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PAPER

Stabilization in a chemotaxis system modelling T-cell dynamics with simultaneous production and consumption of signals

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Abstract

In a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, this manuscript considers the homogeneous Neumann boundary problem for the chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v + u - \alpha uv, \end{cases}$$

with parameter $\alpha > 0$ and with coincident production and uptake of attractants, as recently emphasized by Dallaston et al. as relevant for the understanding of T-cell dynamics.
It is shown that there exists $\delta_\star = \delta_\star(n) > 0$ such that for any given $\alpha \geq \frac{1}{\delta_\star}$ and for any suitably regular initial data satisfying $v(\cdot, 0) \leq \delta_\star$, this problem admits a unique classical solution that stabilizes to the constant equilibrium $(\frac{1}{|\Omega|}\int_\Omega u(\cdot, 0), \ \frac{1}{\alpha})$ in the large time limit.

1. Introduction

The classical Keller-Segel chemotaxis model ([13]),

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v + u - \beta v, \end{cases} \tag{1.1}$$

was originally proposed for the description of slime mould aggregation as a typical process in which cells move upward gradients of a chemoattractant secreted by themselves; in such contexts, u and v denote the respective population densities and signal concentrations. Due to the presence of the chemotactic cross-diffusion term $-\nabla \cdot (u\nabla v)$, this system exhibits a striking feature of destabilization, as analytically captured by results concerning the occurrence of exploding solutions in two- and higher-dimensional settings ([6, 26]). Although partially understood less comprehensively from a mathematical point of view, numerous variants of the prototypical chemotaxis-production system (1.1) arise in various biological application contexts ([7, 17]).

In cases in which, in contrast to the above type of situations, taxis-type movement is directed by a signal which is absorbed upon contact, modelling rather relies on chemotaxis-consumption models such as

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$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v - uv; \end{cases} \tag{1.2}$$

typical examples include the motion of *Escherichia coli* or of *Bacillus subtilis* towards sources of nutrient and oxygen ([3, 14, 24]). Moreover, such nutrient taxis mechanisms actually play a key role in predator-prey interactions in which predators adjust their migration towards areas of higher prey density ([11, 12, 23, 28]). In comparison to the chemotaxis-production system (1.1), the dissipative property of the signal consumption mechanism in the nutrient taxis system (1.2) prevents blow-up to some extent. Indeed, this is reflected in results on global existence of classical solutions in two-dimensional boundary value problems ([20]); in three-dimensional analogues, after all, some global weak solutions with properties of eventual smoothness and stabilization could be constructed ([20]). Here, it might be worth further mentioning that when the diffusion Δu in (1.2) is replaced by any slightly enhanced diffusion $\nabla \cdot (D(u)\nabla u)$ with D satisfying $D(u) \to +\infty$ as $u \to \infty$, the corresponding no-flux initial boundary problem admits globally bounded solutions also in such three-dimensional settings ([10, 27]).

A chemotaxis model with synchronous production and consumption of signals. The immune system protects us from the development of inflammatory diseases, and its establishment and maintenance rely heavily on T-cells which are responsible for recognizing and destroying pathogens that have been infected by viruses. The movement of T-cells within inflamed tissues is driven and controlled by the distribution of chemotactic signalling agents that are chemokines secreted by effector T-cells ([2, 4, 5]). However, unlike the signal mechanisms in (1.1) or (1.2), the effector T-cells do not only produce the chemokine but also absorb this attractant ([2]). In order to capture some essential features of a chemotaxis system with concurrent production and consumption of signals, by neglecting the regulatory T-cell population, we shall here focus on a minimal model for T-cell dynamics recently developed by Dallaston et al. in [2], and we shall subsequently consider the no-flux initial boundary problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + u - \alpha uv, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\alpha > 0$ is a given parameter.

To underline associated challenges, let us recall that essential parts of the literature both on (1.1) and on (1.2) have relied on favourable global structures of both these simple systems ([16, 20, 26]): Sufficiently regular trajectories of the chemotaxis-production system (1.1) are subject to the energy identity

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega u\ln u\right\} + \int_\Omega v_t^2 + \int_\Omega \left|\frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v\right|^2 = 0, \tag{1.4}$$

while smooth solutions of (1.2) in convex domains Ω satisfy the inequality

$$\frac{d}{dt}\left\{2\int_\Omega |\nabla\sqrt{v}|^2 + \int_\Omega u\ln u\right\} + \int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega v|D^2\ln v|^2 + \frac{1}{2}\int_\Omega u\frac{|\nabla v|^2}{v} \le 0 \tag{1.5}$$

throughout evolution. In the simultaneous presence of both production and consumption of signal such as in (1.3), however, none of these structural features appear to persist in any generalized nor weakened form.

