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A NOTE ON THE RELATIVE TRACE FORMULA

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ABSTRACT. This paper deals with the relative trace formula in the case of base change. Two truncations of the kernel are introduced, both based on the ideas of Arthur, and their integrals are shown to be asymptotic to each other. We also consider products of the kernel with automorphic forms, as these appear when comparing trace formulae (see [5]).

0. **Introduction.** Let G be a reductive algebraic group defined over $\mathbb{Q}$ and let E be a finite extension of $\mathbb{Q}$. Given any algebraic group H defined over $\mathbb{Q}$, write $\tilde{H}$ for the Weil restriction $\mathrm{Res}_{E/\mathbb{Q}} H$ of H from E to $\mathbb{Q}$, and $H(\mathbb{A})^1$ for the intersection of the kernels in $H(\mathbb{A})$ of the absolute values of all rational characters of H. In the Selberg-Arthur trace formula for the group $\tilde{G}$ one considers the regular representation R of $\tilde{G}(\mathbb{A})$ on $L^2\big(\tilde{G}(\mathbb{Q}) \setminus \tilde{G}(\mathbb{A})^1\big)$, given by

$$\big(R(y)\phi\big)(x) = \phi(xy), \quad x, y \in \tilde{G}(\mathbb{A})^1,\ \phi \in L^2\big(\tilde{G}(\mathbb{Q}) \setminus \tilde{G}(\mathbb{A})^1\big).$$

Given a function $f \in C_c^\infty\big(\tilde{G}(\mathbb{A})^1\big)$ we can consider the operator

$$R(f) = \int_{\tilde{G}(\mathbb{A})^1} f(y)R(y)\,dy.$$

It is well known that the operator $R(f)$ is an integral operator with kernel

$$K(x,y) = \sum_{\gamma \in \tilde{G}(\mathbb{Q})} f(x^{-1}\gamma y), \quad x, y \in \tilde{G}(\mathbb{A})^1.$$

On can decompose this kernel in two ways: one, the "geometric side," a sum over equivalence classes $\mathfrak{o}$ of conjugacy classes in $\tilde{G}(\mathbb{Q})$, the other, the "spectral side," a sum over equivalence classes of equivalence classes χ of cuspidal representations of Levi components of parabolic subgroups of $\tilde{G}$. This gives an equality

$$\sum_{\mathfrak{o} \in \mathfrak{O}} K_{\mathfrak{o}}(x,y) = \sum_{\chi \in \mathcal{X}} K_\chi(x,y), \quad x, y \in \tilde{G}(\mathbb{A})^1. \tag{0.1}$$

For the Selberg-Arthur trace formula, one considers only the case $x = y$. This gives an equality of two functions on $\tilde{G}(\mathbb{A})^1$ that are left $\tilde{G}(\mathbb{Q})$-invariant. One then truncates both sides of this equality, integrates over $\tilde{G}(\mathbb{Q}) \setminus \tilde{G}(\mathbb{A})^1$, and then (see [1], [2], and many more) adjusts the resulting equality into a more useful form.

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For the relative trace formula, one treats (0.1) as an equality of two functions on $G(\mathbb{A})^1 \times G(\mathbb{A})^1 \subset \tilde{G}(\mathbb{A})^1 \times \tilde{G}(\mathbb{A})^1$, truncates these functions, and integrates the resulting equality over $\big(G(\mathbb{Q}) \setminus G(\mathbb{A})^1\big)^2$. This idea was first carried out in [4] for the group GL_2, quadratic extensions E of $\mathbb{Q}$, and functions f satisfying certain local conditions. The paper [5] extended this to a larger class of functions, and considered the integral of the product of the truncated kernel and an automorphic form. The presence of this automorphic form was necessitated by the fundamental lemma proven in [5].

In this paper we will present an integrated equality using the truncations of Arthur, for arbitrary groups. We will also begin an investigation of the integral of the truncation of the geometric side. The equivalence classes $\mathfrak{o} \subset \tilde{G}(\mathbb{Q})$ will be different from those considered in the Selberg-Arthur trace formula because we need the function

$$K_{\mathfrak{o}}(x,y) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma y), \quad x, y \in G(\mathbb{A})^1$$

to be left $G(\mathbb{Q})$-invariant in both variables. The spectral kernels K_χ that appear in [1] are already bi-left $G(\mathbb{Q})$-invariant, and so do not need to be changed.

We note that although all statements in this paper treat an extension $E/\mathbb{Q}$, they are also true, and can be proved in exactly the same way, for arbitrary finite extensions E/F of number fields.

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1. **Preliminaries.** Let G be a reductive group defined over $\mathbb{Q}$. Fix, for the remainder of this paper, a minimal parabolic subgroup P_0 of G and a Levi component M_0 of P_0, both defined over $\mathbb{Q}$. When we refer to a parabolic subgroup we mean a standard parabolic subgroup. Given a parabolic subgroup P, write M_P for its Levi component containing M_0, N_P for its unipotent radical, and A_P for the split component of the centre of M_P. We write $A_P(\mathbb{R})^0$ for the connected component of the identity in $A_P(\mathbb{R})$. Choose a maximal compact subgroup K of $G(\mathbb{A})$ satisfying the properties on p. 917 of [1]. For P a parabolic subgroup of G, write Φ^P for the roots of (M_P, A) and $\Delta^P \subseteq \Delta$ for the simple roots.

Given an algebraic group H defined over $\mathbb{Q}$, we defined $\tilde{H}$ to be the group $\mathrm{Res}_{E/\mathbb{Q}} H$, also defined over $\mathbb{Q}$. Recall that $\tilde{H}(\mathbb{Q}) = H(E)$ and $\tilde{H}(E)$ is a product of $[E : \mathbb{Q}]$ copies of $H(E)$. There is a natural inclusion $H \subseteq \tilde{H}$ of algebraic groups that on $\mathbb{Q}$-points in the inclusion $H(\mathbb{Q}) \subseteq H(E)$ and on E-points is given by the diagonal map. This lets us treat H as an algebraic subgroup of $\tilde{H}$. The group $\tilde{P}_0$ is a minimal parabolic subgroup of $\tilde{G}$. Furthermore, given a parabolic subgroup P of G, the group $\tilde{P}$ is a (standard) parabolic subgroup of $\tilde{G}$ with Levi component $\tilde{M}_P$ and unipotent radical $\tilde{N}_p$, as expected. The component of the centre of $\tilde{M}_P$ that is split over $\mathbb{Q}$ is $A_P \subseteq \tilde{A}_P$, and the corresponding root system equals that of (M_P, A_P).

