UNIQUENESS CLASSES FOR DIFFERENCE FUNCTIONALS

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1. Introduction. If $\{L_n\}$ is a sequence of linear functionals on a linear space C of functions to the complex numbers, then a subspace $C_1 \subset C$ is a uniqueness class for $\{L_n\}$ if a function f in C_1 is uniquely determined by the sequence $\{L_n(f)\}$ of complex numbers; i.e., if $f \in C_1$ and $L_n(f) = 0$, $n = 0, 1, 2, \ldots$, implies $f = 0$. For example, the class of all functions f analytic at the origin is a uniqueness class for the sequence $\{f^{(n)}(0)\}$ of linear functionals. Gontcharoff **(9)** asked the following question: Suppose, instead of $\{f^{(n)}(0)\}$, we use $\{f^{(n)}(a_n)\}$. How much can we let the sequence $\{a_n\}$ differ from the zero sequence without reducing the size of this uniqueness class? He showed, for example, that if

$$a_n \to 0 \quad \text{and} \quad \sum_{n=0}^{\infty} |a_n - a_{n+1}| < \infty,$$

then the class of all functions f analytic at zero is still a uniqueness class for $\{f^{(n)}(a_n)\}$. In this paper, we shall discuss the analogous question for difference functionals

$$\Delta^n f(a_n) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} f(a_n + k);$$

i.e., if we denote by K_π a maximal uniqueness class for $\{\Delta^n f(0)\}$, then for what sequences $\{a_n\}$ will it be true that K_π is also a uniqueness class for $\{\Delta^n f(a_n)\}$?

The classes of functions with which we shall be dealing will be subclasses of the set K of entire functions of exponential type. This is the class of entire functions f having the property that for some numbers A and τ, $|f(z)| \leqslant Ae^{\tau|z|}$ for all z; for a discussion of this class, see Boas **(1**, pp. 66ff.**)**. If $f \in K$ and

$$f(z) = \sum_{n=0}^{\infty} c_n z^n / n!,$$

then f has a Borel transform F given by

$$F(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$$

Received January 28, 1965. This research was supported by Aerospace Research Laboratories, Office of Aerospace Research, Wright-Patterson Air Force Base under Contract No. AF 33(616)-6503.

analytic at infinity. For f in K with Borel transform F, $\Delta^n f(a_n)$ has the representation

$$\Delta^n f(a_n) = \frac{1}{2\pi i} \int_\Gamma e^{a_n \zeta}(e^\zeta - 1)^n F(\zeta)\, d\zeta, \tag{1.1}$$

where Γ is a simple closed contour such that F is analytic outside and on Γ. If $f(z) = e^{\alpha z}$, then $F(z) = (z - \alpha)^{-1}$, so that in this case

$$\Delta^n f(a_n) = e^{a_n \alpha}(e^\alpha - 1)^n.$$

If G is any domain in the complex plane, let $K[G]$ denote the set of all functions f in K such that the Borel transform F of f is analytic on G', the complement of G. If the set G is the strip $H_\pi = \{z = x + iy \,|\, |y| < \pi\}$, we shall use the special notation K_π for $K[H_\pi]$. Then the class K_π is a uniqueness class for $\{\Delta^n f(0)\}$ and if G_1 is any domain that properly contains H_π, then $K[G_1]$ is not a uniqueness class for this sequence of functionals **(6)**. We denote by S_π the class of all sequences $\{a_n\}$ of complex numbers such that K_π is a uniqueness class for $\{\Delta^n f(a_n)\}$. We restrict our consideration to certain classes of real sequences $\{a_n\}$. The only results known to the author concerning functionals $\{\Delta^n f(a_n)\}$ where the a_n are allowed to be non-real indicate that the nature of associated uniqueness classes differs sufficiently from K_π to make the restriction to real sequences advisable **(4; 7)**.

In the problem studied by Gontcharoff, in order that the sequence $\{a_n\}$ give the same uniqueness class as the zero sequence, it is sufficient that $\{a_n\}$ be "close" to $\{0\}$ in a rather direct sense. The questions in that case were "how close?" and "precisely how should closeness be measured?" In the problem we are studying, the situation is quite different. Membership in S_π is not dependent on the sequence $\{a_n\}$ being close to $\{0\}$ in any naïve sense. This can be illustrated by the following examples, which will be proved later:

$$2, 1, 0, 2, 1, 0, 2, 1, 0, \ldots \in S_\pi,$$
$$1, 2, 0, 1, 2, 0, 1, 2, 0, \ldots \notin S_\pi,$$
$$2, 0, 2, 0, 3, 0, 0, 2, 0, 2, 0, 3, 0, 0, \ldots \in S_\pi,$$
$$2, 0, 3, 0, 0, 3, 0, 0, 2, 0, 3, 0, 0, 3, 0, 0, \ldots \notin S_\pi,$$
$$3, 0, 0, 3, 0, 0, 3, 0, 0, \ldots \in S_\pi,$$
$$2, 0, 2, 0, 2, 0, 2, 0, \ldots \notin S_\pi.$$

In this paper we shall show some basic properties of the set S_π; namely that S_π is closed under translations, i.e., if $b_n = a_n + \alpha$ for some fixed α and each n, then $\{b_n\} \in S_\pi$ if and only if $\{a_n\} \in S_\pi$; that membership in S_π depends only on the ultimate terms of the sequence; and that S_π has a certain symmetry in that the sequence $\{-a_n - n\}$ belongs to S_π if and only if $\{a_n\}$ is in S_π. If $a_n = -n/2$, then $\{a_n\} = \{-a_n - n\}$. This choice of $\{a_n\}$ gives the Stirling functionals, which are known to have K_π as a uniqueness class **(2)**. Other cases of arithmetic sequences $a_n = \beta n$ have also been studied. Buck **(2)** has shown that $\{\beta n\} \notin S_\pi$ if $\beta < -1$ or $\beta > 0$. We shall show that $\{\beta n\} \in S_\pi$ if $-1 \leqslant \beta \leqslant 0$.

The bulk of the results of this paper will be for sequences of integers. The problem is more accessible in this case because of relationships between successive differences. For such sequences, we give a sufficient condition for membership in S_π and a number of results that give some indication as to how close this condition is to being necessary.