Main results. In order to nevertheless describe some dynamical features of (1.3) in comparison to those observed for (1.1) and (1.2), in the present manuscript, we shall build an analysis of (1.3) on tracing the evolution of functionals of the form

$$\int_\Omega u^p\varphi(v),$$

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with the weight functions

$$\varphi(s) := (\delta - s)^{-\kappa} - As, \qquad s \in [0, \delta_0],$$

depending on $p > 1$ through appropriate choices of the parameters $\kappa = \kappa(p) > 0, \delta = \delta(p) > 0, A = A(p) > 0$ and $\delta_0 = \delta_0(p) \in (0, \delta)$ (see Lemma 2.4 below for details). In domains of arbitrary dimension and for initial data which are suitably small with respect to their second component, through accordingly obtained *a priori* estimates we shall thereby discover that regardless of the size in the corresponding first component and hence in stark contrast to (1.1), blow-up can entirely be ruled out and that solutions asymptotically behave in an essentially diffusion-dominated manner:

Theorem 1.1. *Let $n \geq 1$. Then there exists $\delta_\star = \delta_\star(n) > 0$ with the property that whenever $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, given an arbitrary $\alpha \geq \frac{1}{\delta_\star}$ and any initial data (u_0, v_0) which are such that*

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) \text{ is nonnegative with } u_0 \not\equiv 0 \qquad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative,} \end{cases} \tag{1.6}$$

and such that

$$v_0(x) \leq \delta_\star \qquad \text{for all } x \in \Omega, \tag{1.7}$$

one can find a uniquely determined pair of nonnegative functions

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \qquad \text{and} \\ v \in \bigcap_{q>n} C^0([0, \infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \end{cases} \tag{1.8}$$

such that (u, v) solves (1.3) in the classical sense. Moreover,

$$u(\cdot, t) \to \frac{1}{|\Omega|} \int_\Omega u_0 \qquad \text{in } L^\infty(\Omega) \tag{1.9}$$

and

$$v(\cdot, t) \to \frac{1}{\alpha} \qquad \text{in } L^\infty(\Omega) \tag{1.10}$$

as $t \to \infty$.

In order to further underline the strongly smoothing effect of the absorptive action induced by the choice $\alpha > 0$ in (1.3), let us state the following simple consequence of known results for the planar version of (1.2) on solutions to (1.3) for which the initial signal concentration is, unlike the situation covered by Theorem 1.1, conveniently large throughout the domain:

Proposition 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, let $\alpha > 0$, and assume that (1.6) holds with*

$$v_0(x) > \frac{1}{\alpha} \qquad \text{for all } x \in \overline{\Omega}. \tag{1.11}$$

Then (1.3) admits a unique global classical solution within the class of functions specified in (1.8), and furthermore, this solution satisfies (1.9) and (1.10).

As a simple consequence of this result, we can finally make sure that for some suitably chosen initial data, the behaviour in (1.3) for appropriately large $\alpha > 0$ drastically differs from that seen for $\alpha = 0$:

Corollary 1.3. *Let $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^2$. Then there exist radially symmetric functions u_0 and v_0 which satisfy (1.6) and which are such that for $\alpha = 0$, the problem (1.3) admits a solution blowing up in finite time, while for sufficiently large $\alpha > 0$, a global classical solution fulfilling (1.9) and (1.10) can be found. More precisely, for these data, there exist $T_{max} \in (0, \infty)$,*

$$\begin{cases} u^{(0)} \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})) \qquad \text{and} \\ v^{(0)} \in \bigcap_{q>n} C^0([0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})) \end{cases} \tag{1.12}$$

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as well as $\alpha_0 > 0$ *and*

$$\begin{cases} (u^{(\alpha)})_{\alpha>\alpha_0} \subset C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) & \text{and} \\ (v^{(\alpha)})_{\alpha>\alpha_0} \subset \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)) \end{cases} \tag{1.13}$$

such that $u^{(\alpha)} \ge 0$ *and* $v^{(\alpha)} \ge 0$ *for all* $\alpha \in \{0\} \cup (\alpha_0, \infty)$, *that* $(u^{(0)}, v^{(0)})$ *solves (1.3) in* $\Omega \times (0, T_{\max})$ *for* $\alpha = 0$, *with*

$$\limsup_{t\nearrow T_{\max}} \|u^{(0)}(\cdot,t)\|_{L^\infty(\Omega)} = \infty, \tag{1.14}$$

and that for each $\alpha > \alpha_0$, *the pair* $(u^{(\alpha)}, v^{(\alpha)})$ *is a global classical solution of (1.3) satisfying*

$$u^{(\alpha)}(\cdot,t) \to \frac{1}{|\Omega|}\int_\Omega u_0 \quad \text{and} \quad v^{(\alpha)}(\cdot,t) \to \frac{1}{\alpha} \qquad \text{in } L^\infty(\Omega) \tag{1.15}$$

as $t \to \infty$.

2. Global existence

The following statement on local existence and extensibility can be obtained by straightforward adaptation of standard arguments, as detailed for a closely related setting in [8].

Lemma 2.1. *Let* $\alpha > 0$, *and assume (1.6). Then there exist* $T_{\max} \in (0, \infty]$ *and uniquely determined nonnegative functions*

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,T_{max})) \cap C^{2,1}(\overline{\Omega}\times(0,T_{max})) & \text{and} \\ v \in \bigcap_{q>n} C^0([0,T_{max}); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,T_{max})) \end{cases} \tag{2.1}$$

such that (1.3) is satisfied in the classical sense in $\Omega \times (0, T_{max})$, *that*

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \text{for all } t \in (0, T_{max}), \tag{2.2}$$

and that

$$\text{if } T_{max} < \infty, \qquad \text{then}$$
$$\limsup_{t\nearrow T_{max}} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \right\} = \infty. \tag{2.3}$$

A sharpening of the above extensibility criterion (2.3) has been achieved in [1, Lemma 3.2] in a problem framework actually more general than that of (1.3).

