delta_L)_{L\in\mathcal{D}^+}$ so that for all $M \in \mathcal{D}^+$, we have

$$\sum_{L\subset M} \delta_L \leq \varepsilon'_M/6.$$

Here, $\mathcal{D}_{-1} = \{\emptyset\}$ and $\mathcal{D}_0 = \{[0,1)\}$. For each $k \in \mathbb{N}$, and for every $L \in \mathcal{D}_{k-2}$, we have for some $n_L \in \mathcal{U}$, Γ_L is a finite disjoint collection of $\mathcal{D}_{n_L}$ and $|\Gamma_L^*| = |L|$. Additionally, if $\iota(I) = k$, the following hold:

(a) For some $\alpha_k \in \mathbb{N}$, Δ_I is a disjoint collection of $\mathcal{D}^{\alpha_k} \setminus \mathcal{D}^{\alpha_{k-1}}$ and $|\Delta_I| = |I|$. If $k > 1$, then $\alpha_k > a_{k-1}$, and we put $\mathcal{D}^{\alpha_0} = \emptyset$.

(b) For every $J \in \mathcal{D}^+$ with $\iota(J) < k$ and every $M \in \mathcal{D}^{k-2}$, we have

$$\left|\left\langle \widetilde{h}_I, T^{\Gamma_M}\left(|J|^{-1}\widetilde{h}_J\right)\right\rangle\right| \leq \varepsilon'_I/2 \text{ and } \left|\left\langle \widetilde{h}_J, T^{\Gamma_M}\left(|I|^{-1}\widetilde{h}_I\right)\right\rangle\right| \leq \varepsilon'_I/2.$$

We will impose additional conditions. As in the proof of Proposition 5.2, we put $\Gamma_L^+ = \{M \in \mathcal{D}_{n_L+1} : M \subset [\tilde{h}_\emptyset \tilde{h}_L = 1]\}$ and $\Gamma_L^- = \{M \in \mathcal{D}_{n_L+1} : M \subset [\tilde{h}_\emptyset \tilde{h}_L = -1]\}$, for each L. If $L = \emptyset$, put $E_L = \{[0,1)\}$, if $L = [0,1)$, put $E_L = \Gamma_\emptyset$, if $L = L_0^+$, put $E_L = \Gamma_{L_0}^+$, and if $L = L_0^-$, put $E_L = \Gamma_{L_0}^-$. Furthermore, for each $\alpha \in \mathbb{N}$ let $P_\alpha : L_1 \to L_1$, denote the canonical projection onto $\langle\{h_I : I \in \mathcal{D}^\alpha\}\rangle$. We require the following for each $L \in \mathcal{D}_{k-2}$:

(i) The set Γ_L is of the form $E_L(n_L)$.

(ii) $\|P_{\alpha_k}(T_{n_L}^{E_L} - T_\infty^{E_L})P_{\alpha_k}\| \leq \delta_L$.

(iii) $\|P_{\alpha_k}(T_\infty^{E_L} - T_\infty^{\Gamma_L^+})P_{\alpha_k}\| \leq \delta_L$ and $\|P_{\alpha_k}(T_\infty^{E_L} - T_\infty^{\Gamma_L^-})P_{\alpha_k}\| \leq \delta_L$.

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One might jump to the conclusion that the weaker property that for each $k \in \mathbb{N}$, $\lim_{n\in\mathcal{U}} P_k T_n^\Gamma P_k$ exists is sufficient to yield the same result. This is in fact false. We would not know that $T_\infty^\Gamma : L_1 \to L_1$ is well defined as the Haar system is not boundedly complete. In the inductive step, the operators $T_\infty^{E_L}$, $L \in \mathcal{D}_{k-2}$ are used in the choice of $\widetilde{h}_I$, $\iota(I) = k$. Therefore the fact that for each Γ, WOT-$\lim_{n\in\mathcal{U}} T_n^\Gamma = T_\infty^\Gamma$ is necessary.