When P_i occurs as a subscript or superscript of a symbol other than T, we may write i for the subscript instead of P_i. When $i = 0$ a subscript of P_i may be left out altogether. The absence of an expected subscript of a parabolic subgroup indicates that the subscript is actually P_0.

Let $X(M_0)_{\mathbb{Q}}$ denote the group of characters of M_0 defined over $\mathbb{Q}$, and define the real vector spaces

$$\mathfrak{a} = \mathrm{Hom}\big(X(M_0)_{\mathbb{Q}}, \mathbb{R}\big), \quad \tilde{\mathfrak{a}} = \mathrm{Hom}\big(X(\tilde{M}_0)_{\mathbb{Q}}, \mathbb{R}\big).$$

Recall that $X(M_0)_{\mathbb{Q}}$ is canonically isomorphic to $X(A_0)_{\mathbb{Q}}$, so that $\mathfrak{a}$ is canonically isomorphic to $\mathrm{Hom}\big(X(A_0)_{\mathbb{Q}}, \mathbb{R}\big)$. This gives a canonical isomorphism between $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ that commutes with evaluation at roots, because roots are defined as characters on A_0. We will identify $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ under this isomorphism, its dimension is that of A_0. As usual there are continuous homomorphisms $H: G(\mathbb{A}) \longrightarrow \mathfrak{a}$, $\tilde{H}: \tilde{G}(\mathbb{A}) \longrightarrow \tilde{\mathfrak{a}}$, such that

$$e^{\langle \lambda, H(m)\rangle} = \lambda(m), \quad m \in M_0(\mathbb{A}), \lambda \in X(M_0)_{\mathbb{Q}}$$
$$e^{\langle \lambda, \tilde{H}(m)\rangle} = \lambda(m), \quad m \in \tilde{M}_0(\mathbb{A}), \lambda \in X(\tilde{M}_0)_{\mathbb{Q}}.$$

Then for x in $G(\mathbb{A})^1$, and α in Δ,

$$\alpha\big(\tilde{H}(x)\big) = [E:\mathbb{Q}]\alpha\big(H(x)\big).$$

Therefore, with our identification of $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$, the restriction of $\tilde{H}$ to the set $G(\mathbb{A}) \subset \tilde{G}(\mathbb{A})$ equals $[E:\mathbb{Q}]H$.

We will adopt the notation of Arthur; in particular $\tau = \tau_0$ is the characteristic function of the thin Weyl chamber $\mathfrak{a}^+$, $\hat{\tau}$ is the characteristic function of the thick Weyl chamber. We pick a fixed point $T_1 \in -\mathfrak{a}^+$ and a fixed compact set $\omega \subseteq N_0(\mathbb{A})M_0(\mathbb{A})^1$ so the the Siegel set

$$\mathfrak{s}^P(T_1, \omega) = \big\{\mathrm{pak} : p \in \omega, a \in A_0(\mathbb{R})^0 \cap G(\mathbb{A})^1, k \in K, \alpha\big(H(a) - T_1\big) > 0 \text{ for all } \alpha \in \Delta^P\big\}$$

is a fundamental set for $P(\mathbb{Q}) \setminus G(\mathbb{A})^1$ for each parabolic subgroup P of G. Given a parabolic subgroup P of G, the truncated Siegel set $\mathfrak{s}^P(T; T_1, \omega)$ is given by

$$\mathfrak{s}^P(T; T_1, \omega) = \big\{\mathrm{pak} \in \mathfrak{s}^P(T_1, \omega) : \varpi\big(H(a) - T\big) \leq 0 \text{ for all } \varpi \in \hat{\Delta}^P\big\},$$

and the function $F^P(\cdot, T)$ is the characteristic function of the the compact subset of $M_P(\mathbb{Q})N_P(\mathbb{A})A_P(\mathbb{R})^0 \setminus G(\mathbb{A})$ obtained as the projection to $M_P(\mathbb{Q})N_P(\mathbb{A})A_P(\mathbb{R})^0 \setminus G(\mathbb{A})$ of $P(\mathbb{Q})\mathfrak{s}^P(T; T_1, \omega)$. Given parabolic subgroups $P_1 \subseteq P_2$, the function $\sigma_{1,2}$ on $\mathfrak{a}_1$ is the characteristic function of the set of $X \in \mathfrak{a}_1$ such that

(1) $\alpha(X) > 0$ for all $\alpha \in \Delta_1^2$,
(2) $\alpha(X) \leq 0$ for all $\alpha \in \Delta_1 \setminus \Delta_1^2$, and
(3) $\varpi(X) > 0$ for all $\varpi \in \hat{\Delta}_2$; it is also given as an alternating sum over parabolic subgroups on p. 938 of [1].

Lemma 6.4 of [A1], a restatement of reduction theory, implies that for $g \in G(\mathbb{Q})$, the value of $F^P(g, T)$, with respect to the Siegel domain $\mathfrak{s}^P(T; T_1, \omega)$ corresponding to the point T_1, equals the value of $F^{\tilde{P}}(g, [E:\mathbb{Q}]T)$, with repsect to a Siegel domain $\mathfrak{s}^P(T; T_1, \tilde{\omega}) \subset \tilde{G}(\mathbb{A})^1$ corresponding to the point $[E:\mathbb{Q}]T_1$, where $\tilde{\omega} \cap N_0(\mathbb{A})M_0(\mathbb{A})^1 = \omega$. This means that the truncation we use in the next section can be seen as a truncation with respect to either the group G or the group $\tilde{G}$.