For periodic sequences $\{a_n\}$ where $a_{pn+k} = \beta k (k = 0, 1, \ldots, p-1; n = 0, 1, 2, \ldots; \beta > 0, \beta$ not necessarily an integer), we give a necessary condition on β in order that $\{a_n\} \in S_\pi$.

2. Properties of S_π.

THEOREM 1. *Let* $b_n = a_n + \alpha$ $(n = 0, 1, 2, \ldots; \alpha$ *fixed*). *If* $\{a_n\} \in S_\pi$, *then* $\{b_n\} \in S_\pi$.

Proof. Let $\{a_n\} \in S_\pi$ and let $f \in K_\pi$ with $\Delta^n f(b_n) = 0$, $n = 0, 1, 2, \ldots$. Let $g(z) = f(z + \alpha)$. Then $g \in K_\pi$ and $\Delta^n g(a_n) = \Delta^n f(b_n) = 0$, $n = 0, 1, \ldots$. Therefore, since $\{a_n\} \in S_\pi$, $g(z) \equiv 0$, so that $f(z) \equiv 0$.

THEOREM 2. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of real numbers such that for some integer* N, $a_n = b_n$ *for all* $n > N$. *Then* $\{a_n\}$ *and* $\{b_n\}$ *are both in* S_π *or are both not in* S_π.

Proof. We shall use the familiar fact that given two finite sequences

$$\{c_j\}_{j=0}^{k} \quad \text{and} \quad \{d_j\}_{j=0}^{k},$$

there exists a unique polynomial p of degree at most k such that $\Delta^j p(c_j) = d_j$ $(j = 0, 1, \ldots, k)$. Since p is unique, if $d_j = 0$ $(j = 0, 1, \ldots, k)$, then $p = 0$.

Let $\{a_n\} \in S_\pi$ and let $f \in K_\pi$ with $\Delta^n f(b_n) = 0$ $(n = 0, 1, 2, \ldots)$. We wish to show that this implies that $f = 0$. Let p be the unique polynomial of degree at most N such that

$$\Delta^n p(a_n) = \Delta^n f(a_n) \qquad (n = 0, 1, \ldots, N).$$

Let $g(z) = f(z) - p(z)$. Then for $n = 0, 1, \ldots, N$,

$$\Delta^n g(a_n) = \Delta^n f(a_n) - \Delta^n p(a_n) = 0.$$

Since, for $n > N$, $b_n = a_n$, $\Delta^n f(a_n) = 0$ for all such n, and since p is a polynomial of degree at most N, $\Delta^n p(a_n) = 0$ also. Therefore, $\Delta^n g(a_n) = 0$ for $n > N$ and hence $\Delta^n g(a_n) = 0$ $(n = 0, 1, 2, \ldots)$. Since $g \in K_\pi$ and $\{a_n\} \in S_\pi$, this implies that $g = 0$. That is, $f(z) \equiv p(z)$. Then

$$\Delta^n p(b_n) = \Delta^n f(b_n) = 0 \qquad (n = 0, 1, \ldots, N).$$

Therefore $p = 0$, so that $f = 0$ and $\{b_n\} \in S_\pi$.

Now, let $\{a_n\} \notin S_\pi$. Then $\{b_n\} \notin S_\pi$, for otherwise, by the above proof, $\{a_n\}$ would also belong to S_π.

https://doi.org/10.4153/CJM-1966-058-8 Published online by Cambridge University Press

It follows from this theorem that the conditions in the hypotheses of all of the theorems that follow need apply not to the whole sequence, but just to the part beyond some index.

THEOREM 3. *For any sequence* $\{a_n\}$ *of real numbers, the sequence* $\{-a_n - n\}$ *belongs to* S_π *if and only if* $\{a_n\} \in S_\pi$.

Proof. Let $f \in K_\pi$ and let $g(z) = f(-z)$. Then $g \in K_\pi$ since the horizontal strip H_π is symmetric with respect to the origin. Also

$$\begin{aligned}\Delta^n g(-a_n - n) &= \sum_{k=0}^{n} (-1)^{n+k}\binom{n}{k} g(-a_n - n + k)\\ &= \sum_{m=0}^{n} (-1)^m \binom{n}{m} g(-a_n - m)\\ &= (-1)^n \sum_{m=0}^{n} (-1)^{m+n}\binom{n}{m} f(a_n + m)\\ &= (-1)^n \Delta^n f(a_n).\end{aligned}$$

Let $\{a_n\} \in S_\pi$. Then, if $\Delta^n g(-a_n - n) = 0$ $(n = 0, 1, 2, \ldots)$, $\Delta^n f(a_n) = 0$ $(n = 0, 1, 2, \ldots)$, so that $f = 0$, which implies that $g = 0$. Conversely, if $\{a_n\} \notin S_\pi$, there exists a function $f \not\equiv 0$ in K_π such that $\Delta^n f(a_n) = 0$ $(n = 0, 1, 2, \ldots)$. Then $g(z) = f(-z)$ is in K_π, is not the zero function, and has the property that $\Delta^n g(-a_n - n) = 0$ $(n = 0, 1, 2, \ldots)$, so that

$$\{-a_n - n\} \notin S_\pi.$$

3. Arithmetic sequences: $a_n = \beta n$. It has been shown by the author **(6)** that a necessary and sufficient condition that $K[G]$ be a uniqueness class for $\{\Delta^n f(\beta n)\}$ is that the function $W(\zeta) = e^{\beta\zeta}(e^\zeta - 1)$ in (1.1) be univalent on G. Buck **(2)** has shown that $W(\zeta)$ is not univalent on the strip H_π if $\beta > 0$, so that $\{\beta n\} \notin S_\pi$. Using this together with Theorem 3, we also have $\{\beta n\} \notin S_\pi$ for $\beta < -1$. If $-1 \leqslant \beta \leqslant 0$, Kober **(10**, p. 114**)** shows that $W(\zeta)$ is univalent on H_π, so that for these values of β, $\{\beta n\} \in S_\pi$. Thus, we have the following theorem.