We assume that we have completed the construction to finish the proof. Take the isometry A given by $A(h_I \otimes h_L) = \widetilde{h}_I \otimes \widehat{h}_L$ and the norm-one projection P onto the image of A given by

$$P(u) = \sum_{I,L\in\mathcal{D}^+} \langle \widetilde{h}_I \otimes |L|^{-1/q}\widehat{h}_L, u\rangle |I|^{-1}\widetilde{h}_I \otimes |L|^{-1/p}\widehat{h}_L.$$

The operator T is a 1-projectional factor of $S = A^{-1}PTA$, and S is X-diagonal with entries $(S^L)_{L\in\mathcal{D}^+}$ so that for each $L, I, J \in \mathcal{D}^+$, we have

$$\langle h_I, S^L(h_J)\rangle = \langle \widetilde{h}_I, T^{\Gamma_L}\widetilde{h}_J\rangle.$$

We fix $I \neq J \in \mathcal{D}^+$ with $\iota(J) < \iota(I) = k$ and $L \in \mathcal{D}^+$. If $L \in \mathcal{D}^{k-2}$, then by (b), we have

$$\begin{aligned} \left|\left\langle h_I, S^L\left(|J|^{-1}h_J\right)\right\rangle\right| &\le \varepsilon_I'/2 \le \varepsilon_{(I,J)} \text{ and} \\ \left|\left\langle h_J, S^L\left(|I|^{-1}h_I\right)\right\rangle\right| &\le \varepsilon_I'/2 \le \varepsilon_{(J,I)}. \end{aligned} \tag{46}$$

Assume then that $L \in \mathcal{D}_{k'-2}$ with $k' > k$. Let $L_k, \dots, L_{k'-1}, L_{k'} = L$ be a sequence with $L_j \in \mathcal{D}_{j-2}^+$ and each term a direct successor of the one before it. Repeat the argument from equation (43) to deduce that for $k \le j < k'$,

$$\begin{aligned} \left\|P_{\alpha_k}(T^{\Gamma_{L_j}} - T^{\Gamma_{L_{j+1}}})P_{\alpha_k}\right\| \le \left\|P_{\alpha_j}(T^{\Gamma_{L_j}} - T^{\Gamma_{L_{j+1}}})P_{\alpha_j}\right\| &\le 2\delta_{L_j} + \delta_{L_{j+1}}\text{: that is,} \\ \left\|P_{\alpha_k}(T^{\Gamma_L} - T^{\Gamma_{L_k}})P_{\alpha_k}\right\| \le 3\sum_{M\subset L_k}\delta_M &\le \varepsilon_{L_k}'/2 \le \varepsilon_I'/2 \text{ (because } \iota(I) \le \iota(L_k)). \end{aligned}$$

Therefore,

$$\begin{aligned} \left|\left\langle h_I, S^L\left(|J|^{-1}h_J\right)\right\rangle\right| &= \left|\left\langle \widetilde{h}_I, T^{\Gamma_L}\left(|J|^{-1}\widetilde{h}_J\right)\right\rangle\right| = \left|\left\langle \widetilde{h}_I, P_{\alpha_k}T^{\Gamma_L}P_{\alpha_k}\left(|J|^{-1}\widetilde{h}_J\right)\right\rangle\right| \\ &\le \left|\left\langle \widetilde{h}_I, P_{\alpha_k}T^{\Gamma_{L_k}}P_{\alpha_k}\left(|J|^{-1}\widetilde{h}_J\right)\right\rangle\right| + \varepsilon_I'/2 \\ &= \left|\left\langle \widetilde{h}_I, T^{\Gamma_{L_k}}\left(|J|^{-1}\widetilde{h}_J\right)\right\rangle\right| + \varepsilon_I'/2 \\ &= \left|\left\langle h_I, S^{L_k}\left(|J|^{-1}h_J\right)\right\rangle\right| + \varepsilon_I'/2 \overset{equation (46)}{\le} \varepsilon_I'/2 + \varepsilon_I'/2 \le \varepsilon_{(I,J)}. \end{aligned}$$