We define equivalence classes of orbits in $\tilde{G}(\mathbb{Q})$ under left and right multiplication by $G(\mathbb{Q})$ as in [6]: Two elements x, x' of $\tilde{G}(\mathbb{Q})$ are equivalent if their semisimple parts s, s' satisfy $s' \in G(\mathbb{Q})sG(\mathbb{Q})$. Notice that if $P = MN$ is a parabolic subgroup and $\mathfrak{o}$ is an equivalence class in $\tilde{G}(\mathbb{Q})$, then

$$\tilde{P}(\mathbb{Q}) \cap \mathfrak{o} = \big(\tilde{M}_P(\mathbb{Q}) \cap \mathfrak{o}\big)\tilde{N}_P(\mathbb{Q}).$$

Write $\mathfrak{O}$ for the collection of equivalence classes in $\tilde{G}(\mathbb{Q})$.

2. **Truncation of the kernel.** We preserve the earlier notation. Let f be a function in $C_c^\infty\big(\tilde{G}(\mathbb{A})^1\big)$. The integral over $(x,y) \in \big(G(\mathbb{Q}) \setminus G(\mathbb{A})^1\big)^2$ of

$$K(x,y) = \sum_{\gamma \in \tilde{G}(\mathbb{Q})} f(x^{-1}\gamma y)$$

will not converge in general because $K(x,y)$, although continuous, is not bounded. The easiest truncation we can apply to $K(x,y)$ is multiplication by the characteristic function of a compact set. For points $T, T' \in \mathfrak{a}^+$, let

$$\phi_f^{T,T'}(x,y) = K(x,y)F^G(x,T')F^G(y,T),$$

with F^G the compactly-supported function mentioned in the previous section. Recall that we have two expansions of the kernel $K(x,y)$, one as a sum over equivalence classes $\mathfrak{o} \in \mathfrak{O}$,

$$K(x,y) = \sum_{\mathfrak{o} \in \mathfrak{O}} K_{\mathfrak{o}}(x,y), \quad K_{\mathfrak{o}}(x,y) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma y),$$

the other a sum over equivalence classes of representations,

$$K(x,y) = \sum_{\chi \in X} K_\chi(x,y)$$

given in [1] (the set X corresponds to $\tilde{G}$). These lead to two expansions of $\phi_f^{T,T'}$.

Now let θ be a slowly increasing function on $G(\mathbb{Q}) \setminus G(\mathbb{A})^1$, so that θ is left $G(\mathbb{Q})$-invariant and there are positive constants $c, r > 0$ such that

$$|\theta(y)| \le c\|y\|^r$$

with $\|\cdot\|$ a norm on $\tilde{G}(\mathbb{A})$ defined as on p. 918 of [1]. We clearly have two expansions of $\phi_f^{T,T'}(x,y)\theta(y)$.

We must prove the absolute convergence of the integrals of these expansions of $\phi_f^{T,T'}(x,y)\theta(y)$ over $\big(G(\mathbb{Q}) \setminus G(\mathbb{A})^1\big)^2$. Let us deal with the geometric side first. It is well-known that there is a continuous seminorm $\|\cdot\|_0$ on $C_c^\infty\big(\tilde{G}(\mathbb{A})^1\big)$ such that for $f \in C_c^\infty\big(\tilde{G}(\mathbb{A})^1\big)$ and $x, y \in \tilde{G}(\mathbb{A})^1$,

$$\sum_{\gamma \in \tilde{G}(\mathbb{Q})} |f(x^{-1}\gamma y)| \le \|f\|_0 \|x\|^{2N},$$

so since the norm $\|\cdot\|$ is bounded on the truncated Siegel set $\mathfrak{s}^G(T; T_1, \omega)$,

$$\sum_{\mathfrak{o}\in\mathfrak{O}} \int_{(G(\mathbb{Q})\backslash G(\mathbb{A})^1)^2} |\phi_{f,\mathfrak{o}}^{T,T'}(x,y)|\theta(y)\,dx\,dy$$
$$\leq \int_{(G(\mathbb{Q})\backslash G(\mathbb{A})^1)^2} \sum_{\gamma\in\tilde{G}(\mathbb{Q})} |f(x^{-1}\gamma y)| F^G(x,T')F^G(y,T)\theta(y)\,dx\,dy$$

converges.

Convergence for the spectral side is equally simple. Corollary 4.6 of [1] implies that there exists a continuous seminorm $\|\cdot\|$ on $C_c^\infty\big(\tilde{G}(\mathbb{A})^1\big)$ and a positive integer N such that for $x,y \in \tilde{G}(\mathbb{A})^1$ and $f \in C_c^\infty\big(\tilde{G}(\mathbb{A})^1\big)$,

$$\sum_{\chi\in X} |K_\chi(x,y)| \leq \|f\|\,\|x\|^N\|y\|^N.$$

This clearly implies that

$$\sum_{\chi\in X} \int_{(G(\mathbb{Q})\backslash G(\mathbb{A})^1)^2} |\phi_{f,\chi}^{T,T'}(x,y)|\theta(y)\,dx\,dy$$
$$= \sum_{\chi\in X} \int_{(G(\mathbb{Q})\backslash G(\mathbb{A})^1)^2} |K_\chi(x,y)| F^G(x,T')F^G(y,T)\theta(y)\,dx\,dy$$

converges.

For equivalence classes $\mathfrak{o}$ and χ, write

$$J_\mathfrak{o}^{T,T'}(f) = \int_{(G(\mathbb{Q})\backslash G(\mathbb{A})^1)^2} \phi_{f,\mathfrak{o}}^{T,T'}(x,y)\theta(y)\,dx\,dy$$
$$J_\chi^{T,T'}(f) = \int_{(G(\mathbb{Q})\backslash G(\mathbb{A})^1)^2} \phi_{f,\chi}^{T,T'}(x,y)\theta(y)\,dx\,dy$$

The following is therefore true.

