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Abstract

This paper studies affine Deligne–Lusztig varieties in the affine flag manifold of a split group. Among other things, it proves emptiness for certain of these varieties, relates some of them to those for Levi subgroups, and extends previous conjectures concerning their dimensions. We generalize the superset method, an algorithmic approach to the questions of non-emptiness and dimension. Our non-emptiness results apply equally well to the p-adic context and therefore relate to moduli of p-divisible groups and Shimura varieties with Iwahori level structure.

1. Introduction

1.1 This paper, a continuation of [GHKR06], investigates affine Deligne–Lusztig varieties in the affine flag variety of a split connected reductive group G over a finite field $k = \mathbb{F}_q$. The Laurent series field $L = \overline{k}((\varepsilon))$, where $\overline{k}$ is an algebraic closure of k, is endowed with a Frobenius automorphism σ, and we use the same symbol to denote the induced automorphism of $G(L)$. By definition, the affine Deligne–Lusztig variety associated with x in the extended affine Weyl group $\widetilde{W} \cong I\backslash G(L)/I$ and $b \in G(L)$ is

$$X_x(b) = \{g \in G(L)/I : g^{-1}b\sigma(g) \in IxI\}.$$

(See § 1.2 for the notation used here.) We are interested in determining the dimension of $X_x(b)$ and in finding a criterion for when $X_x(b) \neq \emptyset$. These questions are related to the geometric structure of the reduction of certain Shimura varieties with Iwahori level structure: on the special fiber of the Shimura variety we have, on the one hand, the Newton stratification whose strata are indexed by certain σ-conjugacy classes $[b] \subseteq G(L)$ and, on the other hand, the Kottwitz–Rapoport stratification whose strata are indexed by certain elements of $\widetilde{W}$. The affine Deligne–Lusztig variety $X_x(b)$ is related to the intersection of the Newton stratum associated with $[b]$ and the Kottwitz–Rapoport stratum associated with x. See [GHKR06, § 5.10] and the survey papers by Rapoport [Rap05] and the second author [Hai05].

To provide some context, we begin by discussing affine Deligne–Lusztig varieties

$$X_\mu(b) = \{g \in G(L)/K : g^{-1}b\sigma(g) \in K\varepsilon^\mu K\}$$

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in the affine Grassmannian $G(L)/K$. It is known that $X_\mu(b)$ is non-empty if and only if Mazur's inequality is satisfied, that is to say, if and only if the σ-conjugacy class $[b]$ of b is less than or equal to $[\epsilon^\mu]$ in the natural partial order on the set $B(G)$ of σ-conjugacy classes in $G(L)$. This was proved in two steps: first, the problem was reduced to one on root systems (see [KR03]); the reduced problem was then solved for classical split groups by Lucarelli [Luc04] and, recently, for all quasi-split groups by Gashi [Gas08].

A conjectural formula for $\dim X_\mu(b)$ was put forward by Rapoport [Rap05], who pointed out its similarity to a conjecture of Chai's [Cha00] on dimensions of Newton strata in Shimura varieties. In [GHKR06], Rapoport's dimension conjecture was reduced to the superbasic case, which was then solved by Viehmann [Vie06].

We now return to affine Deligne–Lusztig varieties $X_x(b)$ in the affine flag manifold. For some years, a challenging problem has been to 'explain' the emptiness pattern observed in Figures 3 and 4 of the examples in § 14; see also [GHKR06, Reu04]. In other words, for a given b, one wants to understand the set of $x \in \widetilde{W}$ for which $X_x(b)$ is empty. Let us begin by discussing the simplest case, that in which $b = 1$ and x is *shrunken*, by which we mean that it lies in the union of the shrunken Weyl chambers (see § 14 and [GHKR06]). Then Reuman observed in [Reu04] that a simple rule explained the emptiness pattern for $X_x(1)$ in types A_1, A_2 and C_2, and he conjectured that the same might be true in general. Figure 3, the first example of § 14, illustrates how this simple rule depends on the elements $\eta_2(x)$ (respectively, $\eta_1(x)$) in W that label the 'big' (respectively, 'small') Weyl chambers which contain the alcove $x\mathbf{a}$. (See § 9.5 for the definitions of η_1 and η_2 and Conjecture 1.1.3 below for the precise rule.) Computer calculations in [GHKR06] provided further evidence for the truth of Reuman's conjecture. However, although in the rank-two cases there is a simple geometric pattern in each strip between two adjacent Weyl chambers (see Figures 3 and 4), we do not have a closed formula in group-theoretic terms that is consistent with all higher-rank examples we have computed when $x\mathbf{a}$ lies outside the shrunken Weyl chambers, and the emptiness there has remained mysterious.

In this paper we give, among other things, a precise conjecture describing the whole emptiness pattern for any basic b. This is more general in two ways: we no longer require $b = 1$ (although we do require that b be basic), and we no longer restrict attention to shrunken x. To do this, we introduce the new notion of *P-alcove* for any semistandard parabolic subgroup $P = MN$ (see §§ 2 and 3 and, in particular, Definition 2.1.1). Our Conjecture 9.4.2 says the following.

CONJECTURE 1.1.1. Let $[b]$ be a basic σ-conjugacy class. Then $X_x(b) \neq \emptyset$ if and only if for every semistandard $P = MN$ for which $x\mathbf{a}$ is a P-alcove, b is σ-conjugate to an element $b' \in M(L)$ and x and b' have the same image under the Kottwitz homomorphism

$$\eta_M : M(L) \to \Lambda_M.$$

See § 7 for a review of η_M. If $x\mathbf{a}$ is a P-alcove, then in particular $x \in \widetilde{W}_M$, the extended affine Weyl group of M, so that we can speak of $\eta_M(x)$. The condition $\eta_M(x) = \eta_M(b')$ means that x and b' lie in the same connected component of the $\overline{k}$-ind-scheme $M(L)$. Computer calculations support this conjecture, and for shrunken x we show (see Proposition 9.5.5) that the new conjecture reduces to Reuman's. We prove (see Corollary 9.4.1) one direction of this new conjecture, stated as follows.

THEOREM 1.1.2. *Let $[b]$ be basic. Then $X_x(b)$ is empty when Conjecture 1.1.1 predicts it to be.*

It remains a challenging problem to prove that non-emptiness occurs when predicted.

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In fact, Proposition 9.3.1 proves the emptiness of certain $X_x(b)$ even when b is not basic. However, in the non-basic case, there is a second cause of emptiness, stemming from Mazur's inequality. One might hope that these are the only two causes of emptiness, but this is slightly too naive. Mazur's inequality works perfectly for $G(\mathfrak{o})$-double cosets but not for Iwahori double cosets, and it would have to be improved slightly (in the Iwahori case) before it could be applied to give an optimal emptiness criterion. Although we do not yet know how to formulate Mazur's inequalities in the Iwahori case, we have been able to describe the information that they should carry, whatever they end up being; see § 12.

We now turn to the dimensions of non-empty affine Deligne–Lusztig varieties in the affine flag manifold. In [GHKR06] we formulated two conjectures of this kind, and here we will extend both of them (in a way that is supported by computer evidence). For basic b, we have the following conjecture.

Conjecture 1.1.3 (Conjecture 9.5.1(a)). Let $[b]$ be a basic σ-conjugacy class. Suppose that $x \in \widetilde{W}$ lies in the shrunken Weyl chambers. Then $X_x(b) \neq \emptyset$ if and only if

$$\eta_G(x) = \eta_G(b) \quad \text{and} \quad \eta_2(x)^{-1}\eta_1(x)\eta_2(x) \in W \Big\backslash \bigcup_{T \subsetneq S} W_T,$$

and in this case we have

$$\dim X_x(b) = \tfrac{1}{2}(\ell(x) + \ell(\eta_2(x)^{-1}\eta_1(x)\eta_2(x)) - \mathrm{def}_G(b)).$$

Here $\mathrm{def}_G(b)$ denotes the defect of b (see § 9.5). This extends Conjecture 7.2.2 of [GHKR06] from $b = 1$ to all basic b. For an illustration in the case of $G = \mathrm{GSp}_4$ (where the conjecture can be checked as in [Reu04]), see § 14.

Conjecture 9.5.1(b) extends [GHKR06, Conjecture 7.5.1] from translation elements $b = \epsilon^\nu$ to all b. For this we need the following notation: b_{b} will denote a representative of the unique basic σ-conjugacy class whose image in Λ_G is the same as that of b. (Equivalently, $[b_{\mathrm{b}}]$ is at the bottom of the connected component of $[b]$ in the poset $B(G)$.) In this second conjecture, it is the difference of the dimensions of $X_x(b)$ and $X_x(b_{\mathrm{b}})$ that is predicted. It is not required that x be shrunken, but $X_x(b)$ and $X_x(b_{\mathrm{b}})$ are required to be non-empty, and the length of x is required to be big enough. In the conjecture we phrase this last condition rather crudely in the form of $\ell(x) \geqslant N_b$ for some (unspecified) constant N_b that depends on b. However, the evidence from computer calculations suggests that for fixed b, having x such that $X_x(b)$ and $X_x(b_{\mathrm{b}})$ are both non-empty is almost (but not quite!) enough to make our prediction valid for x. It would be very interesting to understand this phenomenon better, although some insight into it has already been provided by Beazley's work on Newton strata for SL(3) (see [Bea09]). In addition, when $\ell(x) \geqslant N_b$, we conjecture that the non-emptiness of $X_x(b)$ is equivalent to that of $X_x(b_{\mathrm{b}})$.

The main theorem of this paper is a version of the Hodge–Newton decomposition which relates certain affine Deligne–Lusztig varieties for the group G to affine Deligne–Lusztig varieties for a Levi subgroup M.

Theorem 1.1.4 (Theorem 2.1.4 and Corollary 2.1.3). *Suppose that $P = MN$ is semistandard and $x\mathbf{a}$ is a P-alcove.*

(a) *The natural map $B(M) \to B(G)$ restricts to a bijection $B(M)_x \to B(G)_x$, where $B(G)_x$ is the subset of $B(G)$ consisting of $[b]$ for which $X_x^G(b)$ is non-empty. In particular, if $X_x^G(b) \neq \emptyset$, then $[b]$ meets $M(L)$.*

(b) *Suppose $b \in M(L)$. Then the canonical closed immersion $X_x^M(b) \hookrightarrow X_x^G(b)$ induces a bijection*

$$J_b^M \backslash X_x^M(b) \xrightarrow{\sim} J_b^G \backslash X_x^G(b),$$

where J_b^G denotes the σ-centralizer of b in $G(L)$ (see § 2).

The second part of this theorem can be proved using the techniques of [Kot03], but it seems unlikely that the same would be true of the first part. In any case, we shall use a different method, obtaining both parts of the theorem as a consequence of the following key result (Theorem 2.1.2), whose precise relation to the Hodge–Newton decomposition is clarified by the commutative diagram (8.1.1).

Theorem 1.1.5. *For any semistandard parabolic subgroup $P = MN$ and any P-alcove $x\mathbf{a}$, every element of IxI is σ-conjugate under I to an element of $I_M x I_M$, where $I_M := M \cap I$.*

It is striking that the notion of P-alcove, discovered in the attempt to understand the entire emptiness pattern for the $X_x(b)$ when b is basic, is also precisely the notion needed for our Hodge–Newton decomposition.

In §§ 10–13 we consider the questions of non-emptiness and dimensions of affine Deligne–Lusztig varieties from an algorithmic point of view. The following statement summarizes Theorem 11.3.1 and Corollary 13.3.2.

Theorem 1.1.6. *There are algorithms, expressed in terms of foldings in the Bruhat–Tits building of $G(L)$, for determining the non-emptiness and dimension of $X_x(b)$.*

These algorithms were used to produce the data that led to and supported our conjectures. The results of these sections imply, in particular, that non-emptiness in the function field and non-emptiness in the p-adic case are equivalent (Corollary 11.3.5). While this was certainly expected to hold, to the best of our knowledge no proof was known before. This equivalence has been used by Viehmann [Vie09] to investigate closure relations for Ekedahl–Oort strata in certain Shimura varieties; our results enabled her to carry over results from the function field case, thus avoiding the heavy machinery of Zink's displays. It seems plausible that the algorithmic description of Theorem 11.3.1 can also be used to show that the dimensions in the function field case and the p-adic case coincide, once a good notion of dimension has been defined for the latter case.

In § 13 we extend Reuman's superset method [Reu04] from $b = 1$ to general b. To that end we introduce (see Definition 13.1.1) the notion of *fundamental alcove* $y\mathbf{a}$. We show that for each σ-conjugacy class $[b]$, there exists a fundamental alcove $y\mathbf{a}$ such that the whole double coset IyI is contained in $[b]$. We then explain why this allows one to use a superset method to analyze the emptiness of $X_x(b)$ for any x.

In addition, we introduce in § 11 a generalization of the superset method. The superset method is based on I-orbits in the affine flag manifold X. It depends on the choice of a suitable representative for b, whose existence is proved in § 13, as mentioned above. On the other hand, [GHKR06] used orbits of $U(L)$, where U is the unipotent radical of a Borel subgroup containing our standard split maximal torus A. The generalized superset method interpolates between these two extremes, being based on orbits of $I_M N(L)$ on X, where $P = MN$ is a standard parabolic subgroup of G. Theorem 11.3.1 and the discussion preceding it explain how the generalized superset method can be used to study dimensions of affine Deligne–Lusztig varieties.

For any standard parabolic subgroup $P = MN$ and any basic $b \in M(L)$, Proposition 12.1.1 gives a formula for the dimension of $X_x(b)$ in terms of dimensions of affine Deligne–Lusztig varieties for M as well as intersections of I-orbits and $N'(L)$-orbits for certain Weyl group conjugates N' of N. This generalizes [GHKR06, Theorem 6.3.1] and is also analogous to [GHKR06, Proposition 5.6.1] but with the affine Grassmannian replaced by the affine flag manifold.

1.2 Notation

We follow the notation of [GHKR06], for the most part. Let k be a finite field with q elements, and let $\overline{k}$ be an algebraic closure of k. We consider the field $L := \overline{k}((\epsilon))$ and its subfield $F := k((\epsilon))$. We write $\sigma : x \mapsto x^q$ for the Frobenius automorphism of $\overline{k}/k$, and we also regard σ as an automorphism of L/F in the usual way, so that

$$\sigma\left(\sum a_n \epsilon^n\right) = \sum \sigma(a_n)\epsilon^n.$$

We write $\mathfrak{o}$ for the valuation ring $\overline{k}[[\epsilon]]$ of L.

Let G be a split connected reductive group over k, and let A be a split maximal torus of G. Write R for the set of roots of A in G. Put $\mathfrak{a} := X_*(A)_{\mathbb{R}}$. Write W for the Weyl group of A in G. Fix a Borel subgroup $B = AU$ containing A with unipotent radical U, and write R^+ for the corresponding set of positive roots, that is, those occurring in U. We denote by ρ the half-sum of the positive roots. For $\lambda \in X_*(A)$ we write ϵ^λ for the element of $A(F)$ obtained as the image of $\epsilon \in \mathbb{G}_m(F)$ under the homomorphism $\lambda : \mathbb{G}_m \to A$.

Let C_0 denote the dominant Weyl chamber, which by definition is the set of $x \in \mathfrak{a}$ such that $\langle \alpha, x\rangle > 0$ for all $\alpha \in R^+$. We denote by $\mathbf{a}$ the unique alcove in the dominant Weyl chamber whose closure contains the origin, and we call it the base alcove. As Iwahori subgroup I we choose the one fixing the base alcove $\mathbf{a}$; I is then the inverse image of the opposite Borel group of B under the projection $K := G(\mathfrak{o}) \longrightarrow G(\overline{k})$. The opposite Borel group arises here because of our convention that ϵ^λ acts on the standard apartment $\mathfrak{a}$ by translation by λ (rather than by translation by the negative of λ), so that the stabilizer in $G(L)$ of $\lambda \in X_*(A) \subset \mathfrak{a}$ is $\epsilon^\lambda K \epsilon^{-\lambda}$. With this convention, the Lie algebra of the Iwahori subgroup stabilizing an alcove $\mathbf{b}$ in the standard apartment is made up of affine root spaces $\epsilon^j \mathfrak{g}_\alpha$ for all pairs (α, j) such that $\alpha - j \leqslant 0$ on $\mathbf{b}$ (with $\mathfrak{g}_\alpha$ denoting the root subspace corresponding to α).

We will often think of alcoves in a slightly different way. Let Λ_G denote the quotient of $X_*(A)$ by the coroot lattice. The apartment $\mathcal{A}$ corresponding to our fixed maximal torus A can be decomposed into a product, $\mathcal{A} = \mathcal{A}_{\mathrm{der}} \times V_G$, where $V_G := \Lambda_G \otimes \mathbb{R}$ and $\mathcal{A}_{\mathrm{der}}$ is the apartment corresponding to $A_{\mathrm{der}} := G_{\mathrm{der}} \cap A$ in the building for G_{der}. By an *extended alcove* we mean a subset of the apartment $\mathcal{A}$ of the form $\mathbf{b} \times c$, where $\mathbf{b}$ is an alcove in $\mathcal{A}_{\mathrm{der}}$ and $c \in \Lambda_G$. Clearly, each extended alcove determines a unique alcove in the usual sense, but not conversely. However, in the following we will often use these terms interchangeably, leaving context to determine which is meant. In particular, we will often write $\mathbf{a}$ in place of $\mathbf{a} \times 0$.

We denote by $\widetilde{W}$ the extended affine Weyl group $X_*(A) \rtimes W$ of G. Then $\widetilde{W}$ acts transitively on the set of all alcoves in $\mathfrak{a}$ and simply transitively on the set of all extended alcoves. Let $\Omega = \Omega_{\mathbf{a}}$ denote the stabilizer of $\mathbf{a}$ when it is viewed as an alcove in the usual (non-extended) sense. We can write an extended (respectively, non-extended) alcove in the form $x\mathbf{a}$ for a unique element $x \in \widetilde{W}$ (respectively, $x \in \widetilde{W}/\Omega$). Of course, this is just another way of saying that we can think of extended alcoves simply as elements of $\widetilde{W}$. Note that we can also describe $\widetilde{W}$ as the quotient $N_G A(L)/A(\mathfrak{o})$. For $x \in \widetilde{W}$, we write ${}^xI = \dot{x}I\dot{x}^{-1}$ where $\dot{x} \in N_G A(L)$ is a lift of x. It is clear that the result is independent of the choice of lift. As usual, a standard parabolic subgroup is one containing B, and a semistandard parabolic subgroup is one containing A. Similarly, a semistandard Levi subgroup is one containing A, and a standard Levi subgroup is the unique semistandard Levi component of a standard parabolic subgroup. Whenever we write $P = MN$ for a semistandard parabolic subgroup, we take this to mean that M is its semistandard Levi component and N is its unipotent radical. Given a semistandard Levi subgroup M of G, we write $\mathcal{P}(M)$ for the set of parabolic subgroups of G admitting M as Levi component. For $P \in \mathcal{P}(M)$, we denote by $\overline{P} = M\overline{N} \in \mathcal{P}(M)$ the parabolic subgroup opposite to P, i.e. $\overline{N}$ is the unipotent radical of $\overline{P}$. We write R_N for the set of roots of A in N. We denote by I_M, I_N and $I_{\overline{N}}$ the intersections of I with M, N and $\overline{N}$, respectively; one then has the Iwahori decomposition $I = I_N I_M I_{\overline{N}}$.