THEOREM 4. *The sequence* $\{\beta n\}$, β *real, belongs to* S_π *if and only if*

$$-1 \leqslant \beta \leqslant 0.$$

In fact, as β decreases through positive values, the size of the largest uniqueness class $K[G] \subset K$ keeps increasing until it equals K_π when $\beta = 0$. Then as β continues to decrease through the negative reals, the corresponding uniqueness class remains constant until β reaches -1, and then decreases in size as β continues to decrease. It was this interesting behaviour that led to the present investigation.

Since

$$\Delta^n f(a_n) = \sum_{k=0}^{n} (-1)^{n+k}\binom{n}{k} f(a_n + k),$$

https://doi.org/10.4153/CJM-1966-058-8 Published online by Cambridge University Press

the difference $\Delta^n f(a_n)$ is determined by values taken by f at points spaced a unit distance apart in the interval $[a_n, a_n + n]$. Let $I_b(a)$ denote the interval $[a, a + b]$. Then for $a_n = \beta n$, the condition that $-1 \leqslant \beta \leqslant 0$ is equivalent to the condition that for each n, $I_k(a_k) \subset I_n(a_n)$ for all $k < n$. Thus Theorem 4 could have been stated using this as the hypothesis. We shall see in later sections that containing relations of this nature play an important role with other sequences $\{a_n\}$.

4. A sufficient condition for sequences of integers. Let A be the set of sequences $\{a_n\}$ of integers such that for each N_j in some subsequence $\{N_j\}$ of the positive integers, $I_n(a_n) \subset I_{N_j}(a_{N_j})$ for all $n < N_j$.

Examples:

$2, 0, 0, 2, 0, 0, \ldots \in A$; $2, 0, 0, 3, 0, 0, 0, 4, 0, 0, 0, 0, \ldots \in A$;

$0, -1, -2, -3, \ldots \in A$; $0, -2, -2, -5, -5, -5, -9, -9, -9, -9, \ldots \in A$; $2, 0, 2, 0, 2, 0, \ldots \notin A$;

$2, 0, 3, 0, 0, 4, 0, 0, 0, \ldots \notin A$; $0, -2, -1, -4, -3, -6, -5, \ldots \notin A$.

Our next theorem states that $A \subset S_\pi$. To prove this, we shall use the following lemma.

LEMMA 1. *Let r and s be non-negative integers with $r < s$. Let $\{a_k\}$ be a sequence of integers having the property that $I_k(a_k) \subset I_s(a_s)$ for all k such that $r \leqslant k \leqslant s$. Let f be any function such that*

$$\Delta^k f(a_k) = 0 \qquad (k = r, r + 1, \ldots, s).$$

Then

$$\Delta^k f(a_s + m) = 0 \qquad (k = r, r + 1, \ldots, s; m = 0, 1, \ldots, s - k).$$

Proof. The condition that $I_k(a_k) \subset I_s(a_s)$ is equivalent to $a_s \leqslant a_k$ and $a_k + k \leqslant a_s + s$ from the definition of $I_b(a)$. Thus we have

$$a_s \leqslant a_k \leqslant a_s + s - k$$

for all k such that $r \leqslant k \leqslant s$. The proof depends on the fact that for any a and j, $\Delta^{j+1} f(a) = \Delta^j f(a + 1) - \Delta^j f(a)$. From this, it follows that

$$\Delta^j f(a) = 0 \text{ and } \Delta^{j+1} f(a - 1) = 0 \text{ imply } \Delta^j f(a - 1) = 0; \tag{4.1}$$

$$\Delta^j f(a) = 0 \text{ and } \Delta^{j+1} f(a) = 0 \text{ imply } \Delta^j f(a + 1) = 0. \tag{4.2}$$

We are given $\Delta^s f(a_s) = 0$ and we have either $a_{s-1} = a_s$ or $a_{s-1} = a_s + 1$. If $\Delta^{s-1} f(a_s) = 0$, then by (4.2), $\Delta^{s-1} f(a_s + 1) = 0$ also. If $\Delta^{s-1} f(a_s + 1) = 0$, then $\Delta^{s-1} f(a_s) = 0$ by (4.1). Thus, in either case, both $\Delta^{s-1} f(a_s) = 0$ and $\Delta^{s-1} f(a_s + 1) = 0$. Now $a_{s-2} = a_s$ or $a_{s-2} = a_s + 1$ or $a_{s-2} = a_s + 2$. In the first case, since $\Delta^{s-1} f(a_s) = 0$ and $\Delta^{s-2} f(a_s) = 0$, then $\Delta^{s-2} f(a_s + 1) = 0$ by (4.2), from which $\Delta^{s-2} f(a_s + 2) = 0$ again by (4.2). In the second case, $\Delta^{s-2} f(a_s) = 0$ by (4.1) and $\Delta^{s-2} f(a_s + 2) = 0$ by (4.2). In the third case

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$\Delta^{s-2}f(a_s + 1) = 0$ by (4.1), from which $\Delta^{s-2}f(a_s) = 0$ also by (4.1). Thus in all three cases,

$$\Delta^{s-2}f(a_s + m) = 0 \qquad (m = 0, 1, 2).$$

In this manner, we can prove the lemma by induction. Suppose that for some q $0 \leqslant q \leqslant s - r$,

$$\Delta^{s-k}f(a_s + m) = 0 \qquad (k = 0, 1, \ldots, q - 1; m = 0, 1, \ldots, k).$$

We have $a_s \leqslant a_{s-q} \leqslant a_s + q$. If $a_{s-q} > a_s$, then $\Delta^{s-q}f(a_{s-q} - 1) = 0$ by (4.1) and if $a_s < a_{s-q} - 1$, $\Delta^{s-q}f(a_{s-q} - 2) = 0$ by (4.1). In this manner (by induction on m) we obtain

$$\Delta^{s-q}f(a_{s-q} - m) = 0 \qquad (m = 0, 1, \ldots, a_{s-q} - a_s).$$

Similarly, using (4.2) and induction, we obtain

$$\Delta^{s-q}f(a_{s-q} + m) = 0 \qquad (m = 1, 2, \ldots, a_s - a_{s-q} + q).$$

Thus $\Delta^{s-q}f(a_s + m) = 0$ $(m = 0, 1, \ldots, q)$. This completes the induction proof.