LEMMA 2.1. *Let $f \in C_c^\infty\big(\tilde{G}(\mathbb{A})\big)$. Then for each $T, T' \in \mathfrak{a}^+$,*

$$\sum_{\mathfrak{o}\in\mathfrak{O}} J_\mathfrak{o}^{T,T'}(f) = \sum_{\chi\in X} J_\chi^{T,T'}(f).$$

Having produced these distributions $J_\mathfrak{o}^{T,T'}, J_\chi^{T,T'}$ so directly, we must put a lot of effort into determining their behaviour with respect to T, T' sufficiently regular in $\mathfrak{a}^+$. For the remainder of this paper, we consider only the "geometric" distributions $J_\mathfrak{o}^{T,T'}$, $\mathfrak{o} \in \mathfrak{O}$.

One aspect of the behaviour of the distribution $J_\mathfrak{o}^{T,T'}$ as T, T' vary is easy to prove. It follows from the following lemma.

LEMMA 2.2. *Let G be a reductive group defined over $\mathbb{Q}$. Suppose that f is a bounded function on $G(\mathbb{A})^1$ of compact support. Then*

(i) There exists a compact set $C \subset G(\mathbb{A})^1$ such that

$$\sum_{\gamma\in G(\mathbb{Q})} f(x^{-1}\gamma y) = 0$$

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if $x,y \in G(\mathbb{Q}) \setminus G(\mathbb{A})$ with x not in the set yC.

(ii) The function

$$e^{-2\rho(H(a))} \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma a)$$

is bounded on $G(\mathbb{A})^1 \times \{a \in A(\mathbb{R})^0 \cap G(\mathbb{A})^1 : \tau\big(H(a) - T_1\big) = 1\}$.

PROOF. (i) is clear because f has compact support. (ii) follows from (i) and Lemma 3.2 of [2]. ∎

We will apply this lemma on the group $\tilde{G}$. Suppose that $K_{\mathfrak{o}}(x,y)$ is nonzero for some x,y in $G(\mathbb{Q}) \setminus G(\mathbb{A})^1$ and some $\mathfrak{o} \in \mathfrak{O}$. Then by the above lemma, there exists a compact set $C \subseteq \tilde{G}(\mathbb{A})^1$ such that $x \in yC$. This implies that there is a point $T_f \in \mathfrak{a}$ such that for every $T \in \mathfrak{a}^+$ and $T' \in T + T_f + \mathfrak{a}^+$, $F^G(y,T) = 1$ implies that $F^G(x,T') = 1$. Therefore we have the following.

LEMMA 2.3. *Let $f \in C_c^\infty\big(\tilde{G}(\mathbb{A})^1\big)$, $\mathfrak{o} \in \mathfrak{O}$. Then there exists a point T_f such that for $T' \in T + T_f + \mathfrak{a}^+$,*

$$K_{\mathfrak{o}}(x,y)F^G(x,T')F^G(y,T) = K_{\mathfrak{o}}(x,y)F^G(y,T).$$

In particular, $J_{\mathfrak{o}}^{T,T'}(f)$ is independent of $T' \in T + T_f + \mathfrak{a}^+$.

Let us call this number $J_{\mathfrak{o}}^T(f)$. It is the integral over $\big(G(\mathbb{Q}) \setminus G(\mathbb{A})^1\big)^2$ of the product of the function $\phi_{f,\mathfrak{o}}^T$ defined by

$$\phi_{f,\mathfrak{o}}^T(x,y) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma y)F^G(y,T), \quad x,y \in G(\mathbb{Q}) \setminus G(\mathbb{A})^1$$

and the function θ.

3. **The geometric side.** Let $f \in C_c^\infty\big(\tilde{G}(\mathbb{A})^1\big)$ and let $\mathfrak{o}$ be an equivalence class in $\tilde{G}(\mathbb{Q})$ as described in Section 1. In this section we show that $J_{\mathfrak{o}}^T(f)$ approximates another function whose behaviour is more explicit.

It is easy to see that if $E = \mathbb{Q}$ then the integral over $\big(G(\mathbb{Q}) \setminus G(\mathbb{A})^1\big)^2$ of $K(x,y)\theta(y)$ converges, and equals

$$\int_{G(\mathbb{A})^1} f(x)\,dx \int_{G(\mathbb{Q})\setminus G(\mathbb{A})^1} \theta(y)\,dy.$$

In this case there is only one double class, so $\mathfrak{O} = \{G(\mathbb{Q})\}$. The number $J_{G(\mathbb{Q})}^T$ approaches the above expression as T becomes increasingly regular. This case is not very interesting, so we will assume in the future that $[E : \mathbb{Q}] \geq 2$.

Define $k_{\mathfrak{o}}^T$, a function on $\big(G(\mathbb{Q}) \setminus G(\mathbb{A})^1\big)^2$ by

$$k_{\mathfrak{o}}^T(x,y) = \sum_{P \subseteq G} (-1)^{\dim(A_P/A_G)} \sum_{\delta,\delta' \in P(\mathbb{Q})\setminus G(\mathbb{Q})} \Big[\sum_{\gamma \in \mathfrak{o} \cap \tilde{M}_P(\mathbb{Q})} \int_{\tilde{N}_P(\mathbb{A})} f\big((\delta' x)^{-1}\gamma n(\delta y)\big)\,dn\Big]\hat{\tau}_P\big(H(\delta y) - T\big).$$

This is essentially the truncated kernel used in [6] and is based on the truncation in [1].

For $T \in \mathfrak{a}^+$, write $d(T) = \min_{\alpha \in \Delta}\big(\alpha(T)\big)$. This function measures the distance from T to the walls of the Weyl chamber. The following is the main result of this paper.

THEOREM 3.1. *Suppose that $[E:\mathbb{Q}] \geq 2$. Let $\varepsilon, k > 0$. There exists a continuous seminorm $\|\cdot\|$ on $C_c^\infty\big(\tilde{G}(\mathbb{A})^1\big)$ such that for all sufficiently regular $T \in \mathfrak{a}^+$ with $d(T) > \varepsilon\|T\|$, the expression*

$$\sum_{\mathfrak{o}\in\mathfrak{O}} \int_{(G(\mathbb{Q})\backslash G(\mathbb{A})^1)^2} |k_{\mathfrak{o}}^T(x,y) - \phi_{f,\mathfrak{o}}^T(x,y)|\theta(y)\,dx\,dy \tag{3.1}$$

is bounded by $\|f\|e^{-k\|T\|}$. In particular, the integral of $k_{\mathfrak{o}}^T$ converges absolutely.