Lemma 2.2. *If* $\alpha > 0$ *and (1.6) holds, and if for some* $p \ge 1$ *fulfilling* $p > \frac{n}{2}$, *we have*

$$\sup_{t\in(0,T_{max})} \|u(\cdot,t)\|_{L^p(\Omega)} < \infty,$$

then $T_{max} = \infty$, *and there exists* $C > 0$ *such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > 0.$$

Now the presence of the consumption term $-uv$ in the second equation from (1.3) implies a basic but important *a priori* information on L^∞ bounds for v.

Lemma 2.3. *Let* $\alpha > 0$, *and assume (1.6). Then*

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \max\left\{ \|v_0\|_{L^\infty(\Omega)}, \frac{1}{\alpha} \right\} \qquad \text{for all } t \in (0, T_{max}). \tag{2.4}$$

Proof. Since for $\overline{v}(x,t) := \mapsto \max\left\{\|v_0\|_{L^\infty(\Omega)}, \frac{1}{\alpha}\right\}$, $(x,t) \in \overline{\Omega}\times[0,\infty)$, we have

$$\overline{v}_t - \Delta\overline{v} - u(x,t) + \alpha u(x,t)\overline{v} = u(x,t)\cdot(\alpha\overline{v} - 1) \ge 0 \qquad \text{in } \Omega\times(0,T_{\max})$$

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according to the inequality $\bar{v} \geq \frac{1}{\alpha}$, and since moreover $\bar{v}(\cdot, 0) \geq v(\cdot, 0)$ in Ω due to the fact that $\bar{v} \geq \|v_0\|_{L^\infty(\Omega)}$, this immediately results from the comparison principle. □

In view of Lemma 2.2, global existence is a consequence of an *a priori* estimate on $\int_\Omega u^p(\cdot, t)$, $t \in (0, T_{\max})$, with some $p > \max\{1, \frac{n}{2}\}$. This will be achieved in Lemma 2.5 below through analysing the evolution undergone by coupled functionals of the form $\int_\Omega u^p \varphi(v)$ with suitable weight functions $\varphi = \varphi(v)$ which enjoy suitable pointwise bounds from below. In comparison to previous related studies in which similar approaches have been pursued (cf., e.g., [19, 20] and [25]), the simultaneous presence of production and consumption of attractants requires a more complex design of these weight functions. We therefore separately address this in the following elementary but crucial lemma.

Lemma 2.4. *Let $p > 1$. Then there exist $\kappa = \kappa(p) > 0$, $\delta = \delta(p) > 0$, $A = A(p) > 0$ and $\delta_0 = \delta_0(p) \in (0, \delta)$ such that letting*

$$\varphi(s) \equiv \varphi^{(p)}(s) := (\delta - s)^{-\kappa} - As, \qquad s \in [0, \delta_0], \tag{2.5}$$

defines a function $\varphi \in C^\infty([0, \delta_0])$ which satisfies

$$\varphi(s) > 0 \qquad \textit{for all } s \in [0, \delta_0] \tag{2.6}$$

and

$$\varphi'(s) < 0 \qquad \textit{for all } s \in [0, \delta_0] \tag{2.7}$$

as well as

$$4p\varphi'^2(s) + p(p-1)^2\varphi^2(s) < 4(p-1)\varphi(s)\varphi''(s) \qquad \textit{for all } s \in [0, \delta_0]. \tag{2.8}$$

Proof. We abbreviate $\xi_0 := \frac{1}{40}$ and $a := 8$, and given $p > 1$, we fix $\kappa = \kappa(p) \in (0, 1]$ small enough such that

$$\frac{65p\kappa}{(p-1)(\kappa+1)} < \frac{1}{4} \tag{2.9}$$

and

$$\frac{p(p-1)a^2\kappa}{4(\kappa+1)} < \frac{1}{4} \tag{2.10}$$

as well as

$$a\kappa < \frac{1}{4}. \tag{2.11}$$

Furthermore, choosing $\delta = \delta(p) \in (0, 1]$ in such a way that

$$\frac{p(p-1)}{4\kappa(\kappa+1)} \cdot \delta^2 < \frac{1}{4}, \tag{2.12}$$

we set

$$A \equiv A(p) := a\kappa\delta^{-\kappa-1} \tag{2.13}$$

as well as

$$\delta_0 \equiv \delta_0(p) := \xi_0\delta, \tag{2.14}$$

and let φ be as defined through (2.5). Then since $\delta_0 \equiv \frac{\delta}{40} < \delta$, it is evident that $\varphi \in C^\infty([0, \delta_0])$, and computing

$$\varphi'(s) = \kappa(\delta - s)^{-\kappa-1} - A, \qquad s \in [0, \delta_0], \tag{2.15}$$

and

$$\varphi''(s) = \kappa(\kappa+1)(\delta - s)^{-\kappa-2}, \qquad s \in [0, \delta_0], \tag{2.16}$$