Recall that for $x \in \widetilde{W}$ and $b \in G(L)$, the affine Deligne–Lusztig variety $X_x(b)$ is defined by

$$X_x(b) := \{g \in G(L)/I : g^{-1}b\sigma(g) \in IxI\}.$$

In what follows, we shall often abuse notation and use the symbols G, P, M and N to denote the corresponding objects over L.

Let $b \in G(L)$. We denote by $[b]$ the σ-conjugacy class of b inside $G(L)$,

$$[b] = \{g^{-1}b\sigma(g) : g \in G(L)\},$$

and for a subgroup $H \subseteq G(L)$ we write

$$[b]_H := \{h^{-1}b\sigma(h) : h \in H\} \subseteq G(L)$$

for the σ-conjugacy class of b under H. Further notation relevant to $B(G)$, such as η_G, will be explained in § 7.

We shall use the symbols $\subset$ and $\subseteq$ interchangeably to mean 'not necessarily strict inclusion'.

2. Statement of the main theorem

2.1 Let $\alpha \in R$. We identify the root group U_α with the additive group $\mathbb{G}_a$ over k, which then allows us to identify $U_\alpha(L) \cap K$ with $\mathfrak{o}$. The root α induces a partial order $\geqslant_\alpha$ on the set of (extended) alcoves in the standard apartment as follows. Given an alcove $\mathbf{b}$, write it as $x\mathbf{a}$ for $x \in \widetilde{W}$. Then, take $k(\alpha, \mathbf{b}) \in \mathbb{Z}$ such that $U_\alpha(L) \cap {}^xI = \epsilon^{k(\alpha,\mathbf{b})}\mathfrak{o}$; in other words, $k(\alpha, \mathbf{b})$ is the unique integer k such that $\mathbf{b}$ lies in the region between the affine root hyperplanes

$$H_{\alpha,k} = \{x \in X_*(A)_{\mathbb{R}} : \langle \alpha, x\rangle = k\} \quad \text{and} \quad H_{\alpha,k-1}.$$

This description shows immediately that $k(\alpha, \mathbf{b}) + k(-\alpha, \mathbf{b}) = 1$. (For instance, we have $k(\alpha, \mathbf{a}) = 1$ if $\alpha > 0$ and $k(\alpha, \mathbf{a}) = 0$ if $\alpha < 0$, reflecting the fact that the fixer I of $\mathbf{a}$ is the inverse image of the opposite Borel group $\overline{B}$ under the projection $G(\mathfrak{o}) \to G(\overline{k})$.) We say that

$$\mathbf{b}_1 \geqslant_\alpha \mathbf{b}_2 \quad \text{if and only if} \quad k(\alpha, \mathbf{b}_1) \geqslant k(\alpha, \mathbf{b}_2).$$

This is a partial order in the weak sense: $\mathbf{b}_1 \geqslant_\alpha \mathbf{b}_2$ and $\mathbf{b}_2 \geqslant_\alpha \mathbf{b}_1$ do not imply $\mathbf{b}_1 = \mathbf{b}_2$. We also define:

$$\mathbf{b}_1 >_\alpha \mathbf{b}_2 \quad \text{if and only if} \quad k(\alpha, \mathbf{b}_1) > k(\alpha, \mathbf{b}_2).$$

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DEFINITION 2.1.1. Let $P = MN$ be a semistandard parabolic subgroup. Let $x \in \widetilde{W}$. We say that $x\mathbf{a}$ is a *P-alcove* if:

(1) $x \in \widetilde{W}_M$; and

(2) for all $\alpha \in R_N$, $x\mathbf{a} \geqslant_\alpha \mathbf{a}$.

We say that $x\mathbf{a}$ is a *strict* P-alcove if, instead of (2), we have:

(2′) for all $\alpha \in R_N$, $x\mathbf{a} >_\alpha \mathbf{a}$.

Note that condition (2) depends only on the image of x in $\widetilde{W}/\Omega$; however, condition (1) depends on x itself. It is important to work with extended alcoves here. One could argue that the above definition is more about elements of the extended affine Weyl group than about extended alcoves, but the term *P-alcove* seemed most convenient anyway.

By the definition of the partial order $\geqslant_\alpha$, the condition (2) is equivalent to

$$\text{for all } \alpha \in R_N, \quad U_\alpha \cap {}^xI \subseteq U_\alpha \cap I \tag{2.1.1}$$

or, likewise, to

$$\text{for all } \alpha \in R_N, \quad U_{-\alpha} \cap {}^xI \supseteq U_{-\alpha} \cap I. \tag{2.1.2}$$

Moreover, under our assumption that $x \in \widetilde{W}_M$, these in turn are equivalent to the condition

$${}^x(N \cap I) \subseteq N \cap I \quad \text{or, equivalently,} \quad {}^x(\overline{N} \cap I) \supseteq \overline{N} \cap I. \tag{2.1.3}$$

(Also, condition (2′) is equivalent to (2.1.1) with the inclusions replaced by strict inclusions.) Indeed, noting that conjugation by $x = \epsilon^\lambda w$ permutes the subgroups U_α with $\alpha \in R_N$, it is easy to see from the (Iwahori) factorization

$$N \cap I = \prod_{\alpha \in R_N} U_\alpha \cap I \tag{2.1.4}$$

that (2.1.1) is equivalent to (2.1.3). For a fixed semistandard parabolic subgroup $P = MN$, the set of alcoves $x\mathbf{a}$ which satisfy (2.1.1) constitutes a union of 'acute cones of alcoves' in the sense of [HN02]. We shall explain this in § 3 below.

Our key result concerns the map

$$\begin{aligned} \phi : I \times I_M x I_M &\to IxI, \\ (i, m) &\mapsto i m \sigma(i)^{-1}. \end{aligned}$$

There is a left action of I_M on $I \times I_M x I_M$ given by $i_M(i, m) = (i i_M^{-1}, i_M m \sigma(i_M)^{-1})$ for $i_M \in I_M$, $i \in I$ and $m \in I_M x I_M$. Let us denote by $I \times^{I_M} I_M x I_M$ the quotient of $I \times I_M x I_M$ by this action of I_M. Denote by $[i, m]$ the equivalence class of $(i, m) \in I \times I_M x I_M$. The map ϕ obviously factors through $I \times^{I_M} I_M x I_M$. We can now state the key result which enables us to prove the Hodge–Newton decomposition.

THEOREM 2.1.2. *Suppose that $P = MN$ is a semistandard parabolic subgroup and $x\mathbf{a}$ is a P-alcove. Then the map*

$$\phi : I \times^{I_M} I_M x I_M \to IxI$$

induced by $(i, m) \mapsto i m \sigma(i)^{-1}$ is surjective. If $x\mathbf{a}$ is a strict P-alcove, then ϕ is injective. In general, ϕ is not injective, but if $[i, m]$ and $[i', m']$ belong to the same fiber of ϕ, then the elements m and m' are σ-conjugate by an element of I_M.

This theorem was partially inspired by Labesse's study of the 'elementary functions' that he introduced in [Lab90].

Let us mention a few consequences. First, consider the quotient $IxI /_\sigma I$, where the action of I on IxI is given by σ-conjugation. We can also form, in a parallel manner, the quotient $I_M x I_M /_\sigma I_M$. Further, let $B(G)_x$ denote the set of σ-conjugacy classes $[b]$ in $G(L)$ which meet IxI. We note that for $G = \mathrm{SL}_3$, all of the sets $B(G)_x$ have been determined explicitly by Beazley [Bea09].

Corollary 2.1.3. *Suppose that $P = MN$ is semistandard and $x\mathbf{a}$ is a P-alcove. Then the following statements hold.*

(a) *The inclusion $I_M x I_M \hookrightarrow IxI$ induces a bijection*

$$I_M x I_M /_\sigma I_M \xrightarrow{\sim} IxI /_\sigma I.$$

(b) *The canonical map $\iota : B(M)_x \to B(G)_x$ is bijective.*

Part (a) follows directly from Theorem 2.1.2. Indeed, the surjectivity of ϕ implies the surjectivity of

$$I_M x I_M /_\sigma I_M \to IxI /_\sigma I.$$

As for the injectivity of the latter, note that if $i \in I$ and $m, m' \in I_M x I_M$ satisfy $im\sigma(i)^{-1} = m'$, then $[i, m]$ and $[1, m']$ belong to the same fiber of ϕ. As for part (b), we will derive it from part (a) in § 8. (In fact, the surjectivity in part (b) follows easily from the surjectivity in Theorem 2.1.2.)

Another consequence is our main theorem, a version of the Hodge–Newton decomposition, given as Theorem 2.1.4 below. For affine Deligne–Lusztig varieties in the affine Grassmannian of a split group, the analogous Hodge–Newton decomposition was proved under unnecessarily strict hypotheses in [Kot03] and in the general case by Viehmann [Vie08, Theorem 1] (see also the paper [MV07] by Mantovan and Viehmann for the case of unramified groups). To state this result, we need to fix a standard parabolic subgroup $P = MN$ and an element $b \in M(L)$. Let $K_M = M \cap K$ where K, as usual, denotes $G(\mathfrak{o})$. For a G-dominant coweight $\mu \in X_*(A)$, the σ-centralizer

$$J_b^G := \{g \in G(L) : g^{-1} b \sigma(g) = b\}$$

of b acts naturally on the affine Deligne–Lusztig variety $X_\mu^G(b) \subset G(L)/K$, which is defined to be

$$X_\mu^G(b) := \{gK \in G(L)/K : g^{-1} b \sigma(g) \in K \epsilon^\mu K\}.$$

Also, J_b^M acts on $X_\mu^M(b) \subset M(L)/K_M$. Now the Hodge–Newton decomposition under discussion asserts the following. Suppose that the Newton point $\overline{\nu}_b^M \in X_*(A)_\mathbb{R}$ is G-dominant and that $\eta_M(b) = \mu$ in Λ_M. Then the canonical closed immersion $X_\mu^M(b) \hookrightarrow X_\mu^G(b)$ induces a bijection

$$J_b^M \backslash X_\mu^M(b) \xrightarrow{\sim} J_b^G \backslash X_\mu^G(b).$$

Of course, if we impose the stricter condition that $\langle \alpha, \overline{\nu}_b^M \rangle > 0$ for all $\alpha \in R_N$, then $J_b^M = J_b^G$ and we get the stronger conclusion of $X_\mu^M(b) \cong X_\mu^G(b)$, yielding what is normally known as the Hodge–Newton decomposition in this context. The version with the weaker condition is essentially a result of Viehmann, who formulated it somewhat differently (see [Vie08, Theorem 2]), in a way that brings out a dichotomy which occurs when G is simple.

In the affine flag variety, it still makes sense to ask how $X_x^G(b)$ and $X_x^M(b)$ are related, for $x \in \widetilde{W}_M$ and $b \in M(L)$. Our Hodge–Newton decomposition below provides some information in this direction.

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Theorem 2.1.4. *Suppose that $P = MN$ is semistandard and $x\mathbf{a}$ is a P-alcove.*

(a) *If $X_x^G(b) \neq \emptyset$, then $[b]$ meets $M(L)$.*

(b) *Suppose $b \in M(L)$. Then the canonical closed immersion $X_x^M(b) \hookrightarrow X_x^G(b)$ induces a bijection*

$$J_b^M \backslash X_x^M(b) \xrightarrow{\sim} J_b^G \backslash X_x^G(b).$$

Note that part (b) implies that if $x\mathbf{a}$ is a P-alcove, then for every $b \in M(L)$ we have $X_x^G(b) = \emptyset$ if and only if $X_x^M(b) = \emptyset$. We will prove Theorem 2.1.4 in § 8 and then, in § 9, derive some further consequences relating to emptiness or non-emptiness of $X_x^G(b)$.

3. *P*-alcoves and acute cones of alcoves

3.1 Let $P = MN$ be a fixed semistandard parabolic subgroup. The aim of this section is to link the new notion of P-alcove to the notion of acute cones, and to help the reader visualize the set of P-alcoves. Let $\mathfrak{P}$ denote the set of alcoves $x\mathbf{a}$ which satisfy the inequalities $x\mathbf{a} \geqslant_\alpha \mathbf{a}$ for all $\alpha \in R_N$. See Figures 1 and 2 for illustrations of the set $\mathfrak{P}$ for G of type C_2 and G_2, respectively.

For each element $w \in W$, we recall the notion of *acute cone* of alcoves $C(\mathbf{a}, w)$, following [HN02]. Given an affine hyperplane $H = H_{\alpha,k} = H_{-\alpha,-k}$, assume α has sign such that $\alpha \in w(R^+)$, i.e. such that α is a positive root with respect to ${}^wB$. Then define the *w-positive* half-space

$$H^{w+} = \{v \in X_*(A)_\mathbb{R} : \langle \alpha, v\rangle > k\},$$

and let H^{w-} denote the other half-space.

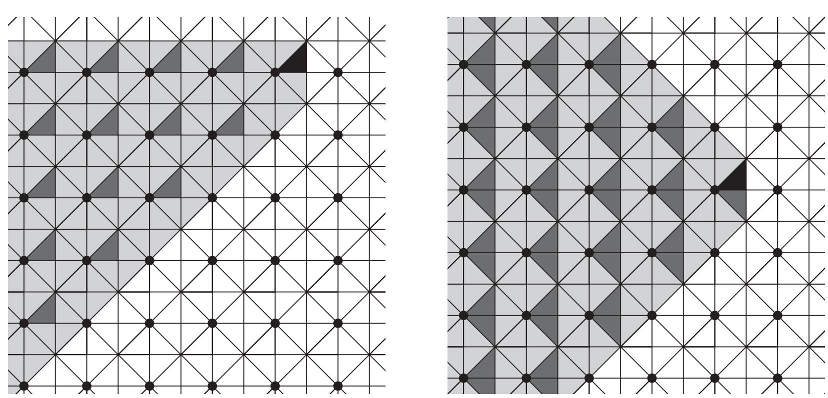

Figure 1. Illustration of the notion of P-alcove for G of type C_2. On the left, $P = {}^{w_0}B$ where w_0 is the longest element in W. On the right, $P = {}^{s_1 s_2 s_1}P'$ where P' is the standard parabolic subgroup $B \cup Bs_2B$. In both cases, the black alcove is the base alcove, the region $\mathfrak{P}$ is in light gray, and the P-alcoves are shown in dark gray.

The acute cone of alcoves $C(\mathbf{a}, w)$ is then defined to be the set of alcoves $x\mathbf{a}$ such that some (or, equivalently, every) minimal gallery joining $\mathbf{a}$ to $x\mathbf{a}$ is in the w-direction. By definition, a gallery $\mathbf{a}_1, \ldots, \mathbf{a}_l$ is *in the w-direction* if for each crossing $\mathbf{a}_{i-1}|_H\, \mathbf{a}_i$, the alcove $\mathbf{a}_{i-1}$ belongs to H^{w-} and $\mathbf{a}_i$ belongs to H^{w+}. By [HN02, Lemma 5.8], the acute cone $C(\mathbf{a}, w)$ is an intersection

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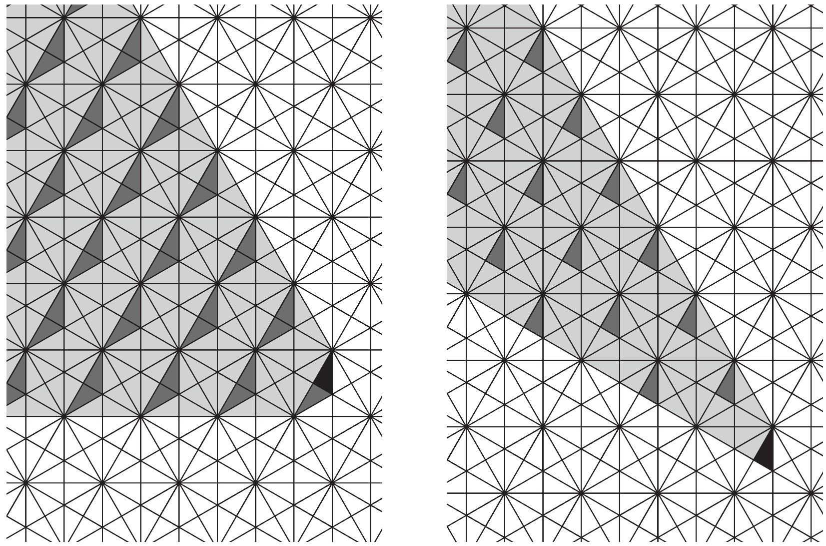

FIGURE 2. Illustration of P-alcoves for G of type G_2: on the left, $P = {}^{s_1 s_2 s_1}(B \cup Bs_2B)$; on the right, $P = {}^{s_2 s_1 s_2 s_1}B$.

of half-spaces:

$$C(\mathbf{a}, w) = \bigcap_{\mathbf{a} \subset H^{w+}} H^{w+}.$$

PROPOSITION 3.1.1. *The set of alcoves $\mathfrak{P}$ is the following union of acute cones of alcoves:*

$$\mathfrak{P} = \bigcup_{w : P \supseteq {}^w B} C(\mathbf{a}, w). \tag{3.1.1}$$

Proof. For any root $\alpha \in R$ and any $k \in \mathbb{Z}$, let $H^+_{\alpha,k}$ denote the unique half-space for $H_{\alpha,k}$ which contains the base alcove $\mathbf{a}$. Note that for any $\alpha \in R$ and $w \in W$, we have

$$H^+_{\alpha,k(\alpha,\mathbf{a})-1} = \begin{cases} H^{w+}_{\alpha,k(\alpha,\mathbf{a})-1} & \text{if } \alpha \in w(R^+), \\ H^{w-}_{\alpha,k(\alpha,\mathbf{a})-1} & \text{if } \alpha \in w(R^-). \end{cases} \tag{3.1.2}$$

Now suppose that $w \in W$ satisfies $P \supseteq {}^w B$ or, in other words, $N \subseteq {}^w U$ or, equivalently, $R_N \subseteq w(R^+)$. Then we see, upon using (3.1.2), that

$$C(\mathbf{a}, w) = \bigcap_{\alpha \in w(R^+)} H^{w+}_{\alpha,k(\alpha,\mathbf{a})-1} = \bigcap_{\alpha \in w(R^+)} H^+_{\alpha,k(\alpha,\mathbf{a})-1}.$$

So the union on the right-hand side of (3.1.1) is

$$\bigcup_{w : R_N \subseteq w(R^+)} \bigcap_{\alpha \in w(R^+)} H^+_{\alpha,k(\alpha,\mathbf{a})-1} \tag{3.1.3}$$

and, in particular, is contained in $\bigcap_{\alpha \in R_N} H^+_{\alpha,k(\alpha,\mathbf{a})-1} = \mathfrak{P}$.