PROOF. In the following, a fixed constant is one independent of both T and f. A fixed compact set will be independent of T, and its dependence on f will be clear from context.

As on pp. 942–943 of [1], the expression (3.1) equals the sum over equivalence classes $\mathfrak{o}$ of the integral over $\big(G(\mathbb{Q})\backslash G(\mathbb{A})^1\big)^2$ of

$$\Big|\sum_{P_1\subsetneq P_2}\sum_{\delta\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} F^1(\delta y, T)\sigma_{1,2}\big(H(\delta y)-T\big)$$
$$\Big(\sum_{\{P:P_1\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/Z)}\sum_{\delta'\in P(\mathbb{Q})\backslash G(\mathbb{Q})}\sum_{\gamma\in\tilde{M}_P(\mathbb{Q})\cap\mathfrak{o}}\int_{\tilde{N}_P(\mathbb{A})} f\big((\delta' x)^{-1}\gamma n(\delta y)\big)\,dn\Big)\Big|.$$

We can therefore clearly bound (3.1) by the sum over equivalence classes and over parabolic subgroups $P_1\subsetneq P_2$ of the integral over $(x,y)\in G(\mathbb{Q})\backslash G(\mathbb{A})^1 \times P_1(\mathbb{Q})\backslash G(\mathbb{A})^1$ of the product of

$$F^1(y,T)\sigma_{1,2}\big(H(y)-T\big)\theta(y) \tag{3.2}$$

and

$$\Big|\sum_{\{P:P_1\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/Z)}\sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})}\sum_{\gamma\in\tilde{M}_P(\mathbb{Q})\cap\mathfrak{o}}\int_{\tilde{N}_P(\mathbb{A})} f\big((\delta x)^{-1}\gamma n y\big)\,dn\Big|. \tag{3.3}$$

Notice that to bound this integral, it would not be sufficient simply to show that x belongs to the product of the projection of y to $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ and a fixed compact set.

We now fix representatives for both x and δ. For each parabolic subgroup $P\supseteq P_1$ pick a set S^P of representatives in $P_1(\mathbb{Q})\backslash G(\mathbb{A})^1$ of $P(\mathbb{Q})\backslash G(\mathbb{A})^1$ contained in $P_1(\mathbb{Q})\backslash P_1(\mathbb{Q})\mathfrak{S}^P(T_1,\omega)$. Given $x\in S^G$, write $(P\backslash G)_x$ for the set of $\delta\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})$ such that δx is in S^P; $(P\backslash G)_x$ is a set of representatives in $P_1(\mathbb{Q})\backslash G(\mathbb{Q})$ of $P(\mathbb{Q})\backslash G(\mathbb{Q})$ that depends on x.

We therefore replace the integral over x in $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ with the integral over $S^G\subseteq G(\mathbb{A})^1$ and replace the sum of δ in (3.3) with the sum over $(P\backslash G)_x$, so that δx lies in S^P. Assume that (3.2) is nonzero. Choose a representative of δx in $\mathfrak{S}^P(T_1,\omega)$ and of y in $\mathfrak{S}^1(T_1,\omega)$. These representatives can be decomposed as $n_P' n^{P'} m' b k'$ and $n_P n^P m a k$, respectively, with $n_P, n_P'\in N_P(\mathbb{A})$, $n^P, n^{P'}\in N^P(\mathbb{A})$, $m,m'\in M(\mathbb{A})^1$ all in fixed compact subsets of their respective domains, with $a,b\in A(\mathbb{R})^0\cap G(\mathbb{A})^1$, $k,k'\in K$ and

$$\tau^1\big(H(a)-T_1\big) = \tau^P\big(H(b)-T_1\big) = 1. \tag{3.4}$$

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As on p. 944 of [1] we can show that

$$\alpha\big(H(a)-T\big)>0 \quad \text{for all } \alpha\in\Delta^2\setminus\Delta^1\supseteq\Delta^P\setminus\Delta^1, \tag{3.5}$$

and so $a^{-1}n^Pma, b^{-1}n^{P'}m'b$ lie in fixed compact sets in $N^P(\mathbb{A})M(\mathbb{A})$. Now, since γ is in $\tilde{M}_P(\mathbb{Q})$ and the contributions of both n_P and N'_P can be absorbed in the integral over $\tilde{N}_P(\mathbb{A})$, we have

$$\int_{\tilde{N}_P(\mathbb{A})} f\big((\delta x)^{-1}\gamma ny\big)\,dn=\int_{\tilde{N}_P(\mathbb{A})} f\big(k'^{-1}(b^{-1}n^{P'}m'b)^{-1}b^{-1}\gamma na(a^{-1}n^Pma)k\big)\,dn.$$

For this to be nonzero, we must have that $b^{-1}\gamma\tilde{N}_P(\mathbb{A})a$ intersects a fixed compact set in $\tilde{G}(\mathbb{A})$ (depending on the support of f) and hence $b^{-1}\gamma a$ lies in a fixed compact set in $\tilde{M}_P(\mathbb{A})^1$. This will imply conditions on both γ and b.

Let $\gamma=\gamma'_n w_s\gamma_a\gamma_n$ be the Bruhat decomposition of γ, with γ_n,γ'_n in $\tilde{N}(\mathbb{A})$, w_s the representative in $M_P(\mathbb{Q})$ of an element s of the Weyl group of (M_P,A) and γ_a is in $\tilde{M}_0(\mathbb{Q})$. The element s is uniquely determined by γ. We know that

$$\begin{aligned} b\gamma a&=b^{-1}\gamma'_n w_s\gamma_a\gamma_n a\\ &=(b^{-1}\gamma'_n b)w_s\big(\gamma_a(w_sbw_s^{-1})^{-1}a\big)(a^{-1}\gamma_n a)\end{aligned}$$

is the Bruhat decomposition of $b^{-1}\gamma a\in\tilde{P}(\mathbb{A})$, and so the element $\gamma_a(w_sbw_s^{-1})^{-1}a$ must be contained in a fixed compact set. Notice that

$$\tilde{H}\big(\gamma_a(w_sbw_s^{-1})^{-1}a\big)=\tilde{H}(a)-s\tilde{H}(b)=[E:\mathbb{Q}]\big(H(a)-sH(b)\big).$$