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we see that according to (2.14) and (2.13),

$$\begin{aligned}\varphi'(s) \le \varphi'(\delta_0) &= \kappa(\delta-\delta_0)^{-\kappa-1} - A \\ &= \kappa(1-\xi_0)^{-\kappa-1}\delta^{-\kappa-1} - a\kappa\delta^{-\kappa-1} \\ &= \kappa\delta^{-\kappa-1}\cdot\Big\{(1-\xi_0)^{-\kappa-1} - a\Big\} \\ &< 0 \qquad \text{for all } s\in[0,\delta_0],\end{aligned}$$

because the inequalities $\xi_0 \le \frac{1}{2}$ and $\kappa \le 1$ ensure that

$$(1-\xi_0)^{-\kappa-1} \le 2^{\kappa+1} \le 2^2 < 8 = a.$$

As thus (2.7) holds, we particularly obtain (2.6) as a consequence thereof, observing that, again by (2.14) and (2.13),

$$\begin{aligned}\varphi(s) \ge \varphi(\delta_0) &= (\delta-\delta_0)^{-\kappa} - A\delta_0 \\ &= (1-\xi_0)^{-\kappa}\delta^{-\kappa} - a\kappa\xi_0\delta^{-\kappa} \\ &= \delta^{-\kappa}\cdot\Big\{(1-\xi_0)^{-\kappa} - a\kappa\xi_0\Big\} \qquad \text{for all } s\in[0,\delta_0],\end{aligned}$$

and that here

$$(1-\xi_0)^{-\kappa} - a\kappa\xi_0 \ge 1 - a\kappa\xi_0 = 1 - \frac{\kappa}{5} > 0$$

due to the restriction that $\kappa \le 1$.

To finally verify (2.8), we first note that by (2.5), (2.15) and (2.16),

$$\begin{aligned}&4p\varphi'^2(s) + p(p-1)^2\varphi^2(s) - 4(p-1)\varphi(s)\varphi''(s) \\ &= 4p\cdot\Big\{\kappa(\delta-s)^{-\kappa-1} - A\Big\}^2 + p(p-1)^2\cdot\Big\{(\delta-s)^{-\kappa} - As\Big\}^2 \\ &\quad -4(p-1)\cdot\Big\{(\delta-s)^{-\kappa} - As\Big\}\cdot\kappa(\kappa+1)(\delta-s)^{-\kappa-2} \\ &= 4p\kappa^2(\delta-s)^{-2\kappa-2} - 8p\kappa A(\delta-s)^{-\kappa-1} + 4pA^2 \\ &\quad + p(p-1)^2(\delta-s)^{-2\kappa} - 2p(p-1)^2As(\delta-s)^{-\kappa} + p(p-1)^2A^2s^2 \\ &\quad -4(p-1)\kappa(\kappa+1)(\delta-s)^{-2\kappa-2} + 4(p-1)\kappa(\kappa+1)As(\delta-s)^{-\kappa-2} \\ &\le I_1(s) + I_2 + I_3(s) + I_4(s) + I_5(s) - J(s) \qquad \text{for all } s\in[0,\delta_0],\end{aligned} \tag{2.17}$$

where

$$I_1(s) := 4p\kappa^2(\delta-s)^{-2\kappa-2}$$

and

$$I_2 := 4pA^2$$

and

$$I_3(s) := p(p-1)^2(\delta-s)^{-2\kappa}$$

as well as

$$I_4(s) := p(p-1)^2A^2s^2$$

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and

$$I_5(s) := 4(p-1)\kappa(\kappa+1)As(\delta-s)^{-\kappa-2}$$

and

$$J(s) := 4(p-1)\kappa(\kappa+1)(\delta-s)^{-2\kappa-2}$$

for $s \in [0, \delta_0]$. Here, once more recalling (2.13) and our definition of a, we can rely on (2.9) to estimate

$$\begin{aligned}
\frac{I_1(s)+I_2}{J(s)} &= \frac{4p\kappa^2(\delta-s)^{-2\kappa-2} + 4pa^2\kappa^2\delta^{-2\kappa-2}}{4(p-1)\kappa(\kappa+1)(\delta-s)^{-2\kappa-2}} \\
&= \frac{p\kappa + 64p\kappa(\delta-s)^{2\kappa+2}\delta^{-2\kappa-2}}{(p-1)(\kappa+1)} \\
&\le \frac{p\kappa + 64p\kappa}{(p-1)(\kappa+1)} \\
&< \frac{1}{4} \qquad \text{for all } s \in [0, \delta_0],
\end{aligned} \tag{2.18}$$

while the fact that $\delta_0 \le \delta \le 1$ ensures that

$$\begin{aligned}
\frac{I_4(s)}{J(s)} &= \frac{p(p-1)^2a^2\kappa^2\delta^{-2\kappa-2}s^2}{4(p-1)\kappa(\kappa+1)(\delta-s)^{-2\kappa-2}} \\
&= \frac{p(p-1)a^2\kappa(\delta-s)^{2\kappa+2}\delta^{-2\kappa-2}s^2}{4(\kappa+1)} \\
&\le \frac{p(p-1)a^2\kappa \cdot \delta^2}{4(\kappa+1)} \\
&\le \frac{p(p-1)a^2\kappa}{4(\kappa+1)} \\
&< \frac{1}{4} \qquad \text{for all } s \in [0, \delta_0]
\end{aligned} \tag{2.19}$$

because of (2.10). Since moreover, for a similar reason,

$$\begin{aligned}
\frac{I_5(s)}{J(s)} &= \frac{a\kappa\delta^{-\kappa-1} \cdot s \cdot (\delta-s)^{-\kappa-2}}{(\delta-s)^{-2\kappa-2}} \\
&= a\kappa(\delta-s)^{\kappa}s \cdot \delta^{-\kappa-1} \\
&\le a\kappa \cdot \delta^{\kappa} \cdot \delta \cdot \delta^{-\kappa-1} \\
&= a\kappa \\
&= 8\kappa \\
&< \frac{1}{4} \qquad \text{for all } s \in [0, \delta_0]
\end{aligned} \tag{2.20}$$