For the opposite inclusion, we set

$$\mathscr{U} = \bigcup_{w : R_N \subseteq w(R^+)} C(\mathbf{a}, w).$$

We will prove the implication

$$x\mathbf{a} \notin \mathscr{U} \implies x\mathbf{a} \notin \mathfrak{P} \tag{3.1.4}$$

by induction on the length ℓ of a minimal gallery $\mathbf{a} = \mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_\ell = x\mathbf{a}$. If $\ell = 0$, there is nothing to show, so we assume $\ell > 0$ and that the implication holds for $y\mathbf{a} := \mathbf{a}_{\ell-1}$.

Assume $x\mathbf{a} \notin \mathscr{U}$. There are two cases to consider. If $y\mathbf{a} \notin \mathscr{U}$, then by induction $y\mathbf{a} \notin \mathfrak{P}$. This means that $y\mathbf{a}$ and $\mathbf{a}$ are on opposite sides of a hyperplane $H_{\alpha,k(\alpha,\mathbf{a})-1}$ for some $\alpha \in R_N$. The same then holds for $x\mathbf{a}$, which shows that $x\mathbf{a} \notin \mathfrak{P}$.

Otherwise, $y\mathbf{a} \in \mathscr{U}$, so that $y\mathbf{a}$ belongs to some $C(\mathbf{a}, w)$ with $R_N \subseteq w(R^+)$. Let $H = H_{\beta,m}$ be the wall separating $y\mathbf{a}$ and $x\mathbf{a}$. Since $x\mathbf{a} \notin C(\mathbf{a}, w)$ and $s_{\beta,m}x\mathbf{a} \in C(\mathbf{a}, w)$, we have that $m \in \{0, \pm 1\}$ and $x\mathbf{a} \in C(\mathbf{a}, s_\beta w)$. Now, if $s_\beta \in W_M$, then $R_N \subseteq s_\beta w(R^+)$ and $x\mathbf{a} \in \mathscr{U}$, which is a contradiction. Thus $\beta \in \pm R_N$, and without loss of generality we may assume $\beta \in R_N$. Now, in passing from $y\mathbf{a}$ to $x\mathbf{a}$, we crossed H in the β-*opposite* direction, where by definition this means that $x(a) - y(a) \in \mathbb{R}_{<0}\beta^\vee$ for any point a in the interior of $\mathbf{a}$. Indeed, if this were not the case, then since $\beta \in w(R^+)$ the crossing $y\mathbf{a} \,|_H\, x\mathbf{a}$ must be in the w-direction, so $x\mathbf{a}$ would belong to $C(\mathbf{a}, w)$ (because $y\mathbf{a}$ does), which is a contradiction.

To conclude, we observe that if $\mathbf{a} = \mathbf{a}_0, \ldots, \mathbf{a}_\ell$ is a minimal gallery and crosses some $H_{\beta,m}$ with $\beta \in R_N$ in the β-opposite direction, then the terminal alcove $\mathbf{a}_\ell$ must actually lie outside $\mathfrak{P}$ (since such a gallery must cross the hyperplane $H_{\beta,k(\beta,\mathbf{a})-1}$). □

4. Reformulation of Theorem 2.1.2

4.1 In the following reformulation of Theorem 2.1.2, we assume that $P = MN$ is semistandard and $x\mathbf{a}$ is a P-alcove. As in Beazley's work [Bea09], it is easier to work with single cosets xI than with double cosets IxI, and the next result allows us to do just that.

Lemma 4.1.1. *Theorem 2.1.2 is equivalent to the following statement: the map*

$$\phi : ({}^xI \cap I) \times^{{}^xI_M \cap I_M} xI_M \to xI$$

given by $(i, m) \mapsto im\sigma(i)^{-1}$ is surjective. Moreover, it is bijective if $x\mathbf{a}$ is a strict P-alcove. In general, if $[i, xj]$ and $[i', xj']$ belong to the same fiber of ϕ, then xj and xj' are σ-conjugate by an element of ${}^xI_M \cap I_M$.

Proof. It is straightforward to verify that the following diagram with vertical inclusion maps is Cartesian.

$$\begin{tikzcd}
({}^xI \cap I) \times^{{}^xI_M \cap I_M} xI_M \arrow[r, "\phi"] \arrow[d] & xI \arrow[d] \\
I \times^{I_M} I_M x I_M \arrow[r, "\phi"] & IxI
\end{tikzcd}$$

The lemma is now clear by appealing to I-equivariance: each element of IxI is σ-conjugate under I to an element of xI, and ϕ is I-equivariant with respect to the action by σ-conjugation on IxI and the action on $I \times^{I_M} I_M x I_M$ given by $i'[i, m] := [i'i, m]$ for $i' \in I$ and $[i, m] \in I \times^{I_M} I_M x I_M$. □

We can now prove the portion of Theorem 2.1.2 relating to the fibers of ϕ. Suppose that $[i_1, xj_1], [i_2, xj_2] \in ({}^xI \cap I) \times^{{}^xI_M \cap I_M} xI_M$ satisfy $i_1 x j_1 \sigma(i_1)^{-1} = i_2 x j_2 \sigma(i_2)^{-1}$. Letting $i := i_2^{-1} i_1$, we see that

$$x^{-1}ix = j_2\sigma(i)j_1^{-1}. \tag{4.1.1}$$

We have the Iwahori decompositions $I = I_{\overline{N}}I_MI_N$ and ${}^xI = {}^xI_{\overline{N}}\,{}^xI_M{}^xI_N$, where $I_N := N \cap I$ and $I_{\overline{N}} := \overline{N} \cap I$. Using our assumption that $x\mathbf{a}$ is a P-alcove, we deduce that

$${}^xI \cap I = I_{\overline{N}}\,({}^xI_M \cap I_M)\,{}^xI_N. \tag{4.1.2}$$

Write $i = i_-i_0i_+$ with $i_- \in I_{\overline{N}}$, $i_0 \in {}^xI_M \cap I_M$ and $i_+ \in {}^xI_N$. Using (4.1.1), we get

$${}^{x^{-1}}i_- \cdot {}^{x^{-1}}i_0 \cdot {}^{x^{-1}}i_+ = {}^{j_2}\sigma(i_-) \cdot j_2\sigma(i_0)j_1^{-1} \cdot {}^{j_1}\sigma(i_+). \tag{4.1.3}$$

By the uniqueness of the factorization of elements in $\overline{N} \cdot M \cdot N$, we get

$${}^{x^{-1}}i_- = {}^{j_2}\sigma(i_-), \tag{4.1.4}$$

$${}^{x^{-1}}i_0 = j_2\sigma(i_0)j_1^{-1}, \tag{4.1.5}$$

$${}^{x^{-1}}i_+ = {}^{j_1}\sigma(i_+). \tag{4.1.6}$$

From (4.1.5), we deduce that xj_1 is σ-conjugate to xj_2 by an element in ${}^xI_M \cap I_M$. This proves the main assertion regarding the fibers of ϕ.

It remains to prove that ϕ is injective when $x\mathbf{a}$ is a strict P-alcove. In that case, conjugation by x is strictly expanding (respectively, contracting) on $I_{\overline{N}}$ (respectively, I_N). In other words, the condition (2.1.1), and hence also (2.1.3), holds with the inclusions replaced by strict inclusions. But then (4.1.4) (respectively, (4.1.6)) can hold only if $i_- = 1$ (respectively, $i_+ = 1$). Thus we have $i = i_0 \in {}^xI_M \cap I_M$, and it follows that $[i_1, xj_1] = [i_2, xj_2]$. This proves the injectivity of ϕ asserted in Theorem 2.1.2.

5. A variant of Lang's theorem for vector groups

5.1 As before, let k denote a finite field with q elements, and let $\overline{k}$ denote an algebraic closure of k. We write σ for the Frobenius automorphism $x \mapsto x^q$ of $\overline{k}$. In this section we will be concerned with an automorphism τ of $\overline{k}$ which is required to be either σ or σ^{-1}. By a τ-space (V, Φ) we shall mean a finite-dimensional vector space V over $\overline{k}$ together with a τ-linear map $\Phi : V \to V$. We do *not* require that Φ be bijective. The category of τ-spaces is abelian and every object in it has finite length. Let (V, Φ) be a simple object in this category. We claim that V is one-dimensional (cf. [KR03, proof of Lemma 1.3]). Since $\ker \Phi$ is a subobject of V, we must have either $\ker \Phi = V$ or $\ker \Phi = 0$. In the first case, $\Phi = 0$, every subspace is a subobject, and therefore simplicity forces V to be one-dimensional. In the second case, Φ is bijective, and a subspace W is a subobject if and only if $\Phi W = W$ or, equivalently, $\Phi^{-1}W = W$. Therefore we may as well assume that $\tau = \sigma$ (since Φ^{-1} is σ-linear if Φ is σ^{-1}-linear). Then, by Lang's theorem for general linear groups over k, our τ-space is a direct sum of copies of $(\overline{k}, \sigma)$; hence, owing to simplicity, it is one-dimensional.

Lemma 5.1.1. *Let (V, Φ) be a τ-space. Then the k-linear map $v \mapsto v - \Phi(v)$ from V to V is surjective.*

Proof. Filter (V, Φ) so that each successive quotient is one-dimensional. Since the desired surjectivity follows from surjectivity of the induced map on the associated graded object, we just need to prove surjectivity when V is one-dimensional. This amounts to the solvability of the equations $x - ax^q = b$ and $x - ax^{1/q} = b$. Solvability of the first equation is obvious, and so too is that of the second after the change of variables $x = y^q$, which leads to the equivalent equation $y^q - ay = b$. □

Corollary 5.1.2. *Let V_0 be a finite-dimensional k-vector space, let $V = V_0 \otimes_k \overline{k}$, and let $M : V \to V$ be a linear map. Then:*

(1) *for every $w \in V$ there exists $v \in V$ such that $\sigma v - Mv = w$; and*
(2) *for every $w \in V$ there exists $v \in V$ such that $v - M\sigma v = w$.*

Proof. The second statement follows from the lemma (with $\tau = \sigma$), and the first follows from the lemma (with $\tau = \sigma^{-1}$) after making the change of variables $v = \sigma^{-1}v'$. □

Remark 5.1.3. We note that the second statement of the corollary can also be proved in the same way as Lang's theorem. However, this method does not handle the first statement of the corollary in the case where M is not bijective.

6. Proof of the surjectivity assertion in Theorem 2.1.2

6.1 The method of successive approximations

We again assume that $x\mathbf{a}$ is a P-alcove. Recall that, by Lemma 4.1.1, we need to prove the surjectivity of the map

$$({}^{x}I \cap I) \times xI_M \to xI$$

given by $(i, m) \mapsto im\sigma(i)^{-1}$. In other words, given an element of xI, we can σ-conjugate it by an element of ${}^{x}I \cap I$ into the set xI_M.

For $n = 0, 1, 2, \ldots$, define the normal subgroup $I_n \subset I$ to be the nth principal congruence subgroup of I. More precisely, let $\mathcal{G}$ denote the Bruhat–Tits parahoric $\mathfrak{o}$-group scheme corresponding to I, so that $\mathcal{G}(\mathfrak{o}) = I$. For $n \geqslant 0$, let I_n denote the kernel of

$$\mathcal{G}(\mathfrak{o}) \twoheadrightarrow \mathcal{G}(\mathfrak{o}/\epsilon^n\mathfrak{o}).$$

Define the normal subgroups $N_n \subset N(\mathfrak{o}) \cap I$, $\overline{N}_n \subset \overline{N}(\mathfrak{o}) \cap I$ and $M_n \subset M(\mathfrak{o}) \cap I$ to be the intersections $I_n \cap N$, $I_n \cap \overline{N}$ and $I_n \cap M$, respectively. For each $n \geqslant 0$, we have the Iwahori factorization

$$I_n = M_n N_n \overline{N}_n = \overline{N}_n N_n M_n.$$

We have the relations

$$ {}^{x}N_n \subseteq N_n \quad \text{and} \quad {}^{x}\overline{N}_n \supseteq \overline{N}_n, \tag{6.1.1}$$

which follow from our assumption that $x\mathbf{a}$ is a P-alcove.

Upon conjugating by x, the decomposition $I = I_M I_N I_{\overline{N}}$ yields ${}^{x}I = {}^{x}I_M\, {}^{x}I_N\, {}^{x}I_{\overline{N}}$. By our assumptions on x, we have

$$ {}^{x}I \cap I = ({}^{x}I_M \cap I_M)\, {}^{x}I_N I_{\overline{N}}.$$

Similarly, for each $n \geqslant 0$ we have

$$ {}^{x}I_n \cap I_n = ({}^{x}M_n \cap M_n)\, {}^{x}N_n\, \overline{N}_n.$$

The next lemma is a key ingredient in the proof of Theorem 2.1.2. Here and in the remainder of this section we shall use the following notation: for $h \in G(L)$, a left superscript ^{h} stands for conjugation by h, and a left superscript ${}^{\sigma}$ means application of σ; so, in particular, for $g, h \in G(L)$ the symbol ${}^{h\sigma}g$ will stand for $h\sigma(g)h^{-1}$ and ${}^{\sigma h}g$ for $\sigma(h)\sigma(g)\sigma(h^{-1})$.

LEMMA 6.1.1. *Fix an element $m \in I_M$ and an integer $n \geqslant 0$.*

(i) *Given $i_- \in \overline{N}_n$, there exists $b_- \in \overline{N}_n$ such that ${}^{(xm)^{-1}}b_- i_- \,{}^{\sigma}b_-^{-1} \in \overline{N}_{n+1}$.*

(ii) *Given $i_+ \in N_n$, there exists $b_+ \in N_n$ such that $b_+ i_+ \,{}^{mx\sigma}b_+^{-1} \in N_{n+1}$.*

Proof. Borrowing the notation of [GHKR06, § 5.3], the group N possesses a finite separating filtration by normal subgroups,

$$N = N[1] \supset N[2] \supset \cdots ,$$

defined as follows. Choose a Borel subgroup B' that contains A and is contained in P; use B' to determine a notion of (simple) positive root for A acting on $\mathrm{Lie}(G)$. Let δ'_N be the cocharacter in $X_*(A/Z)$ (where Z denotes the center of G) which is the sum of the B'-fundamental coweights ϖ_α, where α ranges over the simple B'-positive roots for A that appear in $\mathrm{Lie}(N)$. Then let $N[i]$ be the product of the root groups $U_\beta \subset N$ for β satisfying $\langle \beta, \delta'_N \rangle \geqslant i$. The subgroups $N[i]$ are stable under conjugation by any element in M (as one can check by using the Bruhat decomposition of M with respect to the Borel subgroup $B' \cap M$). The successive quotients $N\langle i \rangle := N[i]/N[i+1]$ are abelian (see [GHKR06]).

We define $N_n[i] := N_n \cap N[i]$ and $N_n\langle i \rangle := N_n[i]/N_n[i+1]$. We define the groups $\overline{N}[i]$, $\overline{N}\langle i \rangle$, $\overline{N}_n[i]$ and $\overline{N}_n\langle i \rangle$ in an analogous manner.

Now we are ready to prove statement (i). Note that the successive quotients $\overline{N}_n\langle i \rangle$ are abelian and that, moreover, $\overline{N}_{n+1}\langle i \rangle$ is a subgroup of $\overline{N}_n\langle i \rangle$ and the quotient

$$\overline{N}_n\langle i \rangle / \overline{N}_{n+1}\langle i \rangle$$

is a vector group over the residue field of $\mathfrak{o}$. Conjugation by $m^{-1} \in I_M$ or x^{-1} preserves $\overline{N}_n$, as well as each $\overline{N}_n[i]$ and $\overline{N}_n\langle i \rangle$ (for x^{-1} we use (6.1.1) above). Hence the map $b_- \mapsto {}^{(xm)^{-1}}b_- \,{}^{\sigma}b_-^{-1}$ induces on each vector group $\overline{N}_n\langle i \rangle / \overline{N}_{n+1}\langle i \rangle$ a map like that considered in Corollary 5.1.2(1). Using that lemma repeatedly on these quotients in a suitable order, we can find $b_- \in \overline{N}_n$ such that

$${}^{(xm)^{-1}}b_- i_- \,{}^{\sigma}b_-^{-1} \in \overline{N}_{n+1},$$

thus verifying part (i).

For part (ii) we use a very similar argument. Conjugation by mx preserves N_n (for x we use (6.1.1) above), as well as each $N_n[i]$ and $N_n\langle i \rangle$. Hence the map $b_+ \mapsto b_+ \,{}^{mx\sigma}b_+^{-1}$ induces on each vector group $N_n\langle i \rangle / N_{n+1}\langle i \rangle$ a map like that considered in Corollary 5.1.2(2). We conclude as in part (i) above. This completes the proof of the lemma. □

Now we continue with the proof of Theorem 2.1.2. The Iwahori subgroup I has the filtration $I \supset I_1 \supset I_2 \supset I_3 \supset \cdots$ by principal congruence subgroups. We want to refine this filtration to a filtration $I = I[0] \supset I[1] \supset I[2] \supset I[3] \supset \cdots$ satisfying the following conditions:

(i) each $I[r]$ is normal in I;

(ii) each $I[r]$ is a semidirect product $I\langle r \rangle I[r+1]$, where $I\langle r \rangle$ is either an affine root subgroup (hence one-dimensional over our ground field k) or else contained in $A(\mathfrak{o})$.

One can construct such filtrations directly by inserting suitable terms into the filtration by principal congruence subgroups. It turns out to be much cleaner, and more useful for other portions of this paper, to take instead a generic Moy–Prasad filtration (see below for a discussion of these filtrations). In any case, we fix one such filtration (which need not have any special properties relative to our chosen $P = MN$).

We start with a P-alcove $x\mathbf{a}$ and an element $y \in xI$. We want to find an element $g \in {}^xI \cap I$ such that $gy\,\sigma(g)^{-1} \in xI_M$. As usual, we do this by successive approximations, σ-conjugating y first into $xI_MI[1]$, then into $xI_MI[2]$, and so on. We have to take care that the elements doing the σ-conjugating approach 1 as $r \to \infty$. Assuming we can do this, if $h^{(r)} \in {}^xI \cap I$ is used to σ-conjugate the appropriate element of $xI_MI[r]$ into $xI_MI[r+1]$, then the convergent product

$$g := \cdots h^{(2)}h^{(1)}h^{(0)}$$

has the desired property.

So we need to show that any element $xi_Mi[r] \in xI_MI[r]$ is σ-conjugate under ${}^xI \cap I$ to an element of $xI_MI[r+1]$ (and that the σ-conjugators can be taken to be small when r is large). Use condition (ii) to decompose $i[r]$ into $i\langle r\rangle i[r+1]$. There are two cases. If $I\langle r\rangle \subset A(\mathfrak{o})$, then we can absorb $i\langle r\rangle$ into i_M, showing that our element already lies in $xI_MI[r+1]$.