Repeating the argument yields $|\langle h_J, S^L(|I|^{-1}h_I)\rangle| \le \varepsilon_{(J,I)}$. To complete the proof, we still need to carry out the inductive construction. In the first step, we may take $\widetilde{h}_\emptyset = h_\emptyset$ (i.e., $\Delta_\emptyset = \{\emptyset\}$), and thus we may take, for example, $\alpha_1 = 1$. Next, we pick $n_\emptyset \in \mathcal{U}$ sufficiently large so that we have $\|P_1(T_{n_\emptyset}^{\{[0,1)\}} - T_\infty^{\{[0,1)\}})P_1\| \le \delta_\emptyset$. We put $\widehat{h}_\emptyset = \sum_{M\in E_\emptyset(n_\emptyset)} h_L$ (i.e., $\Gamma_\emptyset = E_\emptyset(n_\emptyset)$ with $E_\emptyset = \{[0,1)\}$ and $\zeta_M = 1$ for $M \in \Gamma_\emptyset$). The only nontrivial condition to check is (iii), which follows from the fact that $E_\emptyset^* = (\Gamma_\emptyset^+)^*$ and thus $T_\infty^{E_\emptyset} = T_\infty^{\Gamma_\emptyset^+}$. We do not consider the set $\Gamma_\emptyset^-$.

We now present the kth step for $k \ge 2$. Let $I \in \mathcal{D}^+$ with $\iota(I) = k$, and denote by I_0 its immediate predecessor. We will assume that $I = I_0^+$. For each $L \in \mathcal{D}_{k-2}$, we denote its immediate predecessor by L_0. Recall that for each such L, the set E_L has been defined based on whether $L = L_0^+$ or $L = L_0^-$. Consider the following finite sets:

$$\begin{aligned} \mathcal{T} &= \left\{T^{\Gamma_L} : L \in \mathcal{D}^{k-3}\right\} \cup \left\{T_\infty^{E_L} : L \in \mathcal{D}_{k-2}\right\} \subset \mathcal{L}(L_1), \\ G &= \left\{\widetilde{h}_J : \iota(K) < k\right\} \subset L_\infty \text{ and } F = \left\{|J|^{-1}\widetilde{h}_J : \iota(J) < k\right\}. \end{aligned}$$

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By Lemma 4.3, there exists $i_0 \in \mathbb{N}$ so that for any finite disjoint collection $\Delta \subset \mathcal{D}^+$ with $\min \iota(\Delta) \geq i_0$ and any $\theta \in \{-1,1\}^\Delta$, we have that for all $T \in \mathscr{T}$, $g \in G$ and $f \in F$

$$\left|\left\langle g, T(h_\Delta^\theta)\right\rangle\right| \leq |I|\varepsilon_I'/3 \text{ and } \left|\left\langle h_\Delta^\theta, T(f)\right\rangle\right| \leq \varepsilon_I'/3. \tag{47}$$

We pick Δ_I with $\min \iota(\Delta_I) \geq i_0$ and so that (b) is satisfied. The integer α_k is simply chosen so that $P_{\alpha_k}\widetilde{h}_I = \widetilde{h}_I$. It is immediate that condition (b) is satisfied for all $M \in \mathcal{D}^{k-3}$. Later we will show that (b) also holds for $M \in \mathcal{D}^{k-2}$.

In the next step, for each $L \in \mathcal{D}_{k-2}$, we need to pick n_L that defined Γ_L and $\zeta_L \in \{-1,1\}^{\Gamma_L}$. The choice of n_L so that (i) and (ii) are satisfied is easy. However, we wish to ensure that we can additionally achieve condition (iii), and for this we need Lemma 5.4. Consider the relatively compact set $K = \{P_{\alpha_k}T_\infty^\Gamma P_{\alpha_k} : \Gamma$ is a finite disjoint collection of $\mathcal{D}^+\} \subset \mathcal{L}(L_1)$, and take $N(K, 2^{-k}, \varepsilon_I'/6)$ given by Lemma 5.4. For each $L \in \mathcal{D}^{k-2}$, pick $n_L \in \mathscr{U}$ so that (ii) is satisfied as well as $\#E_L(n_L) \geq N(K, 2^{-k}, \varepsilon_I'/6)$. The objective is to pick, for each $L \in \mathcal{D}_{k-2}$, signs $\zeta_L \in \{-1,1\}^{\Gamma_L}$ so that (iii) is satisfied. Repeating, word for word, the argument from the last few paragraphs of the proof of Proposition 5.2, we can do exactly that.