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by (2.11), and since our smallness assumption in (2.12) guarantees that also

$$\begin{aligned}\frac{I_3(s)}{J(s)} &= \frac{p(p-1)^2(\delta-s)^{-2\kappa}}{4(p-1)\kappa(\kappa+1)(\delta-s)^{-2\kappa-2}} \\ &= \frac{p(p-1)(\delta-s)^2}{4\kappa(\kappa+1)} \\ &\le \frac{p(p-1)}{4\kappa(\kappa+1)}\cdot\delta^2 \\ &< \frac{1}{4} \qquad \text{for all } s\in[0,\delta_0],\end{aligned} \tag{2.21}$$

we only need to collect (2.18)-(2.21) to infer (2.8) from (2.17). □

With the above technical preparation at hand, we can perform an essentially straightforward modification of an argument from [20] to identify, given any $p>1$, a smallness condition on v_0 as sufficient to ensure bounds for u with respect to the norm in $L^p(\Omega)$.

Lemma 2.5. *Let $p>1$, and let $\delta_0(p)$ be as provided by Lemma 2.4. Then whenever $\alpha \ge \frac{1}{\delta_0(p)}$ and (u_0,v_0) satisfies (1.6) as well as*

$$v_0(x)\le\delta_0(p) \qquad \textit{for all } x\in\Omega, \tag{2.22}$$

one can find $C=C(p,\alpha,u_0)>0$ such that

$$\int_\Omega u^p(\cdot,t)\le C \qquad \textit{for all } t\in(0,T_{\max}) \tag{2.23}$$

and

$$\int_0^t\int_\Omega u^{p-2}|\nabla u|^2\le C \qquad \textit{for all } t\in(0,T_{\max}). \tag{2.24}$$

Proof. Writing $\delta_0=\delta_0(p)$, in view of Lemma 2.3, the hypothesis $\alpha\ge\frac{1}{\delta_0(p)}$ together with (2.22) ensures that $0\le v(x,t)\le\delta_0$ for all $x\in\Omega$ and $t\in(0,T_{\max})$, whence Lemma 2.4 applies so as to warrant that with $\varphi=\varphi^{(p)}$ as defined there, the functions $\varphi\circ v, \varphi'\circ v$ and $\varphi''\circ v$ are continuous on $\overline{\Omega}\times[0,T_{\max})$ and that

$$\varphi(v)\ge c_1, \quad \varphi'(v)\le -c_2 \quad \text{and} \quad \varphi''(v)\ge c_3 \qquad \text{in } \Omega\times(0,T_{\max}) \tag{2.25}$$

as well as

$$\frac{4p(p-1)\varphi(v)\varphi''(v)-4p^2\varphi'^2(v)-p^2(p-1)^2\varphi^2(v)}{4\varphi''(v)-4p\varphi'(v)}\ge c_4 \qquad \text{in } \Omega\times(0,T_{\max}) \tag{2.26}$$

with some $c_i=c_i(p)>0$, $i\in\{1,2,3,4\}$. To make appropriate use of this information, we go back to (1.3) and compute

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$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u^p\varphi(v) &= p\int_\Omega u^{p-1}\varphi(v)\cdot\Big\{\Delta u-\nabla\cdot(u\nabla v)\Big\}+\int_\Omega u^p\varphi'(v)\cdot\Big\{\Delta v+u-\alpha uv\Big\}\\
&= -p(p-1)\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2-p\int_\Omega u^{p-1}\varphi'(v)\nabla u\cdot\nabla v\\
&\quad +p(p-1)\int_\Omega u^{p-1}\varphi(v)\nabla u\cdot\nabla v+p\int_\Omega u^p\varphi'(v)|\nabla v|^2\\
&\quad -p\int_\Omega u^{p-1}\varphi'(v)\nabla u\cdot\nabla v-\int_\Omega u^p\varphi''(v)|\nabla v|^2\\
&\quad +\int_\Omega u^{p+1}\varphi'(v)\cdot(1-\alpha v)\\
&= -p(p-1)\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2+\int_\Omega u^{p-1}\cdot\Big\{-2p\varphi'(v)+p(p-1)\varphi(v)\Big\}\nabla u\cdot\nabla v\\
&\quad -\int_\Omega u^p\cdot\Big\{\varphi''(v)-p\varphi'(v)\Big\}|\nabla v|^2\\
&\quad +\int_\Omega u^{p+1}\varphi'(v)\cdot(1-\alpha v)\qquad\text{for all } t\in(0,T_{\max}).
\end{aligned}\tag{2.27}
$$