Otherwise, $i\langle r\rangle$ lies in one of the affine root subgroups of I; write α for the ordinary root obtained as the vector part of our affine root. If α is a root in M, then again we absorb $i\langle r\rangle$ into i_M and do not need to σ-conjugate. Otherwise α is a root in N or $\overline{N}$, and in either case we may use the Lang theorem variant (i.e. the appropriate statement in Lemma 6.1.1) to produce an element $h \in {}^xI \cap I$ (suitably small when r is large) such that

$$hxi_Mi\langle r\rangle\sigma(h)^{-1} = xi_Mi'$$

for some $i' \in I[r+1]$. (For example, if $i\langle r\rangle \in N_n$, take $h := {}^xb_+$ where b_+ is the element produced in Lemma 6.1.1(ii) for $m := i_M$ and $i_+ := mi\langle r\rangle m^{-1}$.) Then

$$hxi_Mi\langle r\rangle i[r+1]\sigma(h)^{-1} = xi_Mi'(\sigma(h)i[r+1]\sigma(h)^{-1}) \in xI_MI[r+1],$$

as desired. (We have used here that $I[r+1]$ is normal in I.) Lemma 6.1.1 produces elements h which are suitably small when r is large, so we are done, modulo the information on Moy–Prasad filtrations to be described next.

6.2 Moy–Prasad filtrations

Our reference for Moy–Prasad filtrations is [MP94]. Recall that Moy–Prasad filtrations on I are obtained from points x in the base alcove $\mathbf{a}$. On the Lie algebra this works as follows. The vector space $\mathfrak{g} \otimes_k k[\epsilon, \epsilon^{-1}]$ is graded by the group $X^*(A) \oplus \mathbb{Z}$ (since $\mathfrak{g}$ is graded by $X^*(A)$ and $k[\epsilon, \epsilon^{-1}]$ is graded by $\mathbb{Z}$). (For the moment, k is any field.) The pair $(x, 1)$ gives a homomorphism $X^*(A) \oplus \mathbb{Z} \to \mathbb{R}$, which we use to obtain an $\mathbb{R}$-grading on $\mathfrak{g} \otimes_k k[\epsilon, \epsilon^{-1}]$ as well as an associated $\mathbb{R}$-filtration. We also obtain an $\mathbb{R}$-filtration on the completion $\mathfrak{g}(F)$ of $\mathfrak{g} \otimes_k k[\epsilon, \epsilon^{-1}]$. Thus, for $r \in \mathbb{R}$, the subspace $\mathfrak{g}(F)_{\geqslant r}$ is the completion of the direct sum of the affine weight spaces of weight (with respect to $(x, 1)$) greater than or equal to r, which for the affine weight space $\epsilon^n\mathfrak{a}$ means that $n \geqslant r$, and for an affine weight space $\epsilon^n\mathfrak{g}_\alpha$ (α being an ordinary root) means that $\alpha(x) + n \geqslant r$. Of course, $\mathfrak{g}(F)_{\geqslant 0}$ is the Iwahori subalgebra obtained as the Lie algebra of I.[1] It is clear that $[\mathfrak{g}(F)_{\geqslant r}, \mathfrak{g}(F)_{\geqslant s}] \subset \mathfrak{g}(F)_{\geqslant r+s}$, from which it follows that $\mathfrak{g}(F)_{\geqslant r}$ is an ideal in $\mathfrak{g}(F)_{\geqslant 0}$ whenever r is non-negative.

When r is non-negative, the Moy–Prasad subgroups $G(F)_{\geqslant r}$ of $G(F)$ are by definition the subgroups generated by suitable subgroups of $A(\mathfrak{o})$ and of the various root subgroups in such

[1] We warn that this description is incompatible with the normalization of the correspondence between alcoves and Iwahori subgroups that we are using in this paper: it turns out that $G(F)_{\geqslant 0}$ is really 'opposite' to our Iwahori subgroup I. To get our I, we should instead define $\mathfrak{g}(F)_{\geqslant r}$ to be the completion of the sum of the affine weight spaces of weight (with respect to $(x, -1)$) less than or equal to $-r$.

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a way that the Lie algebra of $G(F)_{\geqslant r}$ ends up being $\mathfrak{g}(F)_{\geqslant r}$. In characteristic zero, the fact that $\mathfrak{g}(F)_{\geqslant r}$ is an ideal in $\mathfrak{g}(F)_{\geqslant 0}$ implies that $G(F)_{\geqslant r}$ is normal in $I = G(F)_{\geqslant 0}$. Moy and Prasad proved normality in the general case from other considerations. In our present situation, where G is split, it is straightforward to prove the normality using commutator relations for the various affine root groups $U_{\alpha+n}$ in $G(F)$.

What does it mean for x to be a *generic* element in the base alcove? For an arbitrary point x in the standard apartment it could accidentally happen that the homomorphism

$$(x, 1) : X^*(A) \oplus \mathbb{Z} \to \mathbb{R}$$

sends two distinct affine weights occurring in $\mathfrak{g} \otimes_k k[\epsilon, \epsilon^{-1}]$ to the same real number. If such an accident never occurs, we say that x is generic. The set of non-generic points in the standard apartment is a locally finite union of affine hyperplanes, including all the affine root hyperplanes but also those obtained by setting any difference of roots equal to an integer. In the case of SL(2), all points in the base alcove except its midpoint are generic. In general, one can at least say that the set of generic points in the base alcove is non-empty and open. When x is generic, then going down the Moy–Prasad filtration strips away affine weight spaces one by one, just as we want.

6.3 A refinement

It is clear that when ${}^xI_M = I_M$ we can do better: we can σ-conjugate any element in xI_M to x using an element of I_M. To see this, we adapt the proof of Lang's theorem to show surjectivity of the map $I_M \to I_M$ given by $h \mapsto h^{-1}\, {}^{x\sigma}h$. Indeed, I_M has a filtration by normal subgroups which are stabilized by $\mathrm{Ad}(x)$, such that our map induces on the successive quotients a finite étale surjective map (take the Moy–Prasad filtration on I_M corresponding to the barycenter of the alcove in the reduced building for $M(L)$ corresponding to I_M). Using the surjectivity just proved, given $i \in I_M$ we find an $h \in I_M$ solving the equation $xix^{-1} = h^{-1}\, {}^{x\sigma}h$. We then have $h(xi)\sigma(h)^{-1} = x$. Thus, we have proved the following proposition.

Proposition 6.3.1. *Suppose $x \in \widetilde{W}_M$ is such that there exists a semistandard parabolic subgroup $P = MN$ having the property that ${}^xI_N \subseteq I_N$, i.e. such that $x\mathbf{a}$ is a P-alcove. Then any element of xI is σ-conjugate to an element of xI_M by an element of ${}^xI \cap I$. If, moreover, ${}^xI_M = I_M$, then we may σ-conjugate any element of xI to x, using an element of ${}^xI \cap I$.*

Given an element $x \in \widetilde{W}_M$ such that ${}^xI_M = I_M$, in general there is no parabolic $P = MN$ such that ${}^xI_N \subseteq I_N$ and ${}^{x^{-1}}I_{\overline{N}} \subseteq I_{\overline{N}}$ (see also the discussion after Definition 7.2.3 below). However, when M is adapted to I in the sense of Definition 13.2.1, such P does exist, as is shown in Proposition 13.2.2.

7. Review of σ-conjugacy classes

7.1 Classification of σ-conjugacy classes

We recall the description of the set $B(G)$ of σ-conjugacy classes in $G(L)$; for details see [Kot85], [Kot97, § 5.1] and [Kot06, § 1.3]. We denote by Λ_G the quotient of $X_*(A)$ by the coroot lattice; this is the algebraic fundamental group of G. We can identify Λ_G with the group of connected components of the loop group $G(L)$. Let $\eta_G \colon G(L) \longrightarrow \Lambda_G$ be the natural surjective homomorphism constructed in [Kot97, § 7] and denoted there by ω_G; it is sometimes called the *Kottwitz homomorphism*. Analogously, we denote by Λ_M the quotient of $X_*(A)$ by the coroot lattice for M and by η_M the corresponding homomorphism.

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If $P = MN$ is a standard parabolic subgroup of G with unipotent radical N and M the unique Levi subgroup containing A, then the set Δ of simple roots for G decomposes as the disjoint union of Δ_M and Δ_N, where Δ_M is the set of simple roots of M and Δ_N is the set of those simple roots for G which occur in the Lie algebra of N. We write A_P (or A_M) for the connected component of the center of M, and we let $\mathfrak{a}_P$ denote the real vector space $X_*(A_P) \otimes \mathbb{R}$. As usual, P determines an open chamber $\mathfrak{a}_P^+$ in $\mathfrak{a}_P$ defined by

$$\mathfrak{a}_P^+ = \{v \in \mathfrak{a}_P : \langle \alpha, v\rangle > 0 \text{ for all } \alpha \in \Delta_N\}.$$

The composition $X_*(A_P) \hookrightarrow X_*(A) \twoheadrightarrow \Lambda_M$, when tensored with $\mathbb{R}$, yields a canonical isomorphism $\mathfrak{a}_P \cong \Lambda_M \otimes \mathbb{R}$. Let Λ_M^+ denote the subset of elements in Λ_M whose image under $\Lambda_M \otimes \mathbb{R} \cong \mathfrak{a}_P$ lies in $\mathfrak{a}_P^+$.

Let $\mathbb{D}$ be the diagonalizable group over F with character group $\mathbb{Q}$. As in [Kot85], an element $b \in G(L)$ determines a homomorphism $\nu_b : \mathbb{D} \to G$ over L whose $G(L)$-conjugacy class depends only on the σ-conjugacy class $[b] \in B(G)$. We can assume that this homomorphism factors through our torus A, and that the corresponding element $\overline{\nu}_b \in X_*(A)_{\mathbb{Q}}$ is dominant. Then $b \mapsto \overline{\nu}_b$ is called the *Newton map* (relative to the group G). Recall that $b \in G(L)$ is called *basic* if ν_b factors through the center $Z(G)$ of G.

We shall use some properties of the Newton map. We can identify the quotient $X_*(A)_{\mathbb{Q}}/W$ with the closed dominant chamber $X_*(A)_{\mathbb{Q}}^+$. The map

$$\begin{aligned} B(G) &\to X_*(A)_{\mathbb{Q}}^+ \times \Lambda_G, \\ b &\mapsto (\overline{\nu}_b, \eta_G(b)) \end{aligned} \tag{7.1.1}$$

is injective [Kot97, § 4.13].

The Newton map is functorial, such that we have a commutative diagram

$$\begin{tikzcd} B(M) \arrow[r] \arrow[d] & B(G) \arrow[d] \\ X_*(A)_{\mathbb{Q}}/W_M \times \Lambda_G \arrow[r] & X_*(A)_{\mathbb{Q}}/W \times \Lambda_G \end{tikzcd} \tag{7.1.2}$$

where, moreover, the vertical arrows, given by '(Newton point, Kottwitz point)', are *injections*. Indeed, the right vertical arrow is the injection (7.1.1). To show that the left vertical arrow is injective, it is enough to prove that if $b_1, b_2 \in M(L)$ have the same Newton point and the same image under η_G, then they have the same image under η_M. We may assume that $b_1, b_2 \in \widetilde{W}_M$; see Corollary 7.2.2 below. For $i = 1, 2$, write $b_i = \epsilon^{\lambda_i} w_i$ for $\lambda_i \in X_*(A)$ and $w_i \in W_M$. Let $Q^\vee$ (respectively, $Q_M^\vee$) denote the lattice generated by the coroots of G (respectively, of M) in $X_*(A)$. The equality $\eta_G(b_1) = \eta_G(b_2)$ means that $\lambda_1 - \lambda_2 \in Q^\vee$. The equality $\overline{\nu}_{b_1} = \overline{\nu}_{b_2}$ implies that $\lambda_1 - \lambda_2 \in Q_M^\vee \otimes \mathbb{R}$. It follows that $\lambda_1 - \lambda_2 \in Q_M^\vee$, and this is what we wanted to prove.

The following lemma is a direct consequence of the commutativity of the diagram above.

Lemma 7.1.1. *Let $M \subset G$ be a Levi subgroup containing A. If $[b']_M \subset [b]$ for some $b' \in M(L)$, then $\overline{\nu}_b = \overline{\nu}_{b',G\text{-dom}}$ as elements of $X_*(A)_{\mathbb{Q}}^+$.*

Here $\overline{\nu}_{b'}$ is the Newton point of b' (viewed as an element of $M(L)$), and $\overline{\nu}_{b',G\text{-dom}}$ denotes the unique G-dominant element of $X_*(A)_{\mathbb{Q}}$ in its W-orbit.

We denote by λ_M the canonical map

$$\lambda_M : \Lambda_M = X^*(Z(\widehat{M})) \to X^*(Z(\widehat{M}))_{\mathbb{R}} = X_*(Z(M))_{\mathbb{R}} \hookrightarrow X_*(A)_{\mathbb{R}}. \tag{7.1.3}$$

This can be identified with the map

$$\Lambda_M \to X_*(A_M)_{\mathbb{Q}} \hookrightarrow X_*(A)_{\mathbb{Q}},$$

where the first arrow is given by averaging the W_M-action. Next, we define the following subsets of $X_*(A)^+_{\mathbb{Q}}$: the subset $\mathcal{N}_G$ consists of all Newton points $\overline{\nu}_b$ with $b \in B(G)$, and $\mathcal{N}^+_M$ consists of the images of elements of Λ^+_M under the map λ_M. We have the equality

$$\mathcal{N}_G = \coprod_{P=MN} \mathcal{N}^+_M, \tag{7.1.4}$$

where the union ranges over all standard parabolic subgroups of G.

This equality results from two facts. First, we are taking the Newton points associated to elements of $B(G)$ and making use of the decomposition

$$B(G) = \coprod_P B(G)_P,$$

where P ranges over standard parabolic subgroups and $B(G)_P$ is the set of elements $[b] \in B(G)$ such that $\overline{\nu}_b \in \mathfrak{a}^+_P$ (see [Kot85, Kot97]); note that elements in $B(G)_P$ can be represented by basic elements in $M(L)$ (see [Kot97, (5.1.2)]). Second, for a basic element b in $M(L)$ (representing, for instance, an element in $B(G)_P$), its Newton point $\overline{\nu}_b$ is the image of $\eta_M(b) \in \Lambda_M$ under λ_M. This follows from the characterization of $\overline{\nu}_b$ in [Kot85, § 4.3] (applied to M in place of G) in conjunction with (7.1.2).

Remark 7.1.2. The right-hand side in (7.1.4) is easy to enumerate for any given group (with the aid of a computer). This fact makes feasible the computer-aided verifications of our conjectures relating to the non-emptiness of $X_x(b)$; see § 9. Moreover, the injectivity of (7.1.1), together with (7.1.4), gives a concrete way of checking whether two elements in $G(L)$ are σ-conjugate.

7.2 Construction of standard representatives for $B(G)$

Here we will define the *standard representatives* of σ-conjugacy classes in the extended affine Weyl group. First, note that the map $G(L) \to B(G)$ induces a map $\widetilde{W} \to B(G)$. Our goal is to find special elements in $\widetilde{W}$ which parametrize the elements of $B(G)$.

Denote by $\Omega_G \subset \widetilde{W}$ the subgroup of elements of length 0. Let $G(L)_{\mathrm{b}}$ and $B(G)_{\mathrm{b}}$ denote, respectively, the sets of basic elements and basic σ-conjugacy classes in $G(L)$. In the following lemma, we collect some standard facts relating the Newton map to the homomorphism $\eta_G\colon G(L) \twoheadrightarrow \Lambda_G$. The connection between these two maps stems from the fact that if $b \in G(L)$ is basic, then the Newton point $\overline{\nu}_b \in X_*(Z(G))_{\mathbb{R}}$ is the image of $\eta_G(b) \in \Lambda_G$ under the canonical map $\lambda_G\colon \Lambda_G \to X_*(A)_{\mathbb{R}}$; see (7.1.3).

Lemma 7.2.1.

(i) *The map η_G induces a bijection $B(G)_{\mathrm{b}} \xrightarrow{\sim} \Lambda_G$.*

(ii) *Elements in $\Omega_G \subset G(L)$ are basic, and the map η_G induces a bijection $\Omega_G \xrightarrow{\sim} \Lambda_G$.*

(iii) *The canonical map $\Omega_G \to B(G)_{\mathrm{b}}$ is a bijection.*

Proof. First, suppose $b \in \Omega_G$. For sufficiently divisible $N > 1$, the element b^N is a translation element which preserves the base alcove and hence belongs to $X_*(Z(G))$. The characterization of ν_b in [Kot85, § 4.3] then shows that b is basic, proving the first assertion in (ii). For statement (i), recall that an isomorphism was constructed in [Kot85, Proposition 5.6], and this was shown in [Kot97, § 7.5] to be induced by η_G. Since η_G is trivial on I and $W_{\mathrm{aff}} \subset G_{\mathrm{sc}}(L)$, statement (i) and the Bruhat–Tits decomposition

$$G(L) = \coprod_{w\tau \in W_{\mathrm{aff}} \rtimes \Omega_G} Iw\tau I$$

together imply that the composition

$$\Omega_G \longrightarrow G(L)_{\mathrm{b}} \xrightarrow{\eta_G} \Lambda_G$$

is surjective. Since this composition is easily seen to be injective, (ii) holds. Part (iii) follows from (i) and (ii). □

Here is a slightly different point of view of the lemma: the basic conjugacy classes are in bijection with Λ_G, the group of connected components of the ind-scheme $G(L)$ (or the affine flag variety), and the bijection is given by just mapping each basic σ-conjugacy class to the connected component it lies in. The key point here is that the Kottwitz homomorphism agrees with the natural map $G(L) \to \pi_0(G(L)) = \Lambda_G$; see [Kot85] and [PR08, § 5].

As a consequence of the lemma (applied to G and its standard Levi subgroups), we have the following corollary.

Corollary 7.2.2. *The map $\widetilde{W} \to B(G)$ is surjective.*

Definition 7.2.3. For $[b] \in B(G)_P \subset B(G)$, we call the representative in $\Omega_M \subseteq \widetilde{W}$ which we get from Lemma 7.2.1(iii) the *standard representative* of $[b]$. Here, 'standard' refers back to our particular choice B of Borel subgroup. If we had made a different choice of Borel subgroup containing A, we would get a different standard representative; all such representatives will be referred to as *semistandard*.

The standard representative $b = \epsilon^\nu v$ therefore satisfies:

(i) $b \in \widetilde{W}_M$, i.e. $v \in W_M$;

(ii) $bI_M b^{-1} = I_M$.

Remark 7.2.4. Let $x \in \Omega_G$ and write $x = \epsilon^\lambda w$ with $\lambda \in X_*(A)$ and $w \in W$; we call λ the *translation part* of x. Then λ is the (unique) dominant minuscule coweight whose image in Λ_G coincides with that of x. Indeed, since x preserves the base alcove $\mathbf{a}$, the transform of the origin by x, namely λ, lies in the closure of the base alcove. This is what it means to be dominant and minuscule.

Now consider *standard* (semistandard is not enough) $P = MN$ and $x \in \Omega_M$. Write $x = \epsilon^\lambda w_M$ with $\lambda \in X_*(A)$ and $w_M \in W_M$. We know that λ is M-dominant and M-minuscule. We claim that $x\mathbf{a}$ is a P-alcove if and only if λ is dominant. Indeed, $x\mathbf{a}$ is a P-alcove if and only if $xI_N x^{-1} \subset I_N$. Now $w_M I_N w_M^{-1} = I_N$, because P was assumed to be standard. So $x\mathbf{a}$ is a P-alcove if and only if $\epsilon^\lambda I_N \epsilon^{-\lambda} \subset I_N$ if and only if $\alpha(\lambda) \geqslant 0$ for all $\alpha \in R_N$ if and only if $\alpha(\lambda) \geqslant 0$ for all $\alpha > 0$.