The final touch that is required to complete the proof is to observe that (b) is now also satisfied for all $L \in \mathcal{D}_{k-2}$. Indeed, for $J \in \mathcal{D}^+$ with $\iota(J) < k$, we have

$$\begin{aligned}
\left|\left\langle \widetilde{h}_I, T^{\Gamma_L}(|J|^{-1}\widetilde{h}_J)\right\rangle\right| &= \left|\left\langle P_{\alpha_k}^*\widetilde{h}_I, T_{n_L}^{E_L}P_{\alpha_k}(|J|^{-1}\widetilde{h}_J)\right\rangle\right| \\
&= \left|\left\langle \widetilde{h}_I, P_{\alpha_k}T_{n_L}^{E_L}P_{\alpha_k}(|J|^{-1}\widetilde{h}_J)\right\rangle\right| \\
&\overset{\text{(ii)}}{\leq} \left|\left\langle \widetilde{h}_I, P_{\alpha_k}T_\infty^{E_L}P_{\alpha_k}(|J|^{-1}\widetilde{h}_J)\right\rangle\right| + \delta_L \\
&\overset{\text{equation(47)}}{\leq} \varepsilon_I'/3 + \delta_L \leq \varepsilon_I'/3 + \varepsilon_I'/6 = \varepsilon_I'/2.
\end{aligned}$$

The same argument yields $|\langle \widetilde{h}_J, T^{\Gamma_L}(|I|^{-1}\widetilde{h}_I)\rangle| \leq \varepsilon_I'/2$. □

6. Projectional factors of scalar operators

In this section, we provide the finishing touches to prove our main result.

Theorem 6.1. *Let X be in $\mathcal{H}^*$ and $\mathcal{H}^{**}$, and let $T : L_1(X) \to L_1(X)$ be a bounded linear operator. Then for every $\varepsilon > 0$, T is a 1-projectional factor with error ε of a scalar operator. In particular, $L_1(X)$ is primary.*

We first need to prove a perturbation result that will allow us to pass from Theorem 5.1 to the conclusion.

Proposition 6.2. *Let X be a Haar system space and $T : L_1(X) \to L_1(X)$ be an X-diagonal operator with entries $(T^L)_{L\in\mathcal{D}^+}$, and let $\varepsilon > 0$. Assume that for all $L, M \in \mathcal{D}^+$ with $L \subset M$, we have $\|T^L - T^M\| \leq \varepsilon|M|^2$. Then $\|T - T^\emptyset \otimes I\| \leq 7\varepsilon$.*

Remark 6.3. Let $n_0 \in \mathbb{N}$, and for each $L \in \mathcal{D}_{n_0}$, let $(\theta_k^L)_{k=0}^{n_0}$ be the signs given by equation (20). Then for scalars $(a_L)_{L\in\mathcal{D}_{n_0}}$, we may write

$$\sum_{L\in\mathcal{D}_{n_0}} a_L|L|^{-1}\chi_L = \Big(\sum_{L\in\mathcal{D}_{n_0}} a_L\Big)h_\emptyset + \sum_{k=1}^{n_0}\sum_{M\in\mathcal{D}_{k-1}}\Big(\sum_{\substack{L\in\mathcal{D}_{n_0}: \\ L\subset M}} \theta_k^L a_L\Big)|M|^{-1}h_M.$$

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We now take an RI space X and translate this into the X setting. For $k = 0, \dots, n_0$, put $\mu_k = \mu_L$ and $\nu_k = \nu_L$ for $L \in \mathcal{D}_k$. Multiply both sides by $\nu_{n_0}^{-1}$ so that for $L \in \mathcal{D}_{n_0}$, we have $|L|^{-1}\nu_{n_0}^{-1} = \mu_L$.

$$\sum_{L\in\mathcal{D}_{n_0}} a_L\mu_L\chi_L = \nu_{n_0}^{-1}\Big(\sum_{L\in\mathcal{D}_{n_0}} a_L\Big)h_\emptyset + \nu_{n_0}^{-1}\sum_{k=1}^{n_0}\nu_{k-1}\sum_{M\in\mathcal{D}_{k-1}}\Big(\sum_{\left\{\substack{L\in\mathcal{D}_{n_0}:\\ L\subset M}\right\}}\theta_k^L a_L\Big)\mu_M h_M. \tag{48}$$

Scalar multiplication may be replaced with tensor multiplication to obtain the same formula (i.e., consider $a_L \otimes \chi_L$, where a_L is, for example, in L_1).