Now if $\gamma\in\tilde{M}_P(\mathbb{Q})$ did not lie in $\tilde{P}_1(\mathbb{Q})$ then by Lemmas 10.2.B and 10.2.C of [3], there would exist a positive root $\alpha\in\Phi^P\setminus\Phi^1$ such that $-s\alpha$ is a positive root β in Φ^P. Then

$$\begin{aligned}\alpha\big(H(a)-sH(b)\big)&=\alpha\big(H(a)\big)-s\alpha\big(H(b)\big)\\ &=\alpha\big(H(a)\big)+\beta\big(H(b^P)\big)\\ &>\alpha\big(H(a)\big)+\beta(T_1)\end{aligned}$$

has a fixed upper bound. This would imply that $\alpha\big(H(a)\big)$ has a fixed upper bound, contradicting (3.5). Therefore γ must lie in $\tilde{P}_1(\mathbb{Q})$, and by Lemma 10.2.B of [3], s permutes the positive roots in $\Phi^2\setminus\Phi^1$. Since $H(a)-sH(b)$ lies in a fixed compact set, there are fixed positive numbers c_α, $\alpha\in\Delta^2\setminus\Delta^1$ so that

$$\alpha\big(H(b)\big)>s^{-1}\alpha\big(H(a)\big)-c_\alpha.$$

Write $\tilde{x}$ for the representative in $\mathfrak{S}^P(T_1,\omega)\subseteq G(\mathbb{A})^1$ that we have chosen for δx. Then for each $\alpha\in\Delta^2\setminus\Delta^1$,

$$\begin{aligned}\alpha\big(H(\tilde{x})-T_2\big)=\alpha\big(H(b)\big)-\alpha(T_2)&>s^{-1}\alpha\big(H(a)\big)-c_\alpha-\alpha(T_2)\\ &>s^{-1}\alpha(T)-\alpha(T_2)-c_\alpha>0,\end{aligned}$$

where T_2 is a fixed sufficiently regular point in $\mathfrak{a}^+$ and T is chosen to be sufficiently regular.

We have shown that our representative $\tilde{x} \in \mathfrak{S}^P(T_1,\omega)$ of δx actually lies in $\mathfrak{S}^2(T_1,\omega)$ and that for each $\alpha \in \Delta^2 \setminus \Delta^1$, $\alpha(H(\tilde{x}) - T_2) > 0$, with T_2 a fixed sufficiently regular point in $\mathfrak{a}^+$. Because $G(\mathbb{A})^1 = P_2(\mathbb{A})\mathfrak{S}^2(T_1,\omega)$, some left multiple of δx by an element of $P_1(\mathbb{Q}) \setminus P_2(\mathbb{Q})$ lies in $S^2 \subseteq S^1$; by reduction theory this element is the trivial class $P_1(\mathbb{Q})$, and so δx lies in S^2. Therefore only δ in $(P_2 \setminus G)_x$ give a nontrivial contribution to (3.3).

Therefore, if (3.2) is nonzero, the integral over x in $G(\mathbb{Q}) \setminus G(\mathbb{A})^1$ of (3.3) equals the integral over x in S^G of

$$\begin{aligned}
&\Big|\sum_{\delta\in(P_2\setminus G)_x}\sum_{\{P:P_1\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/Z)}\sum_{\gamma\in\tilde{P}_1(\mathbb{Q})\cap\tilde{M}_P(\mathbb{Q})\cap\mathfrak{o}}\int_{\tilde{N}_P(\mathbb{A})} f\big((\delta x)^{-1}\gamma n y\big)\,dn\Big| \\
&\qquad\le \sum_{\delta\in(P_2\setminus G)_x}\sum_{\gamma\in\tilde{M}_1(\mathbb{Q})\cap\mathfrak{o}}\Big|\sum_{\{P:P_1\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/Z)}\sum_{\nu\in\tilde{N}_1^P(\mathbb{Q})}\int_{\tilde{N}_P(\mathbb{A})} f\big((\delta x)^{-1}\gamma\nu n y\big)\,dn\Big| \\
&\qquad=\sum_{\delta}\sum_{\gamma}\Big|\sum_{\{P:P_1\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/Z)}\sum_{\zeta\in\tilde{\mathfrak{n}}_1^P(\mathbb{Q})}\int_{\tilde{\mathfrak{n}}_1(\mathbb{A})} f\big((\delta x)^{-1}\gamma e(X) y\big)\psi(X,\zeta)\,dX\Big| \\
&\qquad=\sum_{\delta}\sum_{\gamma}\Big|\sum_{\zeta\in\tilde{\mathfrak{n}}_1^2(\mathbb{Q})'}\int_{\tilde{\mathfrak{n}}_1(\mathbb{A})} f\big((\delta x)^{-1}\gamma e(X) y\big)\psi(X,\zeta)\,dX\Big| \\
&\qquad\le\sum_{\delta}\sum_{\gamma}\sum_{\zeta}\Big|\int_{\tilde{\mathfrak{n}}_1(\mathbb{A})} f\big((\delta x)^{-1}\gamma e(X) y\big)\psi(X,\zeta)\,dX\Big|,
\end{aligned}$$

where $\tilde{\mathfrak{n}}$ is the Lie algebra of $\tilde{N}$, e is the exponential map on $\tilde{\mathfrak{n}}$, $(\cdot,\cdot)$ is an inner product on $\tilde{\mathfrak{n}}$ as in [1], p. 945, and $\tilde{\mathfrak{n}}_1^2(\mathbb{Q})'$ is the set of $\zeta \in \tilde{\mathfrak{n}}_1^2(\mathbb{Q})$ not in any $\tilde{\mathfrak{n}}_1^P(\mathbb{Q})$ for a parabolic subgroup P with $P_1 \subseteq P \subsetneq P_2$. The first equality follows from Poisson summation and the second follows from the binomial theorem (Proposition 1.1 of [1]). It follows, absorbing the sum over δ in the integral over x, that (3.1) is bounded by the sum over parabolic subgroups $P_1 \subsetneq P_2$ of the integral over $(x,y) \in S^2 \times P_1(\mathbb{Q}) \setminus G(\mathbb{A})^1$ of (3.2) times

$$\sum_{\gamma\in\tilde{M}_1(\mathbb{Q})}\sum_{\zeta\in\tilde{\mathfrak{n}}_1^2(\mathbb{Q})'}\Big|\int_{\tilde{\mathfrak{n}}_1(\mathbb{A})} f\big(x^{-1}\gamma e(X) y\big)\psi(X,\zeta)\,dX\Big|. \tag{3.6}$$