Example 7.2.5. Let $G = \mathrm{GL}_n$, let A be the diagonal torus, and let B be the Borel group of upper triangular matrices. In this case, the Newton map is injective; see [Kot06], in particular the last paragraph of § 1.3. We can view the Newton vector ν of a σ-conjugacy class $[b]$ as a descending sequence $a_1 \geqslant \cdots \geqslant a_n$ of rational numbers satisfying an integrality condition. The standard parabolic subgroup $P = MN$ is given by the partition $n_1 + \cdots + n_r$ of n such that the a_i in corresponding batches are equal to each other while the a_i in different batches are different.

The standard representative is (represented by) a block diagonal matrix with r blocks, one for each batch of entries, where the ith block is

$$\begin{pmatrix} 0 & \epsilon^{k_i+1} I_{k'_i} \\ \epsilon^{k_i} I_{n_i-k'_i} & 0 \end{pmatrix} \in \mathrm{GL}_{n_i}(F).$$

Here, we write the entry $a_{n_1+\cdots+n_{i-1}+1}=\cdots=a_{n_1+\cdots+n_i}$ of the ith batch as $k_i+k'_i/n_i$ with $k_i, k'_i \in \mathbb{Z}$ and $0 \leqslant k'_i < n_i$, which is possible by the integrality condition, and we let I_ℓ denote the $\ell \times \ell$ unit matrix. It follows from the definitions that $k_i \geqslant k_{i+1}$ for all $i=1,\ldots,r-1$. We see that the standard representative x of $[b]$ has dominant translation part if and only if for all i with $k'_{i+1} \neq 0$ we have $k_i > k_{i+1}$. Furthermore, this is equivalent to $x\mathbf{a}$ being a P-alcove. If these conditions are satisfied, then $x\mathbf{a}$ is a *fundamental P-alcove* in the sense of Definition 13.1.2.

8. Proofs of Corollary 2.1.3(b) and Theorem 2.1.4

8.1 Assume that $P=MN$ is semistandard and $x\mathbf{a}$ is a P-alcove. There is a commutative diagram as follows.

$$\begin{array}{ccc} I_M x I_M /_\sigma I_M & \xrightarrow{\sim} & IxI /_\sigma I \\ \cong\downarrow & & \cong\downarrow \\ \coprod_{[b'] \in B(M)_x} J^M_{b'} \backslash X^M_x(b') & \longrightarrow & \coprod_{[b] \in B(G)_x} J^G_b \backslash X^G_x(b) \end{array} \tag{8.1.1}$$

Here, for $[b'] \in B(M)_x$, we choose once and for all a representative $b' \in M(L)$; for $[b] \in B(G)_x$, we also choose once and for all a representative $b \in G(L)$. If under $B(M)_x \to B(G)_x$ we have $[b'] \mapsto [b]$, then we choose once and for all $c \in G(L)$ such that $c^{-1} b \sigma(c) = b'$. In that case, our choices yield the map

$$J^M_{b'} \backslash X^M_x(b') \to J^G_b \backslash X^G_x(b),$$
$$m \mapsto cm.$$

Thus we have defined the bottom horizontal arrow.

Next, we define the right vertical arrow. Let an element of $IxI /_\sigma I$ be represented by $y \in IxI$. There is a unique $[b] \in B(G)_x$ such that $y \in [b]$. Write $y = g^{-1} b \sigma(g)$ for some $g \in G(L)$. Then the right vertical map associates to $[y]=[g^{-1} b\sigma(g)]$ the J^G_b-orbit of $gI \in X^G_x(b)$. The left vertical arrow is defined similarly. It is easy to check that both vertical arrows are bijective. It is also clear that the diagram commutes. The bijectivity of the top horizontal arrow (i.e. Corollary 2.1.3(a)) thus implies the surjectivity of the map $B(M)_x \to B(G)_x$ (in Corollary 2.1.3(b)).

We now prove that $B(M)_x \to B(G)_x$ is also injective. Given $b \in M(L)$, regard its Newton point $\overline{\nu}^M_b$ as an element in $X_*(A)^+_{\mathbb{Q}}$, which denotes here the set of M-dominant elements of $X_*(A)_{\mathbb{Q}}$. The map

$$B(M) \to X_*(A)^+_{\mathbb{Q}} \times \Lambda_M,$$
$$b \mapsto (\overline{\nu}^M_b, \eta_M(b))$$

is injective; see (7.1.1). Now suppose that $b_1, b_2 \in B(M)_x$ have the same image in $B(G)_x$. Since $\eta_M(b_1)=\eta_M(x)=\eta_M(b_2)$, by the preceding remark it is enough to show that $\overline{\nu}^M_{b_1} = \overline{\nu}^M_{b_2}$. We claim that our assumption on x forces each $\overline{\nu}^M_{b_i}$ to be not only M-dominant but also G-dominant. Indeed, b_i is σ-conjugate in $M(L)$ to an element in $I_M x I_M$, and since ${}^x(N \cap I) \subseteq N \cap I$, it follows

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that the isocrystal

$$(\mathrm{Lie}\, N(L), \mathrm{Ad}(b_i) \circ \sigma)$$

comes from a crystal (in other words, there is some $\mathfrak{o}$-lattice in Lie $N(L)$ which is carried into itself by the σ-linear map $\mathrm{Ad}(b_i) \circ \sigma$; in fact, when b_i itself lies in $I_M x I_M$, the lattice Lie $N(L) \cap I$ does the job). The slopes of any crystal are non-negative, which, in this situation, means that $\langle \alpha, \overline{\nu}_{b_i}^M \rangle \geqslant 0$ for all $\alpha \in R_N$. This proves our claim. Now, since $\overline{\nu}_{b_1}^M$ and $\overline{\nu}_{b_2}^M$ are conjugate under W (cf. (7.1.2)), they are in fact equal. This completes the proof of Corollary 2.1.3(b).

In light of the diagram (8.1.1), Theorem 2.1.4 follows from Corollary 2.1.3.

9. Consequences for affine Deligne–Lusztig varieties

9.1 In this section we present various consequences of Theorem 2.1.4, as well as some conjectures, relating to the non-emptiness and dimension of $X_x^G(b)$. Our conjectures have been corroborated by ample computer evidence, and we prove some parts of them. The computer calculations were done using the 'generalized superset method', that is, the algorithm implicit in Theorem 11.3.1. This method will be discussed in § 11.

9.2 Translation elements $x = \epsilon^\lambda$

Let us examine the non-emptiness of $X_x(b)$ in a very special case.

Corollary 9.2.1. *Suppose $x = \epsilon^\lambda$. Then $X_x(b) \neq \emptyset$ if and only if $[b] = [\epsilon^\lambda]$ in $B(G)$.*

Proof. There is a choice of Borel subgroup $B' = AU'$ such that $x\mathbf{a}$ is a B'-alcove (λ is B'-dominant for an appropriate choice of B'). Thus, by Theorem 2.1.4 with $M = A$, we see that $X_x^G(b) \neq \emptyset$ if and only if b is σ-conjugate to a translation ϵ^ν with $\nu \in X_*(A)$ and $X_x^A(\epsilon^\nu) \neq \emptyset$. But the latter inequality holds if and only if $\lambda = \nu$. □

Remark 9.2.2. As Lusztig pointed out, the corollary has a simple direct proof in the special case where G is simply connected and $b = 1$. Let $x = \epsilon^\lambda$ and suppose that λ belongs to the coroot lattice. Suppose $g^{-1}\sigma(g) \in IxI$. Since the affine flag variety is of ind-finite type, the Iwahori subgroup ${}^gI$ is fixed by σ^r for some $r > 0$. Thus, $g^{-1}\sigma^r(g) \in I$. On the other hand, $g^{-1}\sigma^r(g) \in IxI \cdots IxI$ (the product of r copies of IxI), which, since the lengths add, is just $I\epsilon^{r\lambda}I$. This intersects I only if $\lambda = 0$.

9.3 A necessary condition for the non-emptiness of $X_x(b)$

We wish to use Theorem 2.1.4 to obtain results about affine Deligne–Lusztig varieties. Clearly, whenever $X_x(b) \neq \emptyset$, x and b must lie in the same connected component of the loop group, i.e. $\eta_G(x) = \eta_G(b)$. Whenever we can use Theorem 2.1.4 to relate $X_x(b)$ to an affine Deligne–Lusztig variety for a Levi subgroup M, we will get a similar necessary condition with respect to η_M. Typically, Λ_M is much larger than Λ_G, so the condition for M will be a much stronger restriction.

However, one has to be careful here, because the intersection of $M(L)$ with the G-σ-conjugacy class $[b]$ will in general consist of several M-σ-conjugacy classes. Here is what we can say.

Proposition 9.3.1. *Fix a σ-conjugacy class $[b]$ in G with Newton vector $\overline{\nu}_b$, and fix an element $x \in \widetilde{W}$. If $X_x^G(b) \neq \emptyset$, then the following holds: if $P = MN$ is a semistandard parabolic subgroup*

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such that $x\mathbf{a}$ is a P-alcove, then $\eta_G(x)=\eta_G(b)$ and

$$\eta_M(x)\in\eta_M(W\overline{\nu}_b\cap\mathcal{N}_M), \tag{9.3.1}$$

where $\mathcal{N}_M$ denotes the image of $B(M)$ in $X_*(A)_{\mathbb{Q}}^{M\text{-dom}}$ under the Newton map.

The set $W\overline{\nu}_b\cap\mathcal{N}_M$ is the finite set of M-dominant elements of $X_*(A)_{\mathbb{Q}}$ that are W-conjugate to $\overline{\nu}_b$ and arise as the Newton point of some element of $M(L)$. See Example 9.3.2 below for a specific example. If b is basic, then the statement of Proposition 9.3.1 simplifies. We will consider the basic case in the next subsection.

Our condition (9.3.1) means that x has the same value under η_M as an element $b'\in M(L)$ with $\overline{\nu}_{b'}^M\in W\overline{\nu}_b$. By the injectivity of the left vertical arrow of (7.1.2), for a fixed $[b]$ there are only *finitely many* σ-conjugacy classes $[b']\in B(M)$ such that $\overline{\nu}_{b'}^M\in W\overline{\nu}_b$ and $\eta_G(b')=\eta_G(b)$. In particular, the condition that $\eta_M(x)=\eta_M(b')$ for some such b' is a condition we can check with a computer.

Proof. Condition (9.3.1) is a direct consequence of Theorem 2.1.4. Indeed, we know from part (a) of Theorem 2.1.4 that $[b]=[b']$ for some $b'\in M(L)$ and $X_x^M(b')\neq\emptyset$, which implies in turn that $\eta_M(x)=\eta_M(b')$. Lemma 7.1.1 then shows that $\overline{\nu}_{b'}^M\in W\overline{\nu}_b$, as desired. $\square$

Example 9.3.2. Let $G=\mathrm{SL}_3$, $P_2=B\cup Bs_2B$ and $P={}^{s_1}P_2$. As in the proposition, write $P=MN$. In terms of matrices, we have

$$M=\begin{pmatrix} * & & * \\ & * & \\ * & & * \end{pmatrix},\qquad N=\begin{pmatrix} 1 & & \\ * & 1 & * \\ & & 1 \end{pmatrix},\qquad I\cap N=\begin{pmatrix} 1 & & \\ \mathfrak{o} & 1 & \epsilon\mathfrak{o} \\ & & 1 \end{pmatrix}.$$

Assume that the Newton vector of b is $\overline{\nu}_b=(1,-1/2,-1/2)$. Then we have $W\overline{\nu}_b\cap\mathcal{N}_M=\{(-1/2,1,-1/2)\}$.

Now consider an element $x=\epsilon^\mu s_1s_2s_1\in\widetilde{W}_M$ where $\mu=(\mu_1,\mu_2,\mu_3)$, and assume that x is a P-alcove, i.e. $\mu_2-\mu_1\geqslant -1$ and $\mu_2-\mu_3\geqslant 1$. The proposition states that $X_x(b)=\emptyset$ unless $(\mu_1+\mu_3,\mu_2)=\eta_M(x)=(-1,1)$. This is equivalent to $\mu_2=1$ since, x being an element of SL_3, we have $\sum\mu_i=0$. Altogether, we find that $X_x(b)=\emptyset$ unless μ is one of the four cocharacters $(-1,1,0)$, $(0,1,-1)$, $(1,1,-2)$ or $(2,1,-3)$.

Note that Proposition 9.3.1 implies that for fixed b and a *proper* parabolic subgroup P, there are only finitely many x such that $x\mathbf{a}$ is a P-alcove and for which $X_x(b)$ can be non-empty.

Proposition 9.3.1 provides an obstruction to the non-emptiness of affine Deligne–Lusztig varieties: (9.3.1) must hold whenever $x\mathbf{a}$ is a P-alcove. In the case where $[b]$ is basic, it seems reasonable to expect that this is the only obstruction; see Conjecture 9.4.2 below. In the general case, it is clear that there are additional obstructions. If b is a translation element, then from [GHKR06, Theorem 6.3.1] we see that whenever $X_x(b)\neq\emptyset$, there exists $w\in W$ such that $x\geqslant {}^wb$ in the Bruhat order. (For general b, one can obtain a similar criterion by passing to a totally ramified extension of L where b splits.) This condition implies, in particular, that for all projections to affine Grassmannians the corresponding affine Deligne–Lusztig variety is non-empty, but it is stronger than that. Nevertheless, as the following example shows, there are still more elements x which give rise to an empty affine Deligne–Lusztig variety.

Example 9.3.3. Let $G=\mathrm{SL}_3$ and $b=\epsilon^\lambda$ with $\lambda=(2,0,-2)$. Let $x=s_{01210120120}=\epsilon^{(3,1,-4)}s_{121}$ (here we write s_{12} for s_1s_2 and so on). Then $x\geqslant b$ (a reduced expression for b is $s_{01210121}$)

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and $x\mathbf{a}$ is not a P-alcove for any proper parabolic subgroup P. However, $X_x(b)=\emptyset$. (See [Reu02, Figure 3.24], which shows the situation for this b.)

9.4 Non-emptiness of $X_x(b)$ for b basic

In this subsection, let b be basic in $G(L)$, in which case Lemma 7.1.1 and the injectivity of the left vertical arrow of (7.1.2) imply the following: if $[b]\cap M(L)\neq\emptyset$ for some semistandard Levi subgroup $M\subseteq G$, then Lemma 7.1.1 gives that $[b]\cap M(L)$ is a single σ-conjugacy class inside M with the same Newton vector as the Newton vector of $[b]$ with respect to G. (On the other hand, the standard representative of $[b]$ with respect to G is not necessarily an element of M and, in particular, is typically different from the standard representative with respect to M.)

Applying Proposition 9.3.1 to the basic case, we get the following corollary.

Corollary 9.4.1. *Let $[b]$ be basic. Suppose $P=MN$ is a semistandard parabolic subgroup such that $x\mathbf{a}$ is a P-alcove. Then $X_x(b)=\emptyset$, unless $[b]$ meets $M(L)$ and $\eta_M(x)=\eta_M(\overline{\nu}_b)$.*

Let us emphasize that $\eta_M(\overline{\nu}_b)$ is really an abbreviation; here it stands for the value under η_M for the unique σ-conjugacy class $[b']\in B(M)$ which satisfies $\eta_G(b')=\eta_G(b)$ and $\overline{\nu}_{b'}^M=\overline{\nu}_b$.

Conjecture 9.4.2. In the preceding corollary, the opposite implication holds as well. In other words, when b is basic, $X_x(b)$ is empty if and only if there exists a semistandard $P=MN$ such that $x\mathbf{a}$ is a P-alcove and $\eta_M(x)\neq\eta_M(\overline{\nu}_b)$.

This conjecture can be checked in the rank 2 cases 'by hand', and in higher-rank cases computer experiments provide further support for it. The conjecture has been confirmed for the simply connected groups (i.e. for $b=1$) of type A_3 and x of length not more than 27, of type A_4 and x of length not more than 17, and of type C_3 and x of length not more than 23; it has also been verified in several cases with b basic but different from 1.

In the remainder of this subsection we discuss some sufficient conditions for the non-emptiness of $X_x(b)$, in the case where b is basic.

Lemma 9.4.3. *Let $x=\epsilon^\lambda w\in\widetilde{W}$ be an element which is not contained in any Levi subgroup. Then*

$$X_x(b)\neq\emptyset\iff\eta_G(x)=\eta_G(b).$$

Here, by 'not contained in any Levi subgroup' we mean that no representative of x in $N_G(A)(L)$ is contained in a Levi subgroup of G associated with a proper semistandard parabolic subgroup of G. Since we consider only Levi subgroups containing the fixed maximal torus A, their (extended affine) Weyl groups are subgroups of the (extended affine) Weyl group of G. In terms of Weyl groups, we can state the condition as follows: the finite part w of x is not contained in any conjugate of a proper parabolic subgroup of W.

If w belongs to the Coxeter conjugacy class of W, then the condition is satisfied. For the symmetric groups, i.e. when G is of type A_n, the converse is also true, as one can see from using disjoint cycle decompositions. For all other types, however, there exist other conjugacy classes which do not meet any (standard) parabolic subgroup of W (see, for instance, [GP00], where these conjugacy classes are called cuspidal; some authors call them elliptic).

Before beginning the proof, we note that similar considerations can be found in [KR03, Proposition 4.1] and [Reu02, § 3.3.4].

Proof. As before, it is clear that $X_x(b) \neq \emptyset$ implies $\eta_G(x) = \eta_G(b)$. On the other hand, given the latter condition, we will show that x is itself σ-conjugate to b or, in other words, that the Newton vector of x is $\overline{\nu}_b$. Our assumption ensures that x is in the right connected component of $G(L)$, so that we only need to prove that x is basic.

In order to show that x is basic, we prove that the Newton vector of x, $\overline{\nu}_x = (1/N)\sum_{i=0}^{N-1} w^i\lambda \in X_*(A)_{\mathbb{Q}}$, is W-invariant. (Here N denotes the order of w in W.) The point $\overline{\nu}_x$ lies in (the closure of) some Weyl chamber; therefore its stabilizer is generated by a subset of the set of simple reflections for this chamber and hence is the Weyl group of some Levi subgroup (or of all of G). On the other hand, w is contained in this stabilizer, so our assumption gives us that the stabilizer of $\overline{\nu}_x$ is in fact W. □

As the proof shows, if G is semisimple, the elements $x \in \widetilde{W}$ which are not contained in any Levi subgroup have finite order in $\widetilde{W}$; cf. [GHKR06, Proposition 7.3.1].

Now let $x \in \widetilde{W}$. If x is not contained in any Levi subgroup, then we can tell whether $X_x(b) = \emptyset$ from the lemma. In general, there is a smallest semistandard Levi subgroup M_- that contains x and a smallest semistandard Levi subgroup $M_+ \supseteq M_-$ such that $x\mathbf{a}$ is a P_+-alcove for some semistandard parabolic subgroup P_+ with Levi part M_+. Both of these statements follow from [Bor91, Proposition 14.22], which says that for (semistandard) parabolic subgroups P_1 and P_2, the subgroup $(P_1 \cap P_2)R_uP_1$ is again a (semistandard) parabolic subgroup; it has Levi part $M_1 \cap M_2$. There may be more than one parabolic P_+ with Levi part M_+ for which $x\mathbf{a}$ is a P_+-alcove and, of course, we may have $M_+ = P_+ = G$.