THEOREM 5. *Suppose* $\{a_n\} \in A$; *i.e. all* a_n *are integers and there exists a subsequence* $\{N_j\}$ *of the positive integers such that for each* j,

$$I_n(a_n) \subset I_{N_j}(a_{N_j}) \qquad \text{for all } n < N_j.$$

Then $\{a_n\} \in S_\pi$.

Proof. Let $\{a_n\} \in A$ and let f in K_π have the property that for each n, $\Delta^n f(a_n) = 0$. We can apply Lemma 1 with $r = 0$ and $s = N_j$ obtaining

$$\Delta^{N_j - k}f(a_{N_j} + m) = 0 \qquad (m = 0, 1, \ldots, k; k = 0, 1, \ldots, N_j) \tag{4.3}$$

for each j. Suppose, first, that the sequence $\{a_{N_j}\}$ is bounded below, say

$$M = \inf_j a_{N_j}.$$

Then, from some value of j on, $a_{N_j} = M$. Thus, from (4.3), we have

$$\Delta^n f(M) = 0 \qquad (n = 0, 1, 2, \ldots).$$

Then for $g(z) = f(z + M)$, $g \in K_\pi$ and $\Delta^n g(0) = 0$ $(n = 0, 1, 2, \ldots)$, so that $g = 0$, which implies that $f = 0$. Now, suppose $\{a_{N_j}\}$ has no lower bound. Then, from the fact that $I_0(a_0) \subset I_{N_j}(a_{N_j})$ for each j, we have

$$a_{N_j} \leqslant a_0 \leqslant a_{N_j} + N_j.$$

From (4.3), taking $k = N_j$, we have

$$f(a_{N_j} + m) = 0 \qquad (m = 0, 1, \ldots, N_j).$$

Since $\{a_{N_j}\}$ has no lower bound, this implies that $f(m) = 0$ for all integers $m \leqslant a_0$. Thus, for $g(z) = f(a_0 - z)$, $\Delta^n g(0) = 0$ $(n = 0, 1, 2, \ldots)$, so that since $g \in K_\pi$, $g = 0$, which implies that $f = 0$.

5. Other sequences of integers. Consider the periodic sequence a_0, 0, 0, ..., 0, a_0, 0, 0, ... of non-negative integers of period p, i.e., $a_n = a_0$ for $n = pk$, $k = 0, 1, 2, \ldots$, and $a_n = 0$ otherwise. Then $\{a_n\} \in A$ if and only if $a_0 \leqslant p - 1$. Now, if we take $a_0 = p$, we find an interesting phenomenon; namely, that $\{a_n\} \in S_\pi$ if and only if p is an odd number. We next deal with a class B of sequences that includes these examples.

We start with any sequence $\{a_n\}$ in A that is bounded below. Since, by Theorem 1, translations $\{a_n + k\}$, k fixed, of $\{a_n\}$ have the same properties with respect to differences as $\{a_n\}$ itself, we can assume that the minimum a_n is zero. Since $\{a_n\} \in A$, there exists a subsequence $\{a_{N_j}\}_{j=1}^{\infty}$ of $\{a_n\}$ such that for each j, $I_n(a_n) \subset I_{N_j}(a_{N_j})$ for all $n < N_j$. Since any subsequence of $\{a_{N_j}\}$ will also have this property, we can assume that for each j, $N_{j+1} - N_j > 1$ and since for all sufficiently large j, $a_{N_j} = 0$, we can assume that this holds for all j. Let $n_0 = 0$ and $n_j = N_j + 1$ $(j = 1, 2, 3 \ldots)$. Let $\{b_n\}$ be any sequence satisfying

$$(5.1) \qquad b_n = a_n, \qquad n \notin \{n_j\}, \qquad n_{j+1} - n_j \leqslant b_{n_j} \leqslant n_{j+2} - n_j - 1.$$

Let B be the class of all such sequences $\{b_n\}$ obtained from all sequences $\{a_n\}$ in A satisfying the conditions stated in this paragraph. *Examples*:

$$2, 0, 2, 0, \ldots \in B; \quad 3, 0, 3, 0, \ldots \in B; \quad 4, 0, 4, 0, \ldots \notin B;$$
$$5, 0, 0, 5, 0, 0, \ldots \in B; \quad 6, 0, 0, 6, 0, 0, \ldots \notin B.$$

In what follows, the notation $\{n_j\}$ will be used only for the subsequence of the non-negative integers associated with some given sequence $\{b_n\}$ in B by the above definition.

We define the class B in this way so that it will have the following property.

LEMMA 2. *Let $\{b_n\} \in B$. If f is any function defined on the integers, then $\Delta^n f(b_n) = 0$ $(n = 0, 1, 2, \ldots)$ if and only if*

$$(5.2) \qquad \begin{aligned} &\text{(i) } \Delta^n f(0) = 0 \text{ for } n \notin \{n_j\}, \\ &\text{(ii) } \Delta^{n_j} f(0) = \frac{(-1)^j f(0)}{\prod_{i=1}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}}} \qquad (j = 1, 2, 3, \ldots). \end{aligned}$$

Proof. For both necessity and sufficiency, for $n_{j-1} < n < n_j$, the conditions of Lemma 1 are fulfilled with $r = n_{j-1} + 1$ and $s = n_j - 1$; so

$$\Delta^{n_j - k - 1} f(b_{n_{j-1}} + m) = 0 \qquad (m = 0, 1, \ldots, k;\ k = 0, 1, \ldots, n_j - n_{j-1} + 2).$$

Since $b_{n_{j-1}} = 0$, this can be written

$$\Delta^{n_j - k} f(m) = 0 \qquad (m = 0, 1, \ldots, k - 1;\ k = 1, 2, \ldots, n_j - n_{j-1} + 1).$$

In particular, from this we have $\Delta^n f(b_n) = 0$ for $n \notin \{n_j\}$ if and only if $\Delta^n f(0) = 0$ for $n \notin \{n_j\}$.