Now change variables, replacing y with a representative in $\mathfrak{S}^1(T_1,\omega)$. It can be written in the form $n_2 n^2 m a k$, with n_2, n^2, and m in fixed compact subsets of $N_2(\mathbb{A})$, $N^2(\mathbb{A})$, and $M(\mathbb{A})^1$, respectively, $k \in K$, and $a \in A(\mathbb{R})^0 \cap G(\mathbb{A})^1$. If we assume that (3.2) is nonzero, then a satisfies

$$\begin{aligned}
\alpha\big(H(a) - T_1\big) &> 0 \quad \text{for all } \alpha \in \Delta^1, \\
\varpi\big(H(a) - T\big) &\le 0 \quad \text{for all } \varpi \in \hat{\Delta}^1, \\
\sigma_{1,2}\big(H(a) - T\big) &= 1.
\end{aligned} \tag{3.7}$$

This change of variables introduces a Jacobian of $e^{-2\rho(H(a))}$. Notice that because of standard properties of $\|\cdot\|$,

$$|\theta(y)| \le c\|y\|^r \le c(\|n_2 n^2 m\|\|a\|)^r \le c'\|a\|^r \le c' e^{r\lambda(H(a))}$$

for some positive constant c' and some weight λ (the highest weight of the representation leading to $\|\cdot\|$). We have bounded (3.2) by a function depending only on a.

Notice that n_2 can be absorbed into X without changing (X,ζ) for $\zeta \in \tilde{\mathfrak{n}}_1^2(\mathbb{Q})$. We can rewrite

$$x^{-1}\gamma e(X)n^2amk = x^{-1}\gamma ae\big(\mathrm{Ad}(a^{-1})X\big)(a^{-1}n^2ma)k,$$

and (3.7) implies that $a^{-1}n^2mak$ lies in a fixed compact set $C \subseteq G(\mathbb{A})^1$. Therefore the integral over $y \in P_1(\mathbb{Q})\backslash G(\mathbb{A})^1$ of the product of (3.2) and (3.6) is bounded by a constant multiple of the integral over $a \in A(\mathbb{R})^0 \cap G(\mathbb{A})^1$ satisfying (3.7) of the sum over γ and ζ of

$$\sup_{y\in C}\Big|\int_{\tilde{\mathfrak{n}}_1(\mathbb{A})} f\Big(x^{-1}\gamma ae\big(\mathrm{Ad}(a^{-1})X\big)y\Big)\psi(X,\zeta)\,dX\Big|e^{r\lambda(H(a))}$$
$$= e^{2[E:\mathbb{Q}]\rho_1(H(a))}\sup_{y\in C}\Big|\int_{\tilde{\mathfrak{n}}_1(\mathbb{A})} f\big(x^{-1}\gamma ae(X)y\big)\psi\big(X,\mathrm{Ad}(a)\zeta\big)\,dX\Big|e^{r\lambda(H(a))}.$$

Next we change variables for x. Choose a representative for $x \in S^2 \subseteq P_1(\mathbb{Q})\setminus G(\mathbb{A})^1$ of the form $n_2'n_1^{2\prime}m'b_1k'$, with n_2' and $n_1^{2\prime}$ in fixed compact subsets of $N_2(\mathbb{A})$ and $N_1^2(\mathbb{A})$, respectively, m' in some predetermined set of representatives for $M_1(\mathbb{Q})\setminus M_1(\mathbb{A})^1$, $b_1 \in A_1(\mathbb{R})^0 \cap G(\mathbb{A})^1$, and $k \in K$. This introduces a Jacobian of $e^{-2\rho_1(H(b_1))} = e^{-2\rho(H(b_1))}$. Now, n_2' can be absorbed into the integral over X, and we can rewrite

$$(n_1^{2\prime}m'b_1k')^{-1}\gamma ae(X)y = k'^{-1}(b_1^{-1}n_1^{2\prime}b_1)^{-1}(b_1^{-1}a_1)(m'^{-1}\gamma a^1)e(X)y,$$

where $a = a_1a^1$ with $a_1 \in A_1(\mathbb{R})^0 \cap G(\mathbb{A})^1$, $a^1 \in A^1(\mathbb{R})^0 \cap G(\mathbb{A})^1$. Notice that k' and y lie in fixed compact sets, and $b_1^{-1}n_1^{2\prime}b_1$ lies in $N_1^2(\mathbb{A})$, $b_1^{-1}a_1$ lies in $A_1(\mathbb{R})^0 \cap G(\mathbb{A})^1$, $m'^{-1}\gamma a^1$ lies in $\tilde{M}_1(\mathbb{A})$, and $e(X)$ lies in $\tilde{N}_1(\mathbb{A})$. Since f is compactly supported, its value at the above expression is zero unless $b_1^{-1}a_1$ is in a fixed compact set. By (3.7), this implies that for each $\alpha \in \Delta_1^2$, $\alpha\big(H(b_1)\big) > 0$, so that $k'(b_1^{-1}n_1^{2\prime}b_1)^{-1}$ remains in a fixed compact set C'.