By Theorem 2.1.4 (and assuming that $[b]$ meets M_+, because otherwise $X_x^G(b) = \emptyset$ again by Theorem 2.1.4), we have

$$X_x^G(b) \neq \emptyset \iff X_x^{M_+}(b) \neq \emptyset \implies \eta_{M_+}(x) = \eta_{M_+}(\overline{\nu}_b).$$

Further (and assuming that $[b]$ meets M_-), the lemma gives us

$$X_x^{M_-}(b) \neq \emptyset \iff \eta_{M_-}(x) = \eta_{M_-}(\overline{\nu}_b).$$

The condition $\eta_{M_-}(x) = \eta_{M_-}(\overline{\nu}_b)$ is quite restrictive, and it becomes more restrictive the smaller M_- is.

So, in terms of proving Conjecture 9.4.2, the case that remains to be considered is where x satisfies the following two conditions: (i) either $[b]$ does not meet M_- or it does and $X_x^{M_-}(b) = \emptyset$; and (ii) $[b]$ meets M_+ and $\eta_{M_+}(x) = \eta_{M_+}(\overline{\nu}_b)$. The conjecture predicts that in this case, $X_x^{M_+}(b) \neq \emptyset$.

9.5 Relationship with Reuman's conjecture

In this subsection, we will formulate a generalization of Reuman's conjecture and prove part of it as a consequence of the results obtained above. To formulate the conjecture, we consider the following maps from $\widetilde{W}$ to W. The map η_1 is just the projection from $\widetilde{W} = W \ltimes X_*(A)$ to W; it is a group homomorphism. To describe the second map, we identify W with the set of Weyl chambers. The map $\eta_2 : \widetilde{W} \to W$ keeps track of the finite Weyl chamber whose closure contains the alcove $x\mathbf{a}$. We define $\eta_2(x) = w$, where w is the unique element in W such that $w^{-1}x\mathbf{a}$ is contained in the dominant chamber (so that the identity element of $\widetilde{W}$ maps to the identity element of W).

We say that $x \in \widetilde{W}$ lies in the shrunken Weyl chambers if $k(\alpha, x\mathbf{a}) \neq k(\alpha, \mathbf{a})$ for all roots α or, equivalently, if $U_\alpha \cap {}^xI \neq U_\alpha \cap I$ for all α. For a subset T of the set S of simple reflections in W, let $W_T \subset W$ denote the subgroup generated by T. Let $\ell(w)$ denote the length of an element $w \in \widetilde{W}$. Finally, recall that we define the *defect* $\mathrm{def}_G(b)$ of an element $b \in G(L)$ to be the F-rank of G minus the F-rank of J_b (cf. [GHKR06]).

Conjecture 9.5.1.

(a) Let $[b]$ be a basic σ-conjugacy class. Suppose that $x \in \widetilde{W}$ lies in the shrunken Weyl chambers. Then $X_x(b) \neq \emptyset$ if and only if

$$\eta_G(x) = \eta_G(b) \quad \text{and} \quad \eta_2(x)^{-1}\eta_1(x)\eta_2(x) \in W \Big\backslash \bigcup_{T \subsetneq S} W_T,$$

and in this case we have

$$\dim X_x(b) = \tfrac{1}{2}(\ell(x) + \ell(\eta_2(x)^{-1}\eta_1(x)\eta_2(x)) - \mathrm{def}_G(b)).$$

(b) Let $[b]$ be an arbitrary σ-conjugacy class, and let $[b_{\mathrm{b}}]$ be the unique basic σ-conjugacy class with $\eta_G(b) = \eta_G(b_{\mathrm{b}})$. Then there exists $N_b \in \mathbb{Z}_{\geqslant 0}$ such that for all $x \in \widetilde{W}$ of length $\ell(x) \geqslant N_b$,

$$X_x(b) \neq \emptyset \iff X_x(b_{\mathrm{b}}) \neq \emptyset,$$

and in this case we have

$$\dim X_x(b) = \dim X_x(b_{\mathrm{b}}) - \tfrac{1}{2}(\langle 2\rho, \nu\rangle + \mathrm{def}_G(b) - \mathrm{def}_G(b_{\mathrm{b}})),$$

where ν denotes the Newton point of b.

Part (b) of this conjecture generalizes [GHKR06, Conjecture 7.5.1]. It fits well with [Bea09, Conjecture 1.0.1] and the qualitative picture of $B(G)_x$ suggested by Beazley's results on SL(3). The term $\langle 2\rho, \nu\rangle$ appearing here can also be interpreted (see § 13) as the length of a suitable semistandard representative of $[b]$ in $\widetilde{W}$.

Using the algorithms discussed in [GHKR06] and in this article, we have obtained ample numerical evidence in support of this conjecture. We performed computations for root systems of types A_2, A_3, A_4, C_2, C_3 and G_2, for a number of choices of b, including cases where b is split, basic or neither of the two, as well as for both of the cases $\eta_G(b) = 0$ and $\eta_G(b) \neq 0$.

The following remark shows that this conjecture is compatible with what we already know about affine Deligne–Lusztig varieties in the affine Grassmannian (see [GHKR06, Vie08]).

Remark 9.5.2. Conjecture 9.5.1 implies Rapoport's dimension formula for affine Deligne–Lusztig varieties $X_\mu(b)$ in the affine Grassmannian when b is basic (and $\mu \in X_*(A)$ dominant). Indeed, if $w_0 \in W$ is the longest element, then we have

$$\dim X_\mu(b) + \ell(w_0) = \sup\{\dim X_x(b) : x \in W\epsilon^\mu W\}.$$

Now, for the longest element $x \in W\epsilon^\mu W$ we have $\eta_1(x) = \eta_2(x) = w_0$, so

$$\eta_2(x)^{-1}\eta_1(x)\eta_2(x) = w_0 \in W \Big\backslash \bigcup_{T \subsetneq S} W_T.$$

Further, by the dimension formula given in the conjecture, the supremum above is equal to

$$\tfrac{1}{2}(\sup\{\ell(x) : x \in W\epsilon^\mu W\} + \ell(w_0) - \mathrm{def}_G(b)).$$

Let X^μ denote the $G(\mathfrak{o})$-orbit of $\epsilon^\mu G(\mathfrak{o})$ in the affine Grassmannian. Since

$$\sup\{\ell(x) : x \in W\epsilon^\mu W\} = \dim X^\mu + \ell(w_0) = \langle 2\rho, \mu\rangle + \ell(w_0),$$

overall we obtain

$$\dim X_\mu(b) = \langle \rho, \mu \rangle - \tfrac{1}{2}\, \mathrm{def}_G(b),$$

which is the desired result.

Let us relate this conjecture to the results of the previous subsection. The relationship relies on the following lemma (which also follows easily from Proposition 3.1.1).

LEMMA 9.5.3. *Let $x \in \widetilde{W}$, and write $w = \eta_2(x) \in W$.*

(a) *If $P = MN \supset {}^wB$ is a parabolic subgroup with $x \in \widetilde{W}_M$, then $x\mathbf{a}$ is a P-alcove.*
(b) *If x is an element of the shrunken Weyl chambers which is a P-alcove for a semistandard parabolic subgroup P, then $P \supset {}^wB$.*

Proof. First, note that by assumption $w^{-1}x\mathbf{a}$ lies in the dominant chamber. This means precisely that ${}^{w^{-1}x}I \cap U \subseteq I \cap U$ (where U denotes the unipotent radical of our Borel subgroup B), so we obtain

$${}^xI \cap N \subseteq {}^xI \cap {}^wU \subseteq {}^w(I \cap U) \subseteq I.$$

This inclusion is what we needed to show for part (a).

Now let us prove (b). Assume that $x\mathbf{a}$ is a P-alcove and write $P = MN$ for the Levi decomposition of P. We need to show that $N \subseteq {}^wU$. Let $\alpha \in R_N$. Then we have

$${}^xI \cap U_\alpha \subsetneq I \cap U_\alpha.$$

(We get $\subsetneq$ rather than just $\subseteq$ because x is in the shrunken Weyl chambers.) This implies, however, that

$${}^xI \cap U_{-\alpha} \supsetneq I \cap U_{-\alpha}.$$

On the other hand, by what we have seen above,

$${}^xI \cap {}^wU \subseteq {}^wI \cap {}^wU \subseteq {}^wU(\epsilon\mathfrak{o}).$$

This shows that $U_{-\alpha} \not\subseteq {}^wU$ and hence $U_\alpha \subseteq {}^wU$, as desired. □

From this lemma, we obtain the following strengthening of the 'only if' direction of Conjecture 9.5.1(a).

PROPOSITION 9.5.4. *Assume that the Dynkin diagram of G is connected. Let b be basic. Let $x \in \widetilde{W}$, and write $x = \epsilon^\lambda v$ with $v \in W$. Assume that $\lambda \neq \overline{\nu}_b$ and $\eta_2(x)^{-1}\eta_1(x)\eta_2(x) \in \bigcup_{T \subsetneq S} W_T$. Then $X_x(b) = \emptyset$.*

Proof. Write $w := \eta_2(x) \in W$. By the previous lemma and our hypothesis, $x\mathbf{a}$ is a P-alcove for a proper parabolic subgroup $P = MN \supset {}^wB$ of G. The only thing we need to check in order to apply Corollary 9.4.1 is that $\eta_{M'}({}^{w^{-1}}x) \neq \eta_{M'}(\overline{\nu}_b)$, where $M' = {}^{w^{-1}}M$. (Recall that the precise meaning of $\eta_{M'}(\overline{\nu}_b)$ is described after Corollary 9.4.1.) But if we had equality here, then $w^{-1}\lambda - \overline{\nu}_b$ would be a linear combination of coroots of M'. On the other hand, $w^{-1}\lambda$ is dominant, and since M' is the Levi component of a proper standard parabolic subgroup, we obtain $\lambda = \overline{\nu}_b$, which is excluded by assumption. □

Why does this imply the 'only if' direction of Conjecture 9.5.1(a)? We need to show that $X_x^G(b) = \emptyset$ if $x\mathbf{a}$ is shrunken and $\eta_2(x)^{-1}\eta_1(x)\eta_2(x)$ belongs to a proper parabolic subgroup of W. Let G_i denote a simple factor of G_{ad}, and let x_i and b_i denote, respectively, the images of x and b in G_i. Choose i such that $\eta_2(x_i)^{-1}\eta_1(x_i)\eta_2(x_i)$ belongs to a proper parabolic subgroup of the

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Weyl group of G_i. It is enough to prove that $X_{x_i}^{G_i}(b_i) = \emptyset$, since this obviously implies $X_x^G(b) = \emptyset$. Therefore we can and will assume, from now on, that $G = G_i$, so that the Dynkin diagram of G is connected. Now write $x = \epsilon^\lambda v$. We claim that if $x\mathbf{a}$ belongs to the shrunken Weyl chambers and $\eta_2(x)^{-1}\eta_1(x)\eta_2(x)$ belongs to a proper parabolic subgroup of W, then $\lambda \neq \overline{\nu}_b$. Suppose instead that $\lambda = \overline{\nu}_b$. Then ϵ^λ belongs to the center of G and $x\mathbf{a} = v\mathbf{a}$. This alcove belongs to the shrunken Weyl chambers only if $\eta_1(x) = v = w_0$. But in that case $\eta_2(x)^{-1}\eta_1(x)\eta_2(x)$ cannot belong to a proper parabolic subgroup of W. This proves our claim, and we may then apply Proposition 9.5.4 to conclude that $X_x^G(b) = \emptyset$.

We conclude this subsection by showing that our Conjecture 9.4.2 implies the validity of the 'if' direction of Conjecture 9.5.1(a).

Proposition 9.5.5. *Assume that Conjecture 9.4.2 holds. Let $x \in \widetilde{W}$ be an element of the shrunken Weyl chambers with $\eta_G(x) = \eta_G(b)$ and*

$$\eta_2(x)^{-1}\eta_1(x)\eta_2(x) \in W \Big\backslash \bigcup_{T \subsetneq S} W_T.$$

Then $X_x(b) \neq \emptyset$.

Proof. It is enough to show that $x\mathbf{a}$ is not a P-alcove for any *proper* parabolic subgroup $P = MN \subset G$. If it were, then by the lemma above we would have $P \supset {}^{\eta_2(x)}B$. But the assumption says precisely that x does not lie in $\widetilde{W}_M$ for such P. □

10. Dimension theory for the groups $I_M N$

10.1 In this section we lay some conceptual foundations for studying the dimensions of affine Deligne–Lusztig varieties $X_x(b)$, where $[b] \in B(G)$ is an arbitrary σ-conjugacy class. These foundations play a key role in the sections that follow.

Let us insert here a remark concerning the notion of dimension: using the usual definition of (Krull) dimension as the supremum of the lengths of chains of irreducible closed subsets, we can speak of the dimension of $X_x(b)$ without knowing anything about these subsets; note, though, that we do know that they are schemes, locally of finite type, over $\overline{k}$ (see [HV08, Corollary 5.5]) and that they are finite-dimensional (as follows from the corresponding result for affine Deligne–Lusztig varieties in the affine Grassmannian). In the proof below, however, it is of crucial importance to work with the inverse image of $X_x(b)$ in $G(L)$ and to assign a 'dimension' to this inverse image as well as to more general ('ind-admissible') subsets of $G(L)$.

In the case where $b = \epsilon^\nu$ for some $\nu \in X_*(A)$, a similar study was carried out in [GHKR06, § 6]. The result was a finite algorithm for computing dimensions (a special case of our Theorem 11.3.1 below). In this section, we introduce a suitable framework of ind-admissible sets and a notion of their dimension that works for general elements b.

Let J be an Iwahori subgroup which is the fixer of an alcove in the standard apartment, and let $P = MN \supset A$ be any parabolic subgroup of G. Let $J_P = J_M N$ (where $J_M := J \cap M$). We will define the ind-admissible subsets of J_P and then establish a 'dimension theory' for them, along the lines of the theory in [GHKR06]. The groups J_P 'interpolate' between the extreme cases I and $A(\mathfrak{o})U(L)$, and, as we shall see, they are precisely adapted to the study of affine Deligne–Lusztig varieties for elements b that are more general than the extreme cases of $b = 1$ or b being a translation element.

Fix any semistandard Borel subgroup contained in P and use it to define the sets of simple roots Δ_M and Δ_N. Fix a coweight λ_0 with $\langle\alpha, \lambda_0\rangle = 0$ for $\alpha \in \Delta_M$ and $\langle\alpha, \lambda_0\rangle > 0$ for $\alpha \in \Delta_N$, and consider the subgroups

$$N(m) := \epsilon^{m\lambda_0}(N \cap J)\epsilon^{-m\lambda_0}, \quad m \in \mathbb{Z}.$$

Compared with [GHKR06, § 5.2], our choice of λ_0 is a little different, but this clearly does not affect the validity of the dimension theory for N as in [GHKR06]. Furthermore, we choose a separated descending filtration $(J_M(m))_{m\in\mathbb{Z}}$ of J_M by normal subgroups, such that $J_M(m) = J_M$ for $m \leqslant 0$ and all the quotients $J_M(m)/J_M(m')$ are finite-dimensional over $\overline{k}$. (For example, we could use a Moy–Prasad filtration.) Finally, we set $J_P(m) := J_M(m)N(m)$ and obtain a separated and exhaustive filtration

$$J_P \supset \cdots \supset J_P(-1) \supset J_P(0) \supset J_P(1) \supset J_P(2) \supset \cdots .$$

The quotients $J_P(m)/J_P(m')$, $m \leqslant m'$, are finite-dimensional varieties over k in a natural way (more precisely, they coincide in a natural way with the set of $\overline{k}$-valued points of a k-variety). Since J_M normalizes each $N(m)$, each $J_P(m)/J_P(m')$ is a fiber bundle over $J_M(m)/J_M(m')$ with fibers $N(m)/N(m')$. We say that a subset $Y \subseteq J_P$ is *admissible* if there are m and m' with $m \leqslant m'$ such that Y is contained in $J_P(m)$ and is the full inverse image under the projection $J_P(m) \to J_P(m)/J_P(m')$ of a locally closed subset of $J_P(m)/J_P(m')$. We say that $Y \subseteq J_P$ is *ind-admissible* if for all m, $Y \cap J_P(m)$ is an admissible subset of J_P. Obviously, admissible subsets are, in particular, ind-admissible.

As in [GHKR06], for an admissible subset $Y \subset J_P(m)$ we can define a notion of dimension,

$$\dim Y := \dim(Y/J_P(m')) - \dim(J_P(0)/J_P(m')),$$

for suitable $m' \geqslant 0$; note that $\dim Y$ is always an element of $\mathbb{Z}$, unless Y is empty. For an ind-admissible subset $Y \subset J_P$, we define

$$\dim Y := \sup\{\dim(Y \cap J_P(-m)) : m \geqslant 0\}.$$

We may sometimes have $\dim Y = +\infty$ (for example, when $Y = J_P$). Of course, in making these definitions, we have made a choice in normalizing things so that $\dim(J_P(0)) = 0$. But, as before, differences

$$\dim Y_1 - \dim Y_2$$

for admissible subsets Y_1 and Y_2 are independent of any such choice.

11. The generalized superset method

11.1 Recall that in [GHKR06, Theorem 6.3.1], the dimension of $X_x(\epsilon^\nu)$ was expressed in terms of the dimensions of intersections of ${}^wU(L)$- and I-orbits in $G(L)/I$ (for $w \in W$). Such intersections can be understood in terms of foldings in the Bruhat–Tits building of $G(L)$ (see [GHKR06, § 6.1]), and in this way we derived an algorithm for computing $\dim X_x(\epsilon^\nu)$. The algorithm led to and supported our conjectures in [GHKR06].

In this section we explain the *generalized superset method*, which extends the above technique from translation elements $b = \epsilon^\nu$ to general b. Correspondingly, it provides the data for the dimensions in the general case, and is of independent interest because it shows that the emptiness patterns coincide in the p-adic and function field cases (see Corollary 11.3.5). The generalized superset method involves intersections of ${}^wI_P$- and I-orbits (for $w \in W$). Such intersections can also be interpreted combinatorially in terms of foldings in the building. For this we need to consider a new notion of retraction that is adapted to I_P-orbits rather than $U(L)$-orbits. We will start with a discussion of these new retractions.

11.2 The retractions ρ_P

Fix a standard parabolic subgroup $P = MN$. Write $I_P = I_M N = (I \cap M(L))N(L)$.