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Also, by induction, (5.2) (ii) is equivalent to

$$\Delta^{n_j}f(0) = \frac{-\Delta^{n_{j-1}}f(0)}{\binom{b_{n_{j-1}}}{n_j - n_{j-1}}} \qquad (j = 1, 2, 3, \ldots). \tag{5.3}$$

Thus, it remains to show that if $\{b_n\} \in B$ and $\Delta^n f(0) = 0$ for $n \notin \{n_j\}$, then

$$\Delta^{n_j}f(b_{n_j}) = 0 \qquad (j = 1, 2, 3, \ldots)$$

if and only if (5.3) is satisfied. Let $\Delta^{n_{j-1}}f(0) = c$ and $\Delta^{n_j}f(0) = d$. Using repeatedly the fact that

$$\Delta^k f(m) = \Delta^k f(m-1) + \Delta^{k+1}f(m-1),$$

we obtain

$$\Delta^{n_{j-1}}f(m) = c \qquad (m = 0, 1, \ldots, n_j - n_{j-1} - 1),$$

$$\Delta^{n_j}f(m) = d \qquad (m = 0, 1, \ldots, n_{j+1} - n_j - 1),$$

and

$$\Delta^{n_j - k}f(k) = d \qquad (k = 0, 1, \ldots, n_j - n_{j-1} - 1).$$

Continuing in the same way, we obtain

$$\Delta^{n_j - k}f(q) = \binom{q}{k}d \qquad (k = 0, 1, \ldots. n_j - n_{j-1} - 1;$$
$$q = k, k+1, \ldots, k + n_{j-1} - n_j - 1).$$

Also

$$\Delta^{n_{j-1}}f(q) = c + \binom{q}{n_j - n_{j-1}}d$$
$$(q = n_j - n_{j-1}, n_j - n_{j-1} + 1, \ldots, b_{n_{j-1}} - 1).$$

Thus, we have

$$\Delta^{n_{j-1}}f(b_{n_{j-1}} - 1) = c + \binom{b_{n_j} - 1}{n_j - n_{j-1}}d$$

and

$$\Delta^{n_{j-1}+1}f(b_{n_{j-1}} - 1) = \binom{b_{n_{j-1}} - 1}{n_j - n_{j-1} - 1}d$$

so that

$$c + \left[\binom{b_{n_{j-1}} - 1}{n_j - n_{j-1}} + \binom{b_{n_{j-1}} - 1}{n_j - n_{j-1} - 1}\right]d = \Delta^{n_{j-1}}f(b_{n_{j-1}})$$

or

$$c + \binom{b_{n_{j-1}}}{n_j - n_{j-1}}d = \Delta^{n_{j-1}}f(b_{n_{j-1}}).$$

Thus

$$d = \frac{-c}{\binom{b_{n_{j-1}}}{n_j - n_{j-1}}} \quad \text{if and only if} \quad \Delta^{n_{j-1}}f(b_{n_{j-1}}) = 0.$$

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Thus, if $\{b_n\} \in B$, for any function f such that $\Delta^n f(b_n) = 0$ ($n = 0, 1, 2, \ldots$) the sequence $\{\Delta^n f(0)\}$ depends only on $f(0)$. We use the following result due to Buck to obtain results for sequences in B.

LEMMA 3. *For a given sequence $\{c_n\}$ of complex numbers, there exists a function f in K_π such that $\Delta^n f(0) = c_n$ ($n = 0, 1, 2, \ldots$) if and only if the function $b(z)$ defined by $\sum_{n=0}^{\infty} c_n z^n$ is analytic at the origin and can be continued analytically along the negative real axis to the interval* $[-1, 0]$.

For a proof of this lemma, see (**3** or **5**).

Given a sequence $\{b_n\}$ in B, let

$$d_j = (-1)^j \left[\prod_{i=1}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}} \right]^{-1} \qquad (j = 1, 2, 3, \ldots) \tag{5.4}$$

and define a function h by

$$h(z) = 1 + \sum_{j=1}^{\infty} d_j z^{n_j}.$$

Suppose h is analytic at the origin and can be continued analytically along the negative real axis to the interval $[-1, 0]$. Then, by Lemma 3, there exists a function f in K_π such that $\Delta^n f(0) = 0$ if $n \notin \{n_j\}$ and $\Delta^{n_j} f(0) = d_j$. By Lemma 2, we then have $\Delta^n f(b_n) = 0$ ($n = 0, 1, \ldots$) and $f \not\equiv 0$ (since $f(0) = 1$). Thus $\{b_n\} \notin S_\pi$. On the other hand, if $h(z)$ is not analytic at the origin or cannot be continued to $[-1, 0]$, then there is no function f in K_π such that $\Delta^n f(0) = 0$ for $n \notin \{n_j\}$ and $\Delta^{n_j} f(0) = d_j$. This implies, by Lemma 2, that there is no function f in K_π satisfying

$$\Delta^n f(b_n) = 0 \quad (n = 0, 1, 2, \ldots) \qquad \text{and} \quad f(0) = 1. \tag{5.5}$$

But then there is no function g in K_π such that $\Delta^n g(b_n) = 0$ ($n = 0, 1, 2, \ldots$) and $g(0) = c \neq 0$, since (5.5) would then be satisfied by $f(z) = c^{-1} g(z)$. Thus $f \in K_\pi$ and $\Delta^n f(b_n) = 0$ ($n = 0, 1, 2, \ldots$) implies that $f(0) = 0$ so that, by Lemma 2, $f = 0$. Therefore, $\{b_n\} \in S_\pi$. We have proved the following lemma.

LEMMA 4. *Let $\{b_n\} \in B$ and let*

$$h(z) = 1 + \sum_{j=1}^{\infty} d_j z^{n_j}$$

where d_j is given by (5.4). *Then $\{b_n\} \in S_\pi$ if and only if $h(z)$ cannot be continued analytically from the origin along the negative real axis to the interval* $[-1, 0]$.

We first consider sequences $\{b_n\}$ in B with

$$b_{n_{j-1}} = n_j - n_{j-1} \qquad (j = 1, 2, 3, \ldots).$$

Then $\{b_n\}$ has the property that if the elements of any subsequence of the subsequence $\{b_{n_j}\}$ were each decreased by 1, the resulting sequence would belong to A and hence necessarily to S_π. We show which of these sequences $\{b_n\}$ belong to S_π and which do not.