Here, the second inequality in (2.25) together with (2.4) shows that

$$
\int_\Omega u^{p+1}\varphi'(v)\cdot(1-\alpha v)\le 0\qquad\text{for all } t\in(0,T_{\max}),\tag{2.28}
$$

and Young's inequality implies that thanks to the positivity of $\varphi''(v)-p\varphi'(v)$ entailed by (2.25), the third to last summand can be controlled according to

$$
\begin{aligned}
&\int_\Omega u^{p-1}\cdot\Big\{-2p\varphi'(v)+p(p-1)\varphi(v)\Big\}\nabla u\cdot\nabla v\\
&\le \int_\Omega u^p\cdot\Big\{\varphi''(v)-p\varphi'(v)\Big\}|\nabla v|^2\\
&\quad +\int_\Omega u^{p-2}\cdot\frac{\big\{-2p\varphi'(v)+p(p-1)\varphi(v)\big\}^2}{4\cdot\big\{\varphi''(v)-p\varphi'(v)\big\}}\cdot|\nabla u|^2\qquad\text{for all } t\in(0,T_{\max}).
\end{aligned}\tag{2.29}
$$

Since

$$
\begin{aligned}
&\frac{\big\{-2p\varphi'(v)+p(p-1)\varphi(v)\big\}^2}{4\cdot\big\{\varphi''(v)-p\varphi'(v)\big\}}-p(p-1)\varphi(v)\\
&=\frac{4p^2\varphi'^2(v)-4p^2(p-1)\varphi(v)\varphi'(v)+p^2(p-1)^2\varphi^2(v)}{4\cdot\big\{\varphi''(v)-p\varphi'(v)\big\}}-p(p-1)\varphi(v)\\
&=\frac{1}{4\cdot\big\{\varphi''(v)-p\varphi'(v)\big\}}\cdot\Big\{4p^2\varphi'^2(v)-4p^2(p-1)\varphi(v)\varphi'(v)+p^2(p-1)^2\varphi^2(v)\\
&\quad -4p(p-1)\varphi(v)\varphi''(v)+4p^2(p-1)\varphi(v)\varphi'(v)\Big\}\\
&=\frac{1}{4\cdot\big\{\varphi''(v)-p\varphi'(v)\big\}}\cdot\Big\{4p^2\varphi'^2(v)+p^2(p-1)^2\varphi^2(v)-4p(p-1)\varphi(v)\varphi''(v)\Big\}\qquad\text{in } \Omega\times(0,T_{\max}),
\end{aligned}
$$

on the basis of (2.26), we can estimate

$$\int_\Omega u^{p-2} \cdot \frac{\big\{-2p\varphi'(v) + p(p-1)\varphi(v)\big\}^2}{4 \cdot \big\{\varphi''(v) - p\varphi'(v)\big\}} \cdot |\nabla u|^2 - p(p-1)\int_\Omega u^{p-2}\varphi(v)|\nabla u|^2$$
$$\le -c_4 \int_\Omega u^{p-2}|\nabla u|^2 \qquad \text{for all } t \in (0, T_{\max}).$$

Combining (2.27) with (2.29) and (2.28) hence leads to the inequality

$$\frac{d}{dt}\int_\Omega u^p\varphi(v) \le -c_4\int_\Omega u^{p-2}|\nabla u|^2 \qquad \text{for all } t \in (0, T_{\max}),$$

meaning that

$$\int_\Omega u^p(\cdot,t)\varphi(v(\cdot,t)) + c_4\int_0^t\int_\Omega u^{p-2}|\nabla u|^2 \le \int_\Omega u_0^p\varphi(v_0) \qquad \text{for all } t \in (0, T_{\max}),$$

and that thus the claim results in view of the uniform positivity property of φ contained in (2.25). □

In view of Lemma 2.2, to ensure global extensibility, it is sufficient to apply the latter to some suitably large but fixed $p > 1$:

Lemma 2.6. *There exists $\delta_\star = \delta_\star(n) > 0$ such that if $\Omega \subset \mathbb{R}^n$ is smoothly bounded, if $\alpha \ge \frac{1}{\delta_\star}$ and if u_0 and v_0 satisfy (1.6) as well as (1.7), then $T_{\max} = \infty$ and*

$$\sup_{t>0} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty \tag{2.30}$$

as well as

$$\int_0^\infty\int_\Omega |\nabla u|^2 < \infty. \tag{2.31}$$

Proof. With $(\delta_0(p))_{p>1}$ taken from Lemma 2.4, we fix $p > 1$ such that $p > \frac{n}{2}$, and let $\delta_\star := \min\{\delta_0(p), \delta_0(2)\}$. Then two applications of Lemma 2.5 yield $c_1 > 0$ and $c_2 > 0$ such that

$$\int_\Omega u^p \le c_1 \qquad \text{for all } t \in (0, T_{\max}),$$

and that

$$\int_0^t\int_\Omega |\nabla u|^2 \le c_2 \qquad \text{for all } t \in (0, T_{\max}),$$

whence the claim becomes a consequence of Lemma 2.2. □

3. Large time behaviour. Proof of Theorem 1.1

Our large time analysis will rely on the following general observation, possibly of independent interest, concerning L^∞ decay as a consequence of uniform continuity in conjunction with a certain averaged decay property.