Lemma 11.2.1. *Let $w \in \widetilde{W}$ and $J_P = {}^{w^{-1}}I_P$. The projection $N_G A(L) \to J_P\backslash G(L)/I$ induces a bijection*

$$\widetilde{W} \cong J_P\backslash G(L)/I.$$

Proof. Because we can conjugate the situation by w^{-1}, we may as well assume that $w = 1$. Since the set $P\backslash G(L)/K$ has only one element, we can identify the double quotient $P\backslash G(L)/I$ with $W_M\backslash W \cong \widetilde{W}_M\backslash\widetilde{W}$. We obtain the following commutative diagram.

$$\begin{tikzcd} \widetilde{W} \arrow[r] \arrow[d, "q"] & I_P\backslash G(L)/I \arrow[d, "p"] \\ \widetilde{W}_M\backslash\widetilde{W} \arrow[r, "\cong"] & P\backslash G(L)/I \end{tikzcd}$$

Now, for $v \in \widetilde{W}$, we have

$$q^{-1}(\widetilde{W}_M v) = \widetilde{W}_M v \cong I_M\backslash M/({}^vI)_M \cong I_P\backslash P/({}^vI \cap P) \cong p^{-1}(PvI).$$

This proves the lemma. □

Denote by ${}^MW$ the set of minimal-length representatives in W of the cosets in $W_M\backslash W$.

Lemma 11.2.2. *Let $\lambda \in X_*(A)$ be such that $\langle\alpha, \lambda\rangle = 0$ for all roots α in M, and let $v \in {}^MW$.*

(i) *All elements of I_M fix the alcove $\epsilon^\lambda v\mathbf{a}$.*
(ii) *If $n \in N$ and λ satisfies $\epsilon^{-\lambda}n\epsilon^\lambda \in {}^vI \cap N$ (which is true whenever λ is sufficiently antidominant with respect to the roots in* Lie N*), then n fixes the alcove $\epsilon^\lambda v\mathbf{a}$.*

Proof. To prove (i), we first note that $({}^vI)_M = I_M$ because v is the minimal-length representative in its W_M-coset. This shows that

$$I_M = {}^{\epsilon^\lambda v}(I \cap {}^{v^{-1}}M) \subseteq {}^{\epsilon^\lambda v}I.$$

Similarly, under the assumption on n made in (ii), we obtain that $n \in {}^{\epsilon^\lambda v}I$. □

Denote by $\mathcal{A}$ the standard apartment of G with respect to our fixed torus A. Let ρ_P be the retraction from the Bruhat–Tits building of $G(L)$ to $\mathcal{A}$, defined as follows. For each alcove $\mathbf{b}$ in the building, all retractions of $\mathbf{b}$ with respect to an alcove of the form $\epsilon^\lambda v\mathbf{a}$, with λ and v as in part (ii) of the lemma, have the same image, say $\mathbf{c}$. Here we must stipulate that λ be sufficiently antidominant (depending on $\mathbf{b}$) with respect to the roots in Lie N. We set

$$\rho_P(\mathbf{b}) = \mathbf{c}.$$

(In fact, we get the same retraction if we retract with respect to any alcove which lies between the root hyperplanes H_α and $H_{\alpha,1}$ for all roots α of M and is sufficiently antidominant for all roots of G lying in N. Compare this also with Rousseau's notion of *cheminée*; see [Rou09, § 9].)

Lemma 11.2.3. *For $g \in I_P$, we have $\rho_P|_{g\mathcal{A}} = g^{-1}$.*

Proof. Clearly, g^{-1} maps $g\mathcal{A}$ to $\mathcal{A}$, and g^{-1} fixes the alcoves $t_\lambda v\mathbf{a}$ for λ sufficiently antidominant. This implies the lemma. □

The group $G(L)$ acts transitively on the set of extended alcoves, and the stabilizer of the base alcove is the Iwahori subgroup I. Therefore we can identify the quotient $G(L)/I$ with the set of extended alcoves.

Proposition 11.2.4. *Let $y \in \widetilde{W}$.*

(i) *We have*

$$I_P yI/I = \rho_P^{-1}(y\mathbf{a});$$

in other words, we can identify ρ_P (as a map from the set of alcoves in the building to the set of alcoves in the standard apartment) with the map $G(L)/I \to I_P\backslash G(L)/I \cong \widetilde{W}$ obtained from Lemma 11.2.1.

(ii) *More generally, let $w \in \widetilde{W}$ and $J_P = {}^{w^{-1}}I_P$. Consider the map*

$$\rho_{P,w} : G(L)/I \to \widetilde{W}, \quad g \mapsto w^{-1}\rho_P(wg).$$

Then

$$J_P yI = \rho_{P,w}^{-1}(y\mathbf{a}).$$

Proof. Part (i) follows from the previous lemma (cf. [BT72, Remarque 7.4.22], which deals with the case where $P = G$). To prove part (ii), combine part (i) with the following commutative diagram.

$$\begin{tikzcd}
G(L) \arrow[r, "\mathrm{proj}"] \arrow[rr, bend left, "\rho_P"] \arrow[d, "w^{-1}\cdot_"'] & I_P\backslash G(L)/I \arrow[r, "\cong"] \arrow[d, "w^{-1}\cdot_"'] & \widetilde{W} \arrow[d, "w^{-1}\cdot_"'] \\
G(L) \arrow[r, "\mathrm{proj}"] & J_P\backslash G(L)/I \arrow[r, "\cong"] & \widetilde{W}
\end{tikzcd}$$

□

In the extreme cases, we get the following: if $P = G$, then ρ_G is just the usual retraction $\rho_{\mathbf{a}}$ with respect to the base alcove; if $P = B$, then we get as ρ_B the retraction with respect to 'a point at infinity in the B-antidominant chamber'. Note that the maps $\rho_{P,w}$ are retractions to the standard apartment, just like the ρ_P but with a different choice of base alcove.

11.3 An algorithm for computing dim $X_x(b)$

In this subsection, we give a formula for the dimensions

$$\dim X_x(b) \cap I_P w\mathbf{a}$$

for any $w \in \widetilde{W}$. The method should be seen as an interpolation between the case where b is a translation element and the case where $b = 1$. See Example 11.3.6, where we discuss how these extreme cases fit into the framework used here.

Let $[b] \in B(G)_P$. From the dimensions $\dim X_x(b) \cap I_P w\mathbf{a}$ we get the dimension of $X_x(b)$, because

$$\dim X_x(b) = \sup_{w\in\widetilde{W}} \dim(X_x(b) \cap I_P w\mathbf{a}). \tag{11.3.1}$$

To show this, observe that

$$\dim X_x(b) = \sup_{v\in\widetilde{W}} \dim(X_x(b)\cap \overline{Iv\mathbf{a}}),$$

where an overbar indicates closure. Now, every $\overline{Iv\mathbf{a}}$ is contained in a finite union of I_P-orbits; in fact,

$$\overline{Iv\mathbf{a}} \subseteq \bigcup_{w\in S_v} I_P w\mathbf{a}$$

where $S_v := \{w\in\widetilde{W} : w\leqslant v\}$. Thus

$$\dim(X_x(b)\cap\overline{Iv\mathbf{a}}) = \sup_{w\in S_v}\dim(X_x(b)\cap\overline{Iv\mathbf{a}}\cap I_Pw\mathbf{a}) \leqslant \sup_{w\in\widetilde{W}}\dim(X_x(b)\cap I_Pw\mathbf{a}),$$

which shows that $\leqslant$ holds in (11.3.1). Since the inequality $\geqslant$ is obviously true, the desired equality follows. Also note that we know a priori that $\dim X_x(b)$ is finite, for example by invoking the finite-dimensionality of affine Deligne–Lusztig varieties in the affine Grassmannian established in [GHKR06, Vie06].

Our result in Theorem 11.3.1 is not a 'closed formula' even for fixed w, because it involves the dimensions of intersections of I- and ${}^{w^{-1}}I_P$-orbits. However, these dimensions can be computed (at least with the aid of a computer) for fixed w. (Here we make use of the interpretation of I_P-orbits in terms of 'foldings'; see Proposition 11.2.4.)

Throughout this subsection we fix a σ-conjugacy class, say $[b]\in B(G)_P\subset B(G)$, and let M denote the Levi component of a standard parabolic subgroup $P=MN$. Denote by $b\in\widetilde{W}_M$ the standard representative of $[b]$ (see Definition 7.2.3). Write $I_P=I_MN$. Then we have $bI_Pb^{-1}=I_P$. Denote by $\nu\in X_*(A)_{\mathbb{Q}}$ the Newton vector for b (where b is considered as an element of $M(L)$). Since b is M-basic, ν is 'central in M' (and, in particular, M-dominant). Let ν_{dom} denote the unique G-dominant element in the W-orbit of ν.

For any $y\in\widetilde{W}$, write $\mathbf{a}_y := y\mathbf{a}$. Let $\rho\in X^*(A)_{\mathbb{Q}}$ denote the half-sum of the positive roots of A in G.

Theorem 11.3.1. *Let $w\in\widetilde{W}$. Then, writing $\tilde{b}=w^{-1}bw$ and denoting by ν the Newton vector of b, we have*

$$\dim(X_x(b)\cap I_Pw\mathbf{a}) = \dim(I\mathbf{a}_x\cap {}^{w^{-1}}I_P\mathbf{a}_{\tilde{b}}) - \langle\rho,\nu+\nu_{\mathrm{dom}}\rangle.$$

Proof. Fix a representative of w in $N_GA(L)$ fixed by σ, and again denote it by w. Then multiplication by w^{-1} defines a bijection

$$X_x(b)\cap I_P\mathbf{a}_w \cong X_x(w^{-1}bw)\cap {}^{w^{-1}}I_P\mathbf{a},$$

which preserves the dimensions. Note that ${}^{w^{-1}}I_P := {}^{w^{-1}}(I_P)$ here.

We write $\tilde{b}=w^{-1}bw$ and consider the map

$$\begin{aligned} f_{\tilde{b}} : {}^{w^{-1}}I_P &\to {}^{w^{-1}}I_P,\\ g &\mapsto g^{-1}\tilde{b}\sigma(g)\tilde{b}^{-1}. \end{aligned}$$

Let

$$\widetilde{X_x(\tilde{b})} = \{g\in G(L) : g^{-1}\tilde{b}\sigma g\in IxI\}.$$

Then $\widetilde{X_x(\tilde{b})}\cap {}^{w^{-1}}I_P = f_{\tilde{b}}^{-1}(IxI\tilde{b}^{-1}\cap {}^{w^{-1}}I_P)$, and so

$$X_x(\tilde{b})\cap {}^{w^{-1}}I_P\mathbf{a} = f_{\tilde{b}}^{-1}(IxI\tilde{b}^{-1}\cap {}^{w^{-1}}I_P)/(I\cap {}^{w^{-1}}I_P).$$

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Lemma 11.3.2. *We have the equality*

$$\dim f_{\tilde{b}}^{-1}(IxI\tilde{b}^{-1} \cap {}^{w^{-1}}I_P) - \dim(IxI\tilde{b}^{-1} \cap {}^{w^{-1}}I_P) = \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle.$$

Proof of lemma. To ease the notation, let us write $J_P := {}^{w^{-1}}(I_P) = ({}^{w^{-1}}I)_{w^{-1}P}$ and $J_M := ({}^{w^{-1}}I)_{{}^wM}$. It is easy to see that $IxI\tilde{b}^{-1} \cap J_P$ is an admissible subset of J_P. It will follow from our arguments below that its preimage under $f_{\tilde{b}}$ is ind-admissible, so that we can define the dimensions of these subsets using the theory from § 10. The left-hand side of the equality is therefore well-defined. We can even make a very convenient choice of filtration on J_M, one which is stable under $\mathrm{Ad}(\tilde{b})$: take the Moy–Prasad filtration $J_M(\bullet)$ on J_M associated to the barycenter of the alcove in the reduced building of $M(L)$ which corresponds to J_M. □

A straightforward calculation shows that we can write the map $f_{\tilde{b}}$ as follows (here $i \in J_M$ and $n \in {}^{w^{-1}}N$):

$$g = in \mapsto g^{-1}\tilde{b}\sigma(g)\tilde{b}^{-1} = i^{-1}\, {}^{\tilde{b}}\sigma(i) \cdot {}^{\tilde{i}}n^{-1}\, {}^{\tilde{b}}\sigma(n)$$

with $\tilde{i} := {}^{\tilde{b}}\sigma(i)^{-1}i$.

The projection $J_P \to J_M$ is an 'ind-admissible fiber bundle' in a sense which the reader will have no trouble making precise (see § 10). The above description of $f_{\tilde{b}}$ indicates how it behaves on the base and on the fibers. Let us analyze the relative dimension of $f_{\tilde{b}}$ by studying the base and the fibers in turn.

First, we consider the base J_M. Since $\tilde{b}$ normalizes J_M, the map $J_M \to J_M$, $i \mapsto i^{-1}\, {}^{\tilde{b}}\sigma(i)$ is surjective and has relative dimension zero. The proof is an adaptation of the proof of Lang's theorem. Indeed, J_M has a filtration by normal subgroups (the $J_M(m)$ for $m \geqslant 0$ in the Moy–Prasad filtration described above) which are stabilized by $\mathrm{Ad}(\tilde{b})$, such that on the finite-dimensional quotients our map $J_M \to J_M$ induces a Lang map, which is finite étale and surjective.

Second, we study the relative dimension of $f_{\tilde{b}}$ 'on the fibers' of $J_P \to J_M$; that is, we fix $\tilde{i} \in J_M$ as above and study the fibers of the map ${}^{w^{-1}}N(L) \to {}^{w^{-1}}N(L)$ given by $n \mapsto {}^{\tilde{i}}n^{-1}\, {}^{\tilde{b}}\sigma(n)$. Fortunately, most of the necessary work was already done in [GHKR06, Proposition 5.3.2]. In fact, that proposition implies that the fiber dimension is (in the notation of [GHKR06])

$$d(\tilde{i}, \tilde{b}) := d(\mathfrak{n}(L), \mathrm{Ad}_{\mathfrak{n}}(\tilde{i})^{-1}\, \mathrm{Ad}_{\mathfrak{n}}(\tilde{b})\sigma) + \mathrm{val}\det \mathrm{Ad}_{\mathfrak{n}}(\tilde{i}),$$

where $\mathfrak{n}$ denotes the Lie algebra of ${}^{w^{-1}}N$. Since $\tilde{i} \in J_M$, the second summand vanishes. Moreover, $\mathrm{Ad}_{\mathfrak{n}}(\tilde{i})^{-1}\, \mathrm{Ad}_{\mathfrak{n}}(\tilde{b}) = \mathrm{Ad}_{\mathfrak{n}}(i^{-1}\tilde{b}\sigma(i))$. Since σ-conjugation induces an isomorphism of F-spaces, we obtain

$$d(\tilde{i}, \tilde{b}) = d(1, \tilde{b}) = \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle;$$

cf. [GHKR06, Proposition 5.3.1].

It is clear that we should be able to put these two pieces of information together (and obtain the stated result that the relative dimension of $f_{\tilde{b}}$ is $\langle \rho, \nu - \nu_{\mathrm{dom}} \rangle$) by looking at the corresponding finite-dimensional situation. However, to make this vague idea convincing, it seems easiest to follow the argument of [GHKR06, Proposition 5.3.2]. First, we correct for the inconvenient fact that $f_{\tilde{b}}$ need not preserve $J_P(0)$. Let $P' := {}^{w^{-1}}P$, $M' := {}^{w^{-1}}M$, $N' := {}^{w^{-1}}N$ and $I' := {}^{w^{-1}}I$. For any $m_1, m_2 \in M'(L)$ which normalize $J_P = I'_{P'}$, define

$$\begin{aligned} f_{m_1,m_2} \colon J_P &\to J_P, \\ g &\mapsto m_1 g^{-1} m_1^{-1} \cdot m_2 \sigma(g) m_2^{-1}. \end{aligned}$$

Note that $f_{\tilde{b}} = f_{1,\tilde{b}}$. Fix $\lambda_0 \in X_*(Z(M'))$ such that $\langle \alpha, \lambda_0 \rangle > 0$ for all $\alpha \in R_{N'}$. Then we may replace $f_{\tilde{b}} = f_{1,\tilde{b}}$ with $f := f_{\epsilon^{t\lambda_0}, \epsilon^{t\lambda_0}\tilde{b}}$ for a suitably large integer t, chosen such that f preserves $J_P(0) = I'_{M'} \cdot N' \cap I'$. Note that f then automatically preserves $J_P(m)$ for each integer $m \geqslant 0$ (we will not need this fact). Denote by $f_0 : J_P(0) \to J_P(0)$ the restriction of f to $J_P(0)$. As in [GHKR06, Proposition 5.3.2], our goal is now to prove the following claim.

Claim. Let $m_1 = \epsilon^{t\lambda_0}$ and $m_2 = \epsilon^{t\lambda_0}\tilde{b}$, and set $f := f_{m_1,m_2}$. If $Y \subset J_P$ is admissible, then $f^{-1}Y$ is ind-admissible and

$$\dim f^{-1}Y - \dim Y = d(m_1, m_2).$$

Continuing to follow the strategy of the proof of [GHKR06, Proposition 5.3.2], we can use [GHKR06, proof of Proposition 5.3.2, Claim 1] to find an $a := \epsilon^{t_1\lambda_0}$ for a large integer t_1 such that

$$c_a J_P(0) \subseteq f J_P(0),$$

where c_a denotes the conjugation map $g \mapsto aga^{-1}$ for $g \in J_P$. Fix this element a once and for all. We prove the following subclaim next.

Subclaim. Suppose that Y is an admissible subset of $c_a J_P(0)$. Then $f_0^{-1}(Y)$ is admissible and

$$\dim f_0^{-1}Y - \dim Y = d(m_1, m_2).$$

Proof of subclaim. At this point, we have to replace the filtration $\{J_P(m)\}_{m\geqslant 0}$ of $J_P(0)$ with one that is better behaved with respect to the morphism f_0. So, for $m \geqslant 0$ let $I'_m \subset I'$ denote the mth principal congruence subgroup of the Iwahori subgroup I'; by convention, $I'_0 = I'$. Let $J_{M,m} := I'_m \cap M'$ and $N'_m := I'_m \cap N'$. Let $J_{P,m} = J_{M,m}N'_m = I'_m \cap P'$. It is clear that J_M normalizes each N'_m, so for each $0 \leqslant m_1 \leqslant m_2$ we have a fiber bundle

$$\pi : J_{P,m_1}/J_{P,m_2} \to J_{M,m_1}/J_{M,m_2}$$

with fiber N_{m_1}/N_{m_2}. Also, using our specific choices of m_1 and m_2 above, it is clear that f_0 preserves $J_{P,m}$ and that, in fact, f_0 induces a well-defined map on the quotients

$$\overline{f} : J_{P,0}/J_{P,m} \to J_{P,0}/J_{P,m}$$

for any $m \geqslant 0$. Here we have used the fact that m_1, m_2 and $J_{P,0}$ each normalize $J_{P,m}$, for all $m \geqslant 0$; see (6.1.1).