Let us additionally observe that for any $1 \le k \le n_0$ and $M \in \mathcal{D}_{k-1}$, we have

$$\begin{aligned}\sum_{\left\{\substack{L\in\mathcal{D}_{n_0}\\ L\subset M}\right\}}|a_L| &= \Big\langle \sum_{\left\{\substack{L\in\mathcal{D}_{n_0}\\ L\subset M}\right\}}|a_L|\mu_L\chi_L, \sum_{\left\{\substack{L\in\mathcal{D}_{n_0}\\ L\subset M}\right\}}\nu_L\chi_L\Big\rangle \\ &\le \Big\|\sum_{\left\{\substack{L\in\mathcal{D}_{n_0}\\ L\subset M}\right\}}|a_L|\mu_L\chi_L\Big\|\Big\|\sum_{\left\{\substack{L\in\mathcal{D}_{n_0}\\ L\subset M}\right\}}\nu_L\chi_L\Big\|_{X^*} \\ &\le \Big\|\sum_{L\in\mathcal{D}_{n_0}} a_L\mu_L\chi_L\Big\|\nu_{n_0}\|\chi_M\|_{X^*} \\ &= \nu_{n_0}\nu_{k-1}^{-1}\Big\|\sum_{L\in\mathcal{D}_{n_0}} a_L\mu_L\chi_L\Big\|.\end{aligned} \tag{49}$$

Proof of Proposition 6.2. For $n = 0, 1, \dots$ consider the auxiliary operator $S_n = \sum_{L\in\mathcal{D}_n} T^L \otimes R^L$, where $R^L : X \to X$ denotes the restriction onto L: that is, $R^L f = \chi_L f$. We observe that

$$\begin{aligned}\|S_n - S_{n+1}\| &= \Big\|\sum_{L\in\mathcal{D}_n} T^L \otimes \big(R^{L^+} + R^{L^-}\big) - \sum_{L\in\mathcal{D}_n}\big(T^{L^+}\otimes R^{L^+} + T^{L^-}\otimes R^{L^-}\big)\Big\| \\ &\le \sum_{L\in\mathcal{D}_n}\big(\|T^L - T^{L^+}\|\|R_{L^+}\| + \|T^L - T^{L^-}\|\|R^{L^-}\|\big) \\ &\le 2\varepsilon\sum_{L\in\mathcal{D}_n}|L|^2 = \varepsilon 2^{-n+1}.\end{aligned}$$

In particular, for all $n \in \mathbb{N}$, we have

$$\|T^\emptyset \otimes I - S_n\| = \|S_0 - S_n\| \le 4\varepsilon. \tag{50}$$

By equation (28), to estimate $\|T - T^\emptyset \otimes I\|$, it is sufficient to consider vectors of the form $f \otimes g$, with $f \in B_{L_1}$, $g = \sum_{L\in\mathcal{D}_{n_0}} a_L\mu_L\chi_L$, and $\|\sum_{L\in\mathcal{D}_{n_0}} a_L\mu_L\chi_L\| = 1$. By equation (48),

$$f\otimes g = \nu_{n_0}^{-1}\Big(\sum_{L\in\mathcal{D}_{n_0}} a_L\Big)f\otimes h_\emptyset + \nu_{n_0}^{-1}\sum_{k=1}^{n_0}\nu_{k-1}\sum_{M\in\mathcal{D}_{k-1}}\Big(\sum_{\left\{\substack{L\in\mathcal{D}_{n_0}:\\ L\subset M}\right\}}\theta_k^L a_L\Big)f\otimes\mu_M h_M. \tag{51}$$

From equation (50), it follows that

$$\|(T - T_\emptyset \otimes I)(f\otimes g)\| \le 4\varepsilon + \|(T - S^{n_0})(f\otimes g)\|.$$