Lemma 3.1. *Let $t_0 \in \mathbb{R}$ and $\psi : \overline{\Omega} \times [t_0, \infty) \to \mathbb{R}$ be bounded, uniformly continuous and such that*

$$\int_t^{t+1}\int_\Omega |\psi|^q \to 0 \qquad \text{as } t \to \infty \tag{3.1}$$

with some $q > 0$. Then

$$\psi(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty. \tag{3.2}$$

Proof. Suppose that (3.2) was false. Then since the uniform continuity of ψ on $\overline{\Omega} \times [t_0, \infty)$ implies that $(\psi(\cdot,t))_{t \ge t_0}$ is equi-continuous, the Arzelá-Ascoli theorem would provide $\psi_\infty \in C^0(\overline{\Omega})$ and a sequence

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$(t_k)_{k\in\mathbb{N}} \subset [t_0,\infty)$ such that $t_k \to \infty$ and

$$\psi(\cdot,t_k) \to \psi_\infty \qquad \text{in } L^\infty(\Omega)$$

as $k\to\infty$ and that $\psi_\infty \not\equiv 0$. Accordingly taking $x_0 \in \overline{\Omega}$ such that $c_1 := |\psi_\infty(x_0)| > 0$, we could then find $k_0 \in \mathbb{N}$ such that $|\psi(x_0,t_k)| \ge \frac{c_1}{2}$ for all $k \ge k_0$, whereupon again relying on uniform continuity, we could fix $R>0$ and $\tau \in (0,1)$ such that $|\psi(x,t)| \ge \frac{c_1}{4}$ for all $x \in B_R(x_0)\cap\Omega$, any $t \in (t_k, t_k+\tau)$ and each $k \ge k_0$. Then, however,

$$\int_{t_k}^{t_k+1}\int_\Omega |\psi|^q \ge \int_{t_k}^{t_k+\tau}\int_{B_R(x_0)\cap\Omega} |\psi|^q \ge \frac{c_1\tau}{4}\cdot \big|B_R(x_0)\cap\Omega\big| \qquad \text{for all } k \ge k_0,$$

which is incompatible with (3.2) due to the fact that $|B_R(x_0)\cap\Omega|>0$ by smoothness of $\partial\Omega$. □

In order to satisfy the requirements concerning uniform continuity in the previous lemma, we note that standard parabolic theory ensures that L^∞ bounds entail Hölder estimates in the following sense:

Lemma 3.2. *Let $\delta_\star$ be as in Lemma 2.6, let $\alpha \ge \frac{1}{\delta_\star}$, and assume (1.6) and (1.7). Then there exist $\theta \in (0,1)$ and $C>0$ such that*

$$\|u\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \le C \qquad \text{for all } t \ge 1 \tag{3.3}$$

and

$$\|v\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \le C \qquad \text{for all } t \ge 1 \tag{3.4}$$

Proof. In view of the boundedness properties of u and v asserted by Lemmas 2.6 and 2.3, an application of standard heat semigroup estimates to the second equation in (1.3) ([8]) shows that $\nabla v \in L^\infty((0,\infty); L^\infty(\Omega;\mathbb{R}^n))$. Relying on this and again using the boundedness of u, (3.3) can be obtained from [18, Theorem 1.3] (cf., e.g., [21, Lemma 6.1] for a related precedent). Likewise, the assertion (3.4) can be concluded from the boundedness of u and ∇v through [18, Theorem 1.3]. □

Based on the above two lemmas, the weak decay information contained in (2.31) can be turned into a statement on L^∞ stabilization in the first solution component:

Lemma 3.3. *If $\delta_\star$ is as in Lemma 2.6, and if $\alpha \ge \frac{1}{\delta_\star}$ and (u_0, v_0) satisfies (1.6) and (1.7), then*

$$u(\cdot,t) \to \frac{1}{|\Omega|}\int_\Omega u_0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t\to\infty. \tag{3.5}$$

Proof. According to a Poincaré inequality, there exists $c_1>0$ such that

$$\int_\Omega \left|\psi - \frac{1}{|\Omega|}\int_\Omega \psi\right|^2 \le c_1 \int_\Omega |\nabla\psi|^2 \qquad \text{for all } \psi \in W^{1,2}(\Omega),$$

whence using (2.2), we find that

$$\int_0^t\int_\Omega \left|u(x,t) - \frac{1}{|\Omega|}\int_\Omega u_0\right|^2 \le c_1 \int_0^t\int_\Omega |\nabla u|^2 \qquad \text{for all } t>0.$$

From Lemma 2.6, we thus infer that

$$\int_0^\infty\int_\Omega \left|u - \frac{1}{|\Omega|}\int_\Omega u_0\right|^2 < \infty,$$

so that since $\overline{\Omega}\times[1,\infty) \ni (x,t) \mapsto u(x,t) - \frac{1}{|\Omega|}\int_\Omega u_0$ is uniformly continuous thanks to Lemma 3.2, it is sufficient to apply Lemma 3.1. □

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Our large time analysis of the second solution component will be based on the following elementary relaxation feature which actually does not rely on largeness of α nor on smallness of v_0 and is thus enjoyed also by possibly existing non-global solutions to (1.3).