Now choose a large positive integer m such that Y comes from a locally closed subset $\overline{Y}$ of $J_{P,0}/J_{P,m}$. Consider the commutative diagram

$$\begin{tikzcd}
J_{P,0} \arrow[r, "f_0"] \arrow[d, "p"'] & J_{P,0} \arrow[d, "p"'] \\
J_{P,0}/J_{P,m} \arrow[r, "\overline{f}"] \arrow[d, "\pi"'] & J_{P,0}/J_{P,m} \arrow[d, "\pi"'] \\
J_{M,0}/J_{M,m} \arrow[r, "\overline{f}_M"] & J_{M,0}/J_{M,m}
\end{tikzcd}$$

where p is the canonical projection, π is the fiber bundle described above, and $\overline{f}$ and $\overline{f}_M$ are the morphisms induced by f_0. Note that $f_0^{-1}Y = p^{-1}\overline{f}^{-1}\overline{Y}$, which shows that $f_0^{-1}Y$ is admissible. Note also that since $Y \subseteq c_a J_P(0) \subseteq f J_P(0)$, the subset $\overline{Y}$ is contained in the image of $\overline{f}$, so our dimension formula is a consequence of the identity

$$\dim \overline{f}^{-1}\overline{Y} - \dim \overline{Y} = d(m_1, m_2).$$

The latter equality follows easily from our earlier considerations of the base and fiber of the fiber bundle π: the map $\overline{f}_M$ is surjective of relative dimension zero, and the relative dimension of $\overline{f}$ on locally closed subsets of the fibers of π over $\pi(\overline{Y})$ is given by $d(m_1, m_2)$; see the proof of [GHKR06, Proposition 5.3.2, Claim 3]. This proves our subclaim.

As in [GHKR06, proof of Proposition 5.3.2], our claim follows from the subclaim. Write $d(m_1, m_2) =: d$. If $Y \subset J_P$ is any admissible subset, then we have shown that $f^{-1}Y \cap {a_1}^{-1} J_P(0) a_1$ is admissible of dimension $\dim Y + d$ for any $a_1 \in Z(M')(F)$ such that $a_1 Y a_1^{-1} \subseteq a J_P(0) a^{-1}$. Let t_0 be sufficiently large so that $a_t := \epsilon^{t\lambda_0}$ satisfies $a_t Y a_t^{-1} \subseteq a J_P(0) a^{-1}$ for all $t \geqslant t_0$. For all such t we have proved that $f^{-1}Y \cap a_t^{-1} J_P(0) a_t$ is admissible of dimension $\dim Y + d$. This is enough to establish the claim and hence also the lemma. □

Remark 11.3.3. The proof of Lemma 11.3.2 shows that $f_{\tilde{b}} : J_P \to J_P$ is *surjective*.

Now let

$$d(x, \tilde{b}, {}^{w^{-1}}I_P) := \dim(I\mathbf{a}_x \cap {}^{w^{-1}}I_P \mathbf{a}_{\tilde{b}}).$$

We have a dimension-preserving bijection

$$I\mathbf{a}_x \cap {}^{w^{-1}}I_P\mathbf{a}_{\tilde{b}} \cong (IxI\tilde{b}^{-1} \cap {}^{w^{-1}}I_P)/({}^{w^{-1}}I_P \cap {}^{\tilde{b}}I)$$

given by right multiplication by $\tilde{b}^{-1}$, so that

$$d(x, \tilde{b}, {}^{w^{-1}}I_P) = \dim IxI\tilde{b}^{-1} \cap {}^{w^{-1}}I_P - \dim {}^{w^{-1}}I_P \cap {}^{\tilde{b}}I.$$

Let $\rho_N \in X^*(A)_{\mathbb{Q}}$ denote the half-sum of the roots in R_N.

LEMMA 11.3.4. *Consider* $c_{\tilde{b}} : {}^{w^{-1}}I_P \to {}^{w^{-1}}I_P$, $g \mapsto \tilde{b}g\tilde{b}^{-1}$. *Then*

$${}^{w^{-1}}I_P \cap {}^{\tilde{b}}I = c_{\tilde{b}}({}^{w^{-1}}I_P \cap I)$$

and hence

$$\dim({}^{w^{-1}}I_P \cap I) - \dim({}^{w^{-1}}I_P \cap {}^{\tilde{b}}I) = \langle 2\rho_N, \nu\rangle.$$

Proof. Like the previous lemma, this can be proved by looking at the projection $J_P \to J_M$ and then separately computing the contribution from the base J_M (which is 0) and the contribution from the fibers (which is $\langle 2\rho_N, \nu\rangle$; see [GHKR06]). □

Altogether, we now have

$$\begin{aligned}
&\dim X_x(b) \cap I_P\mathbf{a}_w \\
&\quad = \dim f_{\tilde{b}}^{-1}(IxI\tilde{b}^{-1} \cap {}^{w^{-1}}I_P) - \dim I \cap {}^{w^{-1}}I_P \\
&\quad = \dim IxI\tilde{b}^{-1} \cap {}^{w^{-1}}I_P - \dim I \cap {}^{w^{-1}}I_P + \langle \rho, \nu - \nu_{\mathrm{dom}}\rangle \\
&\quad = d(x, \tilde{b}, {}^{w^{-1}}I_P) + \dim {}^{w^{-1}}I_P \cap {}^{\tilde{b}}I - \dim I \cap {}^{w^{-1}}I_P + \langle \rho, \nu - \nu_{\mathrm{dom}}\rangle \\
&\quad = d(x, \tilde{b}, {}^{w^{-1}}I_P) + \langle \rho, \nu - \nu_{\mathrm{dom}}\rangle - \langle 2\rho_N, \nu\rangle \\
&\quad = d(x, \tilde{b}, {}^{w^{-1}}I_P) - \langle \rho, \nu + \nu_{\mathrm{dom}}\rangle,
\end{aligned}$$

where in the final step we have used the equality $\langle \rho, \nu\rangle = \langle \rho_N, \nu\rangle$. This is what we wanted to show. □

Together with the description (Proposition 11.2.4) of ${}^{w^{-1}}I_P$-orbits in $G(L)/I$ as fibers of a certain retraction of the building, Theorem 11.3.1 gives us an algorithm to determine, for a given w, whether the intersection $X_x(b) \cap I_P w\mathbf{a}$ is empty or non-empty; cf. [GHKR06, § 6.1]. If this information were available for all w, we could decide whether $X_x(b)$ is non-empty (and compute its dimension from the dimensions of all these intersections). As noted above, it is clear that all affine Deligne–Lusztig varieties are finite-dimensional, so that the supremum of $\dim(X_x(b) \cap I_P w\mathbf{a})$ is attained for some w. It does not seem easy to give a bound for the length of w depending on x and b.

The theorem allows us to compare the function field case with the p-adic case. For $b \in \widetilde{W}$, in analogy with $X_x(b)$ defined above, we have an 'affine Deligne–Lusztig set' $X_x(b)_{\mathbb{Q}_p}$ inside $G(\widehat{\mathbb{Q}}_p^{\mathrm{ur}})/I$, where I denotes the corresponding Iwahori subgroup.

Corollary 11.3.5. *Let $x \in \widetilde{W}$ and $b \in \widetilde{W}$. Then $X_x(b) \neq \emptyset$ if and only if $X_x(b)_{\mathbb{Q}_p} \neq \emptyset$.*

Proof. One can check that, as far as the non-emptiness is concerned, the proof of Theorem 11.3.1 works without any changes in the p-adic case. The combinatorial properties of the retractions that describe the intersections occurring there coincide in the function field and p-adic cases. □

Even for the dimensions, it is plausible to expect that arguments like those in the proof of Theorem 11.3.1 can be used in the p-adic case, *once a viable notion of dimension has been defined*.

Example 11.3.6. As examples, let us consider the following extreme cases.

(i) $P = B$. Then $I_P = A(\mathfrak{o})U$, and $b = \epsilon^\nu \in B(G)_B$ where $\nu \in X_*(A)$ is a regular dominant translation element. This case was considered in [GHKR06], and the above formula is the same as [GHKR06, (6.3.3) and (6.3.4)].

(ii) $P = G$. Then $I_P = I$, and $b \in \Omega_G$ is a basic σ-conjugacy class. In this case, the dimension formula reads

$$\dim X_x(b) \cap Iw\mathbf{a} = \dim I\mathbf{a}_x \cap {}^{w^{-1}}I\mathbf{a}_{w^{-1}bw}$$

(since ν is central in G). This case is the one analyzed by Reuman in [Reu04] for $b = 1$ and low-rank groups. So let us take $b = 1$ (for other basic b the situation is analogous). We have

$$\begin{aligned} X_x(1) \neq \emptyset &\iff \exists w \in \widetilde{W} \text{ such that } IxI \cap {}^{w^{-1}}I \neq \emptyset \\ &\iff \exists w \in \widetilde{W} \text{ such that } \rho_G^{-1}(x) \cap \rho_{G,w}^{-1}(1) \neq \emptyset. \end{aligned}$$

There are two ways to reformulate this. The algorithmic description in the spirit of the above amounts to

$$X_x(1) \neq \emptyset \iff \exists w \in \widetilde{W} \text{ such that } 1 \in \rho_{G,w}(IxI).$$

On the other hand, we also obtain

$$X_x(1) \neq \emptyset \iff \exists w \in \widetilde{W} \text{ such that } x \in \rho_G(Iw^{-1}IwI),$$

which leads to the 'folding method' used by Reuman, since $Iw^{-1}IwI/I$, as a set of alcoves in the building, is exactly the set of alcoves which can be reached by a gallery of type $i_r, \ldots, i_1, i_1, \ldots, i_r$ (for a fixed reduced expression $w = s_{i_1} \cdots s_{i_r}$). See also § 13.

Remark 11.3.7. The dimension formula in Example 11.3.6(ii) can be interpreted in terms of structure constants for the affine Hecke algebra. Let H denote the affine Hecke algebra over $\mathbb{Z}[v, v^{-1}]$ corresponding to the extended affine Weyl group $\widetilde{W}$, and let $T_x \in H$ denote the standard basis element corresponding to $x \in \widetilde{W}$. Define the parameter $q := v^2$, and consider the structure constants $C(x, y, z) \in \mathbb{Z}[q]$, for $x, y, z \in \widetilde{W}$, defined by the following equality in H:

$$T_x T_y = \sum_z C(x, y, z) T_z.$$

Then it is straightforward to check that

$$\dim I\mathbf{a}_x \cap {}^{w^{-1}}I\mathbf{a}_{w^{-1}bw} = \deg_q C(x, w^{-1}b^{-1}, w^{-1}).$$

(By convention, we set $\deg_q 0 := -\infty = \dim \emptyset$.) Determining the structure constants $C(x, w^{-1}b^{-1}, w^{-1})$ also involves a 'folding algorithm', so this does not give an essentially different way of computing dimensions of affine Deligne–Lusztig varieties. However, it does give some insight into the inherent complexity of the algorithm.

12. Reduction to the basic case and a finite algorithm

12.1 One drawback of Theorem 11.3.1 is that it does not produce a *finite* algorithm for computing the non-emptiness or dimension of $X_x^G(b)$. In this section, we explain how we can at least find a finite algorithm which reduces the non-emptiness and dimension of $X_x^G(b)$ to the non-emptiness and dimension of a finite number of related varieties $X_y^{M'}(\tilde{b})$ with $\tilde{b}$ basic in M'.

Using Theorem 11.3.1, we usually have to check an infinite number of orbit intersections to determine whether a given $X_x(b)$ is empty or not. However, for b basic, we have proved the emptiness predicted by Conjecture 9.4.2 in Corollary 9.4.1. Why are we confident that Conjecture 9.4.2 also correctly predicts non-emptiness? In order to confirm the non-emptiness of $X_x(b)$ in a case where it is expected, it is sufficient for the computer to detect a single non-empty intersection $I\mathbf{a}_x \cap {}^{w^{-1}}I\mathbf{a}_{w^{-1}bw}$ for some w, and in practice the computer does detect one (as far as we have been able to check). In other words, concerning the non-emptiness question for b basic, in practice the algorithm always terminates in finitely many steps, and in this way we are able to generate a complete emptiness/non-emptiness picture, at least when $\ell(x)$ is small enough for the computer to handle.

Let $P = MN$ denote a standard parabolic subgroup. Suppose that $b \in \Omega_M \subset M(L)$ is the standard representative of a basic σ-conjugacy class in $M(L)$, and let $\nu = \overline{\nu}_b^M$ denote its Newton vector.

Recall that ${}^MW$ denotes the set of minimal-length representatives of the cosets in $W_M \backslash W$. Note that $P\backslash G(L)/I \cong {}^MW$.

From now on, we fix an element $w \in {}^MW$. Write $M' = {}^{w^{-1}}M$, $N' = {}^{w^{-1}}N$ and $P' = {}^{w^{-1}}P$. Let $\tilde{b} := {}^{w^{-1}}b \in \Omega_{M'}$. Note that $I_{M'} = {}^{w^{-1}}(M \cap {}^wI) = {}^{w^{-1}}(M \cap I)$ is an Iwahori subgroup of M'. Let e_0 denote the base point of the affine flag variety $G(L)/I$ and let e_0' denote the base point in $M'(L)/I_{M'}$.

We consider the map

$$\begin{aligned} \alpha_w : Pwe_0 &\to M'(L)/I_{M'}, \\ mnwe_0 &\mapsto {}^{w^{-1}}me_0', \end{aligned}$$

which is easily seen to be well-defined and surjective. Fix $m \in M(L)$ and write $m' := {}^{w^{-1}}m \in M'(L)$. The map $mnwe_0 \mapsto {}^{w^{-1}}n$ determines a bijection

$$\alpha_w^{-1}(m'e_0') = N'/N' \cap I. \tag{12.1.1}$$

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We warn the reader that α_w is not a morphism of ind-schemes; however, its restriction to the inverse image of any connected component of $M'(L)/I_{M'}$ is a morphism of ind-schemes.

Now for $x \in \widetilde{W}$ and w, b as above, define the finite set

$$S_P(x,w) := \{y \in \widetilde{W}_{M'} : N'\mathbf{a}_y \cap I\mathbf{a}_x \neq \emptyset\}.$$

Note that $N'\mathbf{a}_y \cap I\mathbf{a}_x \neq \emptyset$ if and only if $I_{P'}\mathbf{a}_y \cap I\mathbf{a}_x \neq \emptyset$. For a given x, there are only finitely many y such that the latter holds; see Proposition 11.2.4.

The following proposition is an analogue of part of [GHKR06, Proposition 5.6.1].

PROPOSITION 12.1.1.

(1) *The map α_w restricts to give a surjective map*

$$\beta_w : X_x^G(b) \cap Pwe_0 \longrightarrow \bigcup_{y \in S_P(x,w)} X_y^{M'}(\tilde{b}). \tag{12.1.2}$$

(2) *Assume $X_x^G(b) \cap Pwe_0 \neq \emptyset$. For a fixed $m' \in M'(L)$ such that $m'e_0' \in X_y^{M'}(\tilde{b})$, set*

$$b' := m'^{-1}\tilde{b}\sigma(m') \in I_{M'}yI_{M'}.$$

Then the fiber $\beta_w^{-1}(m'e_0')$ is a locally finite-type algebraic variety having dimension

$$\dim \beta_w^{-1}(m'e_0') = \dim(I\mathbf{a}_x \cap\ N'\mathbf{a}_y) - \langle \rho, \nu + \nu_{\mathrm{dom}} \rangle,$$

a number which depends on y but not on $m'e_0'$.

(3) *We have*

$$\dim X_x^G(b) = \sup_{w,y\,:\,y \in S_P(x,w)} \{\dim(I\mathbf{a}_x \cap\ {}^{w^{-1}}N\mathbf{a}_y) + \dim(X_y^{{}^{w^{-1}}M}({}^{w^{-1}}b))\} - \langle \rho, \nu + \nu_{\mathrm{dom}} \rangle.$$

The proposition implies that, modulo knowledge of certain basic cases (i.e. the $X_y^{M'}(\tilde{b})$), there is a finite algorithm for determining the non-emptiness and dimension of $X_x^G(b)$. Conjecture 9.4.2 predicts a finite algorithm for determining the non-emptiness of each $X_y^{M'}(\tilde{b})$. Thus, in effect, it predicts a finite algorithm for determining the non-emptiness of $X_x^G(b)$ itself.

COROLLARY 12.1.2. *We have $X_x^G(b) \neq \emptyset$ if and only if there exist $w \in {}^MW$ and $y \in S_P(x,w)$ with $X_y^{M'}(\tilde{b}) \neq \emptyset$.*

Proof of proposition. It is clear that α_w sends the left-hand side of (12.1.2) into the right-hand side. If $m'e_0' \in X_y^{M'}(\tilde{b})$, then the isomorphism (12.1.1) restricts to give an isomorphism

$$\beta_w^{-1}(m'e_0') = f_{b'}^{-1}(IxIb'^{-1} \cap N')/N' \cap I. \tag{12.1.3}$$

Here $b' := m'^{-1}\tilde{b}\sigma(m')$ and we define

$$\begin{aligned} f_{b'} : N' &\to N', \\ n' &\mapsto n'^{-1}b'\sigma(n')b'^{-1}. \end{aligned}$$

Since $f_{b'}$ is surjective (see Remark 11.3.3) and $IxI \cap N'b' \neq \emptyset$, we see that β_w is surjective, which proves (1). Also, the fibers of β_w are algebraic varieties that are locally of finite type, and their dimension can be computed from (12.1.3) using the method of the proof of Theorem 11.3.1. This proves (2). Finally, (3) follows from (1) and (2). □

Remark 12.1.3. For affine Deligne–Lusztig varieties in the affine Grassmannian, it is known that $X_\mu^G(b) \neq \emptyset$ if and only if $[b] \in B(G,\mu)$ (cf. [Gas08, Kot03, KR03, Luc04, Win05]).

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The condition $[b] \in B(G, \mu)$ means that $\eta_G(b) = \mu$ in Λ_G and $\overline{\nu}_b \leqslant \mu$ ('Mazur's inequality'). For $X_x^G(b)$, where we take $b \in \Omega_M$ as before, one might ask for the analogues of 'Mazur's inequalities', meaning a family of congruence conditions and inequalities imposed on x, b and $\overline{\nu}_b$ which hold if and only if $X_x^G(b)$ is non-empty. In light of the above proposition, we see that whatever Mazur's inequalities end up being, they should hold if and only if there exists $w \in {}^M W$ such that for some $y \in \widetilde{W}_{w^{-1}M}$ we have

$$ {}^{w^{-1}}Ny \cap IxI \neq \emptyset \quad \text{and} \quad X_y^{w^{-1}M}({}^{w^{-1}}b) \neq \emptyset. $$

In view of Conjecture 9.4.2, the second inequality above should be understood as a family of congruence conditions. The first one should correspond to a family of inequalities and congruence conditions between $x, y \in \widetilde{W}$. Taken together, the inequalities will be somewhat stronger than the condition $y \leqslant x$ in the Bruhat order on $\widetilde{W}$.

13. Fundamental alcoves and the superset method

13.1 Fundamental alcoves

We now single out some alcoves that will be used to generalize Reuman's superset method [Reu04] to all σ-conjugacy classes in $G(L)$.

DEFINITION 13.1.1. For $x \in \widetilde{W}$ we say that $x\mathbf{a}$ is a *fundamental alcove* if every element of IxI is σ-conjugate under I to x.

Equivalently, the alcove $x\mathbf{a}$ is fundamental if every element of xI is σ-conjugate under ${}^xI \cap I$ to x.

Now let $P = MN$ be