Let f be any function in K_π such that

$$\Delta^n f(b_n) = 0 \qquad (n = 0, 1, 2, \ldots).$$

Then, by Lemma 2,

$$\Delta^{n_j} f(0) = (-1)^j \left[\prod_{i=1}^{j} \binom{n_i - n_{i-1}}{n_i - n_{i-1}} \right]^{-1} f(0) = (-1)^j f(0),$$

so that

$$h(z) = \sum_{j=0}^{\infty} (-1)^j z^{n_j}.$$

Since the sequence of coefficients of this power series has only a finite number of distinct values, we can apply a theorem of Szegö (**8**, p. 324), which says that such a power series represents either a function $P(z)/(1 - z^m)$ for some polynomial P and some non-negative integer m, or else a function that cannot be continued beyond $|z| = 1$, its circle of convergence. In the latter case, $h(z)$ cannot be analytic at $z = -1$; so $\{b_n\} \in S_\pi$ by Lemma 4.

If $h(z) = P(z)/(1 - z^m)$, then the sequence of coefficients of the power series is periodic from some point on. If it is of period p, then p divides m. Let q be the number of non-zero coefficients in each period. Then, for some N,

$$\begin{aligned} h(z) &= \sum_{j=0}^{N-1} (-1)^j z^{n_j} + (-1)^N \sum_{k=0}^{\infty} \sum_{j=0}^{q-1} (-1)^{j+kq} z^{n_{N+j}+kp} \\ &= \sum_{j=0}^{N-1} (-1)^j z^{n_j} + (-1)^N \sum_{k=0}^{\infty} (-1)^{kq} z^{kp} \sum_{j=0}^{q-1} (-1)^j z^{n_{N+j}} \\ &= \sum_{j=0}^{N-1} (-1)^j z^{n_j} + (-1)^N \frac{\sum_{j=0}^{q-1} (-1)^j z^{n_{N+j}}}{1 - (-1)^q z^p}. \end{aligned}$$

Thus $h(z)$ is analytic on $[-1, 0]$ except possibly at $z = -1$. If $p + q$ is odd, then $h(z)$ is analytic at $z = -1$, so that $\{b_n\} \notin S_\pi$. If p $+ q$ is even, then $h(z)$ is analytic at $z = -1$ if and only if

$$\sum_{j=0}^{q-1} (-1)^j z^{n_{N+j}} = 0$$

when $z = -1$; i.e., if and only if

$$\sum_{j=0}^{q-1} (-1)^{j+n_{N+j}} = 0.$$

If q is an odd number, then this expression has an odd number of terms, each ± 1, so the sum cannot be zero, in which case $\{b_n\} \in S_\pi$. If q is even, then the sum is zero if and only if exactly half of the numbers $\{j + n_{N+j}\}_{j=0}^{q-1}$ are odd and half are even. Thus, for p and q both even, $\{b_n\} \notin S_\pi$ if and only if this is the case.

We collect these results in the following theorem.

THEOREM 6. *Let* $\{b_n\} \in B$ *and suppose* $b_{n_{j-1}} = n_j - n_{j-1}$ $(j = 1, 2, 3, \ldots)$. *Then* $\{b_n\} \notin S$ *if and only if the following two conditions are satisfied:*

(a) *There exist integers* N *and* q *such that* $j \geqslant N$ *implies that* $b_{n_{j+q}} = b_{n_j}$.

(b) *If* $n_{N+q} - n_N = p$, *then either* (i) $p + q$ *is odd, or* (ii) p *and* q *are both even and exactly half of the numbers* $\{j + n_j\}_{j=N}^{N+q-1}$ *are even.*

Thus every sequence in B with

$$b_{n_{j-1}} = n_j - n_{j-1} \qquad (j = 1, 2, 3, \ldots)$$

which is not ultimately periodic is in S_π. Examples of periodic sequences covered by Theorem 6 (the bar indicates the last element in the first period) are:

$$\left.\begin{array}{l} 2, 0, 3, 0, \bar{0}, \ldots \\ 2, 0, 3, 0, 0, 3, 0, \bar{0}, \ldots \\ 2, 0, 3, 0, 0, 4, 0, 0, 0, 3, 0, \bar{0}, \ldots \end{array}\right\} \notin S_\pi \quad \left.\begin{array}{l} 3, 0, \bar{0}, \ldots \\ 2, 0, 2, 0, 3, 0, \bar{0}, \ldots \\ 2, 0, 3, 0, 0, 3, 0, 0, 4, 0, 0, \bar{0}, \ldots \end{array}\right\} \in S_\pi.$$

We remark that of the sequences $\{b_n\}$ in B with $b_{n_j} = n_{j+1} - n_j$, uncountably many are in S_π and only countably many are not in S_π.

THEOREM 7. *Let* $\{b_n\} \in B$ *with* $b_{n_{j-1}} \geqslant n_j - n_{j-1} + 1$ *for each* j. *Let the sequence* $\{n_j - n_{j-1}\}$ *be bounded. Then* $\{b_n\} \notin S_\pi$.

Proof. Let M be an upper bound for $\{n_j - n_{j-1}\}$. Let $f \in K_\pi$ with $\Delta^n f(b_n) = 0$ $(n = 0, 1, 2, \ldots)$. Then by Lemma 2, $\Delta^n f(0) = 0$ for $n \notin \{n_j\}$ and

$$\Delta^{n_j} f(0) = (-1)^j \left[\prod_{i=1}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}}\right]^{-1} f(0).$$

Now, for each i, we have $2 \leqslant n_i - n_{i-1} \leqslant M$ and $n_i - n_{i-1} + 1 \leqslant b_{n_{i-1}}$. Therefore

$$\binom{b_{n_{i-1}}}{n_i - n_{i-1}} \geqslant \binom{n_i - n_{i-1} + 1}{n_i - n_{i-1}} \geqslant \binom{3}{2} = 3.$$

Also,

$$n_j = \sum_{i=1}^{j} (n_i - n_{i-1}) \leqslant Mj.$$

Hence

$$\limsup_{n\to\infty} |\Delta^n f(0)|^{1/n} = \limsup_{j\to\infty} \left\{\left[\prod_{i=1}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}}\right]^{-1} f(0)\right\}^{1/n_j}$$

$$\leqslant \limsup_{j\to\infty} \left[\prod_{i=1}^{j} 3\right]^{-1/n_j} \leqslant \lim_{j\to\infty} (3^j)^{-1/Mj} = 3^{-1/M} < 1.$$

Therefore

$$h(z) = 1 + \sum_{j=0}^{\infty} \left[\prod_{i=1}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}}\right]^{-1} z^{n_j}$$

is analytic in a disk containing the interval $[-1, 0]$ so that $\{b_n\} \notin S_\pi$.