Lemma 3.4. *Let $\alpha > 0$, and assume (1.6). Then there exists $C > 0$ such that*

$$\int_0^t \int_\Omega u \cdot \left(v - \frac{1}{\alpha}\right)^2 \le C \qquad \text{for all } t \in (0, T_{max}). \tag{3.6}$$

Proof. An integration by parts using the second equation in (1.3) shows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \left(v - \frac{1}{\alpha}\right)^2 &= \int_\Omega \left(v - \frac{1}{\alpha}\right) \cdot \left\{\Delta v - \alpha u \cdot \left(v - \frac{1}{\alpha}\right)\right\} \\ &= -\int_\Omega |\nabla v|^2 - \alpha \int_\Omega u \cdot \left(v - \frac{1}{\alpha}\right)^2 \\ &\le -\alpha \int_\Omega u \cdot \left(v - \frac{1}{\alpha}\right)^2 \qquad \text{for all } t \in (0, T_{max}).\end{aligned}$$

Therefore,

$$\frac{1}{2}\int_\Omega \left(v(\cdot,t) - \frac{1}{\alpha}\right)^2 + \alpha \int_0^t \int_\Omega u \cdot \left(v - \frac{1}{\alpha}\right)^2 \le \frac{1}{2}\int_\Omega \left(v_0 - \frac{1}{\alpha}\right)^2 \qquad \text{for all } t \in (0, T_{max}),$$

which implies (3.6). □

Using that the weight function u appearing in (3.6) eventually admits a uniform pointwise lower bound due to Lemma 3.3, we can combine Lemma 3.4 with the uniform continuity property implied by Lemma 3.2 to assert stabilization also in the signal concentration.

Lemma 3.5. *Let $\delta_\star$ be as in Lemma 2.6, let $\alpha \ge \frac{1}{\delta_\star}$, and suppose that (1.6) and (1.7) hold. Then*

$$v(\cdot, t) \to \frac{1}{\alpha} \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty. \tag{3.7}$$

Proof. As we are assuming that $u_0 \not\equiv 0$, letting $c_1 := \frac{1}{2|\Omega|}\int_\Omega u_0$ defines a positive constant which due to Lemma 3.3 has the property that

$$u(x,t) \ge c_1 \qquad \text{for all } x \in \Omega \text{ and } t > t_0$$

with some $t_0 > 0$. Therefore,

$$c_1 \int_{t_0}^t \int_\Omega \left(v - \frac{1}{\alpha}\right)^2 \le \int_{t_0}^t \int_\Omega u \cdot \left(v - \frac{1}{\alpha}\right)^2 \le \int_0^\infty \int_\Omega u \cdot \left(v - \frac{1}{\alpha}\right)^2 \qquad \text{for all } t > t_0,$$

whence using Lemma 3.4, we obtain that since under the current hypotheses, we already know that $T_{max} = \infty$,

$$\int_{t_0}^\infty \int_\Omega \left(v - \frac{1}{\alpha}\right)^2 < \infty.$$

Based on the uniform continuity property of $v - \frac{1}{\alpha}$ implied by (3.4), by means of Lemma 3.1, this yields (3.7). □

Both parts of our claim concerning the large time behaviour in (1.3) have thereby been completed:

Proof of Theorem 1.1. We only need to combine Lemma 2.6 with Lemma 2.1, and collect the outcomes of Lemmas 3.3 and 3.5. □

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4. Blow-up prevention in planar domains. Proofs of Proposition 1.2 and Corollary 1.3

In the particular case when $n = 2$, a simple reduction to known results on (1.2) extends the results from Theorem 1.1 to situations in which v_0 is suitably large throughout Ω:

Proof of Proposition 1.2. Since $v_0 - \frac{1}{\alpha}$ is nonnegative, according to a known argument the problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0, \\ w_t = \Delta w - \alpha u w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad w(x,0) = v_0(x) - \frac{1}{\alpha}, & x \in \Omega, \end{cases} \tag{4.1}$$

possesses a uniquely determined classical solution (u, w) with

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \text{and} \\ w \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \end{cases}$$

and with $u \geq 0$ and $w > 0$ in $\overline{\Omega} \times [0, \infty)$, and this solution additionally satisfies

$$u(\cdot,t) \to \frac{1}{|\Omega|} \int_\Omega u_0 \quad \text{and} \quad w(\cdot,t) \to 0 \qquad \text{in } L^\infty(\Omega)$$

as $t \to \infty$; for the special case $\alpha = 1$, this can be found detailed in [9], while a proof for arbitrary $\alpha > 0$ can be obtained by straightforward modification thereof. Setting $v := w + \frac{1}{\alpha}$, we thus obtain a pair (u, v) of nonnegative functions fulfilling (1.8) as well as (1.9) and (1.10). □

In radially symmetric settings, also on the basis of known approaches from the literature, we can thereby make sure that the behaviour in (1.3) for large $\alpha > 0$ substantially deviates from that when $\alpha = 0$:

Proof of Corollary 1.3. A straightforward modification of the reasonings in either [6] or [15] provides $T_{\max} \in (0, \infty)$ as well as nonnegative, radially symmetric functions $u^{(0)}$ and $v^{(0)}$ such that (1.12) and (1.14) hold and that (1.3) with $\alpha = 0$ is classically solved in $\Omega \times (0, T_{\max})$; in view of the strong maximum principle, we may assume upon replacing $T_{\max}$ with $\frac{1}{2} T_{\max}$ and t with $t - \frac{1}{2} T_{\max}$ here if necessary that $(u_0, v_0) := (u^{(0)}, v^{(0)})(\cdot, 0)$ satisfies (1.6) with $v_0 > 0$ in $\overline{\Omega}$. Accordingly, $\alpha_0 := \frac{1}{\inf_{x \in \Omega} v_0(x)}$ is well defined and positive, and therefore, we may apply Proposition 1.2 to see that for each $\alpha > \alpha_0$, the problem (1.3) possesses a global classical solution $(u^{(\alpha)}, v^{(\alpha)})$ fulfilling (1.13) and (1.15). □

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