Because f is compactly supported and C, C' are compact, there exist non-negative functions f_1, f_2, and f_3 with $f_1 \in C_c^\infty\big(A_1(\mathbb{R})^0 \cap G(\mathbb{A})^1\big)$, $f_2 \in C_c^\infty\big(\tilde{M}_1(\mathbb{A})^1\big)$, and f_3 a Schwartz function on $\tilde{\mathfrak{n}}_1^2(\mathbb{A})$, such that

$$\sup_{\substack{y\in C\\ y'\in C'}}\Big|\int_{\tilde{\mathfrak{n}}_1(\mathbb{A})} f\big(y'^{-1}g_1g_2e(X)y\big)\psi(X,\zeta)\,dX\Big|$$
$$\leq f_1(g_1)f_2(g_2)f_3(\zeta),\quad g_1 \in A_1(\mathbb{R})^0 \cap G(\mathbb{A})^1,\ g_2 \in \tilde{M}_1(\mathbb{A})^1,\ \zeta \in \tilde{\mathfrak{n}}_1^2(\mathbb{A}).$$

(These functions can be chosen to vary continuously with f.) Then (3.1) is bounded by a fixed constant multiple of the sum over $P_1 \subsetneq P_2$ of the integral over $a \in A(\mathbb{R})^0 \cap G(\mathbb{A})^1$ satisfying (3.7) of the product of the three terms

$$e^{r\lambda(H(a))}e^{2([E:\mathbb{Q}]-1)\rho_1(H(a))}\int_{A_1(\mathbb{R})^0\cap G(\mathbb{A})^1} e^{-2\rho(H(b_1))}f_1(b_1^{-1}a_1)\,db_1, \tag{3.8}$$

$$e^{-2\rho^1(H(a))}\int_{M_1(\mathbb{Q})\backslash M_1(\mathbb{A})^1}\sum_{\gamma\in\tilde{M}_1(\mathbb{Q})} f_2(m'^{-1}\gamma a^1)\,dm', \tag{3.9}$$

and

$$\sum_{\zeta \in \tilde{\mathfrak{n}}_1^2(\mathbb{Q})'} f_3\big(\mathrm{Ad}(a)\zeta\big). \tag{3.10}$$

The expression (3.8) equals

$$e^{(r\lambda+2([E:\mathbb{Q}]-2)\rho_1)(H(a))}$$

times the integral over $A_1(\mathbb{R})^0 \cap G(\mathbb{A})^1$ of $e^{2\rho(H(b_1))}f_1(b_1)$, a continuous seminorm of f. By Lemma 2.2 and the finiteness of the volume of $M_1(\mathbb{Q}) \setminus M_1(\mathbb{A})^1$, the expression (3.9) converges and is another continuous seminorm of f.

We will bound (3.10) and conclude our estimates as on p. 1248 of [2]. Let $N(f_3)$ be the smallest positive integer such that the support of f_3 in $\tilde{\mathfrak{n}}_1^2(\mathbb{Q})$ is contained in $\tilde{\mathfrak{n}}_1^2\big(N(f_3)^{-1}\mathbb{Z}\big)$. Because of (3.7), if n is an arbitrarily large number, then for T sufficiently regular (3.10) is bounded by

$$\sum_{\zeta \in \tilde{\mathfrak{n}}_1^2(N(f_3)^{-1}\mathbb{Z})} \|\mathrm{Ad}(a)\zeta\|^{-n}.$$

As on p. 1248 of [2] we find that for n sufficiently large this is bounded by a constant multiple of

$$N(f_3)^n \prod_{\alpha\in\Delta^2} e^{-k_\alpha \alpha(H(a))},$$

with each k_α non-negative, and for $\alpha \in \Delta^2 \setminus \Delta^1$, k_α is a fixed quotient of n. Choose n so large that

$$r\lambda + 2([E:\mathbb{Q}]-2)\rho_1 + (2k/\varepsilon)\sum_{\beta\in\Delta_1^2}\beta - \sum_{\alpha\in\Delta^2} k_\alpha \alpha,$$

when written as a linear combination of simple roots, has all coefficients corresponding to $\alpha \in \Delta^2 \setminus \Delta^1$ negative, where k and ε are the fixed positive numbers in the statement of the theorem.

Write $H(a)$ as

$$\Big(\sum_{\beta\in\Delta_1^2} t_\beta \varpi_\beta^\vee + H^*\Big) - \Big(\sum_{\delta\in\Delta_0^1} r_\delta \delta^\vee\Big) + T,$$

with H^* in $\mathfrak{a}_2$. Because of (3.7), we know that the numbers t_β and r_δ are nonnegative, and H^* belongs to a compact subset of $-\mathfrak{a}_2^+$ whose volume can be bounded by a polynomial $\prod_{\beta\in\Delta_1^2} p(t_\beta)$ in the numbers t_β. Then an elementary calculation shows that

$$e^{(r\lambda+2([E:\mathbb{Q}]-2)\rho_1)(H(a))} \prod_{\alpha\in\Delta^2} e^{-k_\alpha\alpha(H(a))} \leq c \prod_{\beta\in\Delta_1^2} e^{-(2k/\varepsilon)(t_\beta+\beta(T))}$$

for some fixed constant c depending only on T_1. The integral over all a satisfying (3.7) of this is the product of the volume of

$$\left\{ a \in A^1(\mathbb{R})^0 \cap G(\mathbb{A})^1 : \begin{array}{ll} \alpha\big(H(a)-T_1\big) > 0 & \text{for all } \alpha\in\Delta^1 \\ \varpi\big(H(a)-T\big) \leq 0 & \text{for all } \varpi\in\hat{\Delta}^1 \end{array} \right\} \tag{3.11}$$

and

$$\prod_{\beta\in\Delta_1^2}\left(e^{-(2k/\varepsilon)\beta(T)}\int_0^\infty p(t_\beta)e^{-(2k/\varepsilon)t_\beta}\right)\leq c'e^{-(2k/\varepsilon)d(T)}<c'e^{-2k\|T\|}$$

where c' is some fixed fixed constant and the second inequality follows from our restrictions on T. The volume of the set (3.11) is bounded by a polynomial in $\|T\|$, so that the integral of the product of (3.2) times (3.6) is bounded by the product of three continuous seminorms of f with a constant multiple of $e^{-k\|T\|}$. Then we have proven our results, with $\|f\|$ the sum over $P_1 \subsetneq P_2$ of this constant multiple of the product of these three seminorms. ■

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