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Thus for sequences $\{b_n\}$ in $A \cup B$ with $\min_n b_n = 0$ and with $\{n_j - n_{j-1}\}$ bounded, if for each j, $b_{n_{j-1}} < n_j - n_{j-1}$, then $\{b_n\} \in S_\pi$; if $b_{n_{j-1}} > n_j - n_{j-1}$, then $\{b_n\} \notin S_\pi$, and if $b_{n_j} = n_j - n_{j-1}$, then $\{b_n\}$ may or may not be in S_π, the answer being given by Theorem 6. We now consider certain sequences in B with the sequence $\{n_j - n_{j-1}\}$ approaching infinity.

THEOREM 8. *Let $\{b_n\} \in B$ and suppose that*

$$\lim_{j\to\infty} (n_j - n_{j-1}) = \infty$$

and that the sequence $\{b_{n_j} - (n_j - n_{j-1})\}$ is bounded. Then $\{b_n\} \in S_\pi$.

Proof. Let the integer M be an upper bound for the sequence

$$\{b_{n_j} - (n_j - n_{j-1})\}.$$

If we can show that

$$\limsup_{j\to\infty} \left[\prod_{i=1}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}}\right]^{1/n_j} = 1, \tag{5.6}$$

then the radius of convergence of the series for $h(z)$ in Lemma 4 is 1. But from the hypothesis that

$$\lim_{j\to\infty} (n_j - n_{j-1}) = \infty,$$

the circle of convergence of this series is a natural boundary for $h(z)$ (**8**, p. 376) so that h cannot be continued to the interval $[-1, 0]$. Thus we shall have established that $\{b_n\} \in S_\pi$.

It remains to prove (5.6). Since each

$$\binom{b_{n_{i-1}}}{n_i - n_{i-1}} \geqslant 1,$$

we have

$$\limsup_{j\to\infty} \left[\prod_{i=1}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}}\right]^{1/n_j} \geqslant 1.$$

We shall show that it is also less than or equal to 1. Since $(n_i - n_{i-1}) \to \infty$ and for each i, $b_{n_{i-1}} - (n_i - n_{i-1}) \leqslant M$, we can choose N such that $i \geqslant N$ implies that $b_{n_{i-1}} > 2M$. Then, for $i \geqslant N$,

$$\binom{b_{n_{i-1}}}{n_i - n_{i-1}} = \binom{b_{n_{i-1}}}{b_{n_{i-1}} - (n_i - n_{i-1})} \leqslant \binom{b_{n_{i-1}}}{M} \leqslant \binom{M + n_i - n_{i-1}}{M}.$$

Thus

$$\begin{aligned}\limsup_{j\to\infty} \left[\prod_{i=1}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}}\right]^{1/n_j} &= \limsup_{j\to\infty} \left[\prod_{i=N}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}}\right]^{1/n_j} \\ &\leqslant \limsup_{j\to\infty} \left[\prod_{i=N}^{j} \binom{M + n_i - n_{i-1}}{M}\right]^{1/n_j} \\ &= \limsup_{j\to\infty} \left[\prod_{i=1}^{j} \binom{M + n_i - n_{i-1}}{M}\right]^{1/n_j}.\end{aligned}$$

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Now

$$\binom{M+n_i-n_{i-1}}{M} = \frac{(M+n_i-n_{i-1})}{M}\frac{(M+n_i-n_{i-1}-1)}{(M-1)}\cdots\frac{(n_i-n_{i-1}+1)}{1}$$
$$\leqslant (n_i-n_{i-1}+1)^M$$

so that

$$\limsup_{j\to\infty}\left[\prod_{i=1}^{j}\binom{M+n_i-n_{i-1}}{M}\right]^{1/n_j} \leqslant \limsup_{j\to\infty}\left[\prod_{i=1}^{j}(n_i-n_{i-1}+1)\right]^{M/n_j}.$$

Since the arithmetic mean $1+(n_j/j)$ of the numbers $(n_i-n_{i-1}+1)$ $(i=1,2,\ldots,j)$ is greater than or equal to their geometric mean, we have

$$\limsup_{j\to\infty}\left[\prod_{i=1}^{j}\binom{M+n_i-n_{i-1}}{M}\right]^{1/n_j} \leqslant \limsup_{j\to\infty}(1+(n_j/j))^{jM/n_j}.$$

Since

$$\lim_{x\to 0}(1+(1/x))^x = 1$$

and since $(n_j-n_{j-1})\to\infty$ implies that

$$\lim_{j\to\infty}(j/n_j) = 0,$$

we have the desired result.

Finally, we show that for $\{b_n\}$ in B with $(n_j-n_{j-1})\to\infty$, if b_{n_j} increases too fast, then $\{b_n\}\notin S_\pi$.

THEOREM 9. *Let* $\{b_n\}\in B$ *with*

$$\lim_{j\to\infty}(n_j-n_{j-1}) = \infty \quad \textit{and} \quad \lim_{j\to\infty}(b_{n_{j-1}}/(n_j-n_{j-1})) = \alpha > 1.$$

Then $\{b_n\}\notin S_\pi$.