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We next evaluate T and S_{n_0} on $f \otimes g$. Since T is X-diagonal, we have

$$T(f \otimes g) \overset{equation (51)}{=} \nu_{n_0}^{-1}\Big(\sum_{L \in \mathcal{D}_{n_0}} a_L\Big)(T^{\emptyset} f) \otimes h_{\emptyset}$$
$$+ \nu_{n_0}^{-1} \sum_{k=1}^{n_0} \nu_{k-1} \sum_{M \in \mathcal{D}_{k-1}} \Big(\sum_{\substack{\{L \in \mathcal{D}_{n_0}: \\ L \subset M\}}} \theta_k^L a_L \Big)(T^M f) \otimes \mu_M h_M .$$

For the other valuation, note that for $L \in \mathcal{D}_{n_0}$, we have $S_{n_0}(f \otimes \mu_L \chi_L) = (T^L f) \otimes \mu_L \chi_L$. Therefore,

$$S_{n_0}(f \otimes g) \overset{equation (48)}{=} \nu_{n_0}^{-1}\Big(\sum_{L \in \mathcal{D}_{n_0}} a_L (T^L f)\Big) \otimes h_{\emptyset}$$
$$+ \nu_{n_0}^{-1} \sum_{k=1}^{n_0} \nu_{k-1} \sum_{M \in \mathcal{D}_{k-1}} \Big(\sum_{\substack{\{L \in \mathcal{D}_{n_0}: \\ L \subset M\}}} \theta_k^L a_L (T^L f) \Big) \otimes \mu_M h_M .$$

Therefore,

$$\big\|(T - S_{n_0})(f \otimes g)\big\|$$
$$\le \nu_{n_0}^{-1} \sum_{L \in \mathcal{D}_{n_0}} |a_L| \underbrace{\|T^{\emptyset} - T^L\|}_{\le \varepsilon} + \nu_{n_0}^{-1} \sum_{k=1}^{n_0} \nu_{k-1} \sum_{M \in \mathcal{D}_{k-1}} \sum_{\substack{\{L \in \mathcal{D}_{n_0}: \\ L \subset M\}}} |a_L| \underbrace{\|T^M - T^L\|}_{\le \varepsilon |M|^2}$$
$$\overset{equation (49)}{\le} \nu_{n_0}^{-1} \varepsilon \underbrace{\Big\| \sum_{L \in \mathcal{D}_{n_0}} a_L \mu_L \chi_L \Big\|}_{=1} \nu_{n_0} \underbrace{\nu_0^{-1}}_{=1}$$
$$+ \varepsilon \nu_{n_0}^{-1} \sum_{k=1}^{n_0} \nu_{k-1} \sum_{M \in \mathcal{D}_{k-1}} |M|^2 \Big\| \sum_{L \in \mathcal{D}_{n_0}} a_L \mu_L \chi_L \Big\| \nu_{n_0} \nu_{k-1}^{-1}$$
$$= \varepsilon + \varepsilon \sum_{k=1}^{n_0} \sum_{M \in \mathcal{D}_{k-1}} |M|^2 = \varepsilon + \varepsilon \sum_{k=1}^{n_0} \frac{2^{k-1}}{2^{2k-2}} \le 3\varepsilon.$$

In conclusion, $\|(T - T^{\emptyset} \otimes I)(f \otimes g)\| \le 4\varepsilon + 3\varepsilon$. □

We give the proof of the main result.

Proof of Theorem 6.1. Recall that by virtue of Proposition 2.3, being an approximate 1-projectional factor is a transitive property, during which the compounded errors are under control. We successively apply Theorem 4.1, Theorem 5.1 and Proposition 6.2 to find a bounded linear operator $S : L_1 \to L_1$ so that T is a 1-projectional factor with error ε of $S \otimes I : L_1(X) \to L_1(X)$. By Theorem 2.9, S is a 1-projectional factor with error ε of a scalar operator $\lambda I : L_1 \to L_1$, and therefore $S \otimes I : L_1(X) \to L_1(X)$ is a 1-projectional factor with error ε of $\lambda I : L_1(X) \to L_1(X)$. Finally, T is a 1-projectional factor with error 2ε of $\lambda I : L_1(X) \to L_1(X)$. Thus, our claim follows from Proposition 2.4. □

7. Final discussion

Characterising the complemented subspaces of L_1 and those of $C(K)$ remain the most prominent problems in the study of decompositions of classical Banach spaces. This motivates in particular the study of biparameter spaces, especially those with an L_1 or $C(K)$ component. For example, the proof of primariness for each such type of space presents a different challenge and therefore an opportunity to extract new information on the structure of L_1 or $C(K)$ and their operators. Here is a list of classical biparameter spaces for which primariness remains unresolved:

(a) $L_p(L_1)$ for $1 < p \leq \infty$.
(b) $L_p(L_\infty) \simeq L_p(\ell_\infty)$ for $1 \leq p < \infty$.
(c) $\ell_p(C(K))$ for a compact metric space K and $1 \leq p \leq \infty$.
(d) $L_p(C(K))$ for a compact metric space K and $1 \leq p \leq \infty$.
(e) $C(K, \ell_p)$ for a compact metric space K and $1 \leq p \leq \infty$.
(f) $C(K, L_p)$ for a compact metric space K and $1 \leq p < \infty$.

It is noteworthy that all the other biparameter Lebesgue spaces $\ell_p(\ell_q)$ [13], $\ell_p(L_q)$ [9], $\ell_p(L_1)$ [10], $L_p(L_q)$ [11], $L_p(\ell_q)$ [12] and $\ell_\infty(L_q)$ [37] ($1 < p, q < \infty$) are known to be primary. The space $L_1(C[0, 1])$ resists the approach of this paper, but perhaps some of the tools developed here could be of some use. If this were to be resolved, it is conceivable that techniques from [21] might be useful in transcending the separability barrier to show that $L_1(L_\infty)$ is primary. Such methods night also be useful in the investigation of whether for nonseparable RI space $X \neq L_\infty$, $L_1(X)$ is primary. In more generality, one may ask for what types of Banach spaces X the spaces $L_1(X)$, $L_p(X)$, $H_1(X)$ and $H_p(X)$ are primary.

For any two rearrangement-invariant Banach function spaces X and Y on $[0, 1]$, one can define the biparameter space $X(Y)$ as the space of all functions $f : [0, 1]^2 \to \mathbb{C}$ for which $f(s, \cdot) \in Y$ for all $s \in [0, 1]$ and $g = g_f : [0, 1] \to \mathbb{R}$, $s \mapsto \|f(s, \cdot)\|_Y$ is in X. The norm of f in $X(Y)$ would then be $\|f\|_{X(Y)} = \|g_f\|_X$. It would be interesting to formulate general conditions on X and Y, which imply that $X(Y)$ is primary or has the factorisation property (formulated below) with respect to some basis.

The above list may be expanded to the tri-parameter spaces, in which setting there has been little progress.

It is natural to study general conditions under which an operator T on a Banach space is a factor of the identity. A bounded linear operator T on a Banach space X with a Schauder basis $(e_n)_n$ is said to have *large diagonal* if $\inf_n |e_n^*(Te_n)| > 0$. If every operator on X with large diagonal is a factor of the identity, then we say that X has the *factorisation property*. The study of the factorisation property and that of primariness are closely related. Our proof does not directly show that the spaces under investigation have the factorisation property. We may therefore ask: for what Haar system spaces X and Y does the biparameter Haar system $(h_I \otimes h_L)_{(I,L)\in\mathcal{D}^+\times\mathcal{D}^+}$ have the factorisation property in $X(Y)$?

Acknowledgements. We would like to thank the anonymous referee for very helpful suggestions on how to improve the presentation of our paper.

Funding statement. The first author was supported by the Austrian Science Foundation (FWF) under Grant Number Pr.Nr. P32728. The second author was supported by NSERC Grant RGPIN-2021-03639. The third author was supported by the Austrian Science Foundation (FWF) under Project Numbers P344114 and I-5231. The fourth author was supported by the National Science Foundation under Grant Numbers DMS-1764343 and DMS-2054443.

Conflict of Interest. None.

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https://doi.org/10.1017/fms.2022.25 Published online by Cambridge University Press