Proof. It is only necessary to show that

$$\limsup_{j\to\infty}\left[\prod_{i=1}^{j}\binom{b_{n_{i-1}}}{n_i-n_{i-1}}\right]^{1/n_j} > 1$$

since this will imply that the radius of convergence of the series for $h(z)$ is greater than 1, which will imply that h is analytic on $[-1, 0]$. Since

$$\lim_{i\to\infty}(b_{n_{i-1}}/(n_i-n_{i-1})) = \alpha > 1,$$

if we take α_1 with $1<\alpha_1<\alpha$, there exists an integer N such that for $i>N$, $(b_{n_{i-1}}/(n_i-n_{i-1}))>\alpha_1$. Then, for $i>N$,

$$\binom{b_{n_{i-1}}}{n_i-n_{i-1}} = \frac{b_{n_{i-1}}(b_{n_{i-1}}-1)\ldots(b_{n_{i-1}}-(n_i-n_{i-1})+1)}{(n_i-n_{i-1})(n_i-n_{i-1}-1)\ldots\qquad 1}$$
$$\geqslant \left[\frac{b_{n_{i-1}}}{n_i-n_{i-1}}\right]^{n_i-n_{i-1}} > \alpha_1^{\,n_i-n_{i-1}}.$$

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Thus

$$\prod_{i=N}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}} \geqslant \alpha_1^{n_N - n_{N-1}} \alpha_1^{n_{N+1} - n_M} \ldots \alpha_1^{n_j - n_{j-1}} = \alpha_1^{n_j - n_{N-1}}$$

and

$$\limsup_{j\to\infty} \left[\prod_{i=1}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}} \right]^{1/n_j} = \limsup_{j\to\infty} \left[\prod_{i=N}^{j} \binom{b_{n_{i-1}}}{n_i - n_{i-1}} \right]^{1/n_j}$$
$$\geqslant \limsup_{j\to\infty} (\alpha_1^{n_j - n_{N-1}})^{1/n_j} = \alpha_1 > 1.$$

Thus, for $\{b_n\}$ in B with $(n_j - n_{j-1}) \to \infty$, membership in S_π is related to how fast $\{b_{n_j}\}$ increases in relation to $\{n_j - n_{j-1}\}$.

Examples:

$3, 0, 4, 0, 0, 5, 0, 0, 0, 6, 0, 0, 0, 0, \ldots \in S_\pi$ by Theorem 8;
$3, 0, 5, 0, 0, 7, 0, 0, 0, 9, 0, 0, 0, 0, \ldots \notin S_\pi$ by Theorem 9.

6. Periodic sequences of the form $a_{pn+k} = \beta k$. We next give a necessary condition on a periodic sequence $\{a_n\}$ with

(6.1) $$a_{pn+k} = \beta k \qquad (k = 0, 1, \ldots, p-1; n = 0, 1, 2, \ldots, \beta > 0)$$

in order for it to belong to S. This condition is $\beta < (p-1)^{-1}$. In terms of containing relations among the intervals $I_n(a_n)$, this is equivalent to the condition that for each n, $I_m(a_m) \subset I_{pn}(a_{pn})$ for all $m < pn$. It seems probable that this condition is also sufficient, but it has not been proved.

THEOREM 10. *Let $\{a_n\}$ be defined by* (6.1). *If $\beta > (p-1)^{-1}$, then $\{a_n\} \notin S_\pi$.*

Proof. We show this by finding distinct points ζ_1 and ζ_2 having the following properties:

(i) $|\mathrm{Im}(\zeta_1)| < \pi \quad (i = 1, 2)$,
(ii) $(e^{\zeta_1} - 1)^p = (e^{\zeta_2} - 1)^p$,
(iii) $e^{\beta k \zeta_1}(e^{\zeta_1} - 1)^k = e^{\beta k \zeta_2}(e^{\zeta_2} - 1)^k \quad (k = 1, 2, \ldots, p-1)$.

Then, letting $f(z) = e^{\zeta_1 z} - e^{\zeta_2 z}$ will give us a non-zero function f in K_π such that

$$\Delta^{pn+k} f(\beta k) = e^{\beta k \zeta_1}(e^{\zeta_1} - 1)^{pn+k} - e^{\beta k \zeta_2}(e^{\zeta_2} - 1)^{pn+k} = 0$$
$$(k = 0, 1, \ldots, p-1; n = 0, 1, 2, \ldots).$$

Buck **(2)** showed that for $\beta > 0$, the curve γ given by

$$\zeta = u + iv, \quad u = \log(\sin \beta v) - \log(\sin(\beta + 1)v), \ |v| < \pi/(\beta + 1),$$

is mapped onto the part of the negative real axis from $-\beta^\beta(\beta + 1)^{-(\beta+1)}$ to $-\infty$ by $W(\zeta) = e^{\beta\zeta}(e^\zeta - 1)$. The curve γ is symmetric with respect to the real axis, crosses the real axis at $\log(\beta/(\beta + 1))$ and the imaginary axis at $\pm\pi i/(2\beta + 1)$, and is asymptotic to the lines $v = \pm\pi i/(\beta + 1)$ as $u \to +\infty$. Since $W(\bar{\zeta}) = \overline{W(\zeta)}$ and the image of γ under W is real, complex conjugate

https://doi.org/10.4153/CJM-1966-058-8 Published online by Cambridge University Press

points on γ are mapped into the same point. Thus by taking ζ_1 on γ and $\zeta_2 = \bar{\zeta}_1$, properties (i) and (iii) above are satisfied. Let

$$\zeta_1 = \{\log[\sin(\pi/p)/\sin((\beta + 1)\pi/\beta p)]\} + i\pi/\beta p.$$

Since $\beta > (p - 1)^{-1}$; $\beta p - \beta > 1$; so $\beta p > \beta + 1$, which implies that $\pi/\beta p < \pi/(\beta + 1)$. Therefore $\zeta_1 \in \gamma$. Also

$$\tan(\arg(e^{\zeta_1} - 1)) = \frac{\sin(\pi/\beta p)\sin(\pi/p)}{\sin(\pi/p)\cos(\pi/\beta p) - \sin((\beta + 1)\pi/\beta p)}$$

$$= -\frac{\sin(\pi/p)}{\cos(\pi/p)} = -\tan(\pi/p).$$

This implies that $\arg(e^{\zeta_1} - 1) = -(\pi/p) + n\pi$ for some integer n, so that $\arg(e^{\zeta_1} - 1)^p = (pn - 1)\pi$. Therefore $(e^{\zeta_1} - 1)^p$ is real and since

$$(e^{\bar{\zeta}_1} - 1)^p = \overline{(e^{\zeta_1} - 1)^p},$$

we have $(e^{\bar{\zeta}_1} - 1)^p = (e^{\zeta_1} - 1)^p$. Therefore, if $f(z) = e^{\zeta_1 z} - e^{\bar{\zeta}_1 z}$, then $f \in K_\pi$ and $\Delta^n f(a_n) = 0$ $(n = 0, 1, 2, \ldots)$, so that $\{a_n\} \notin S_\pi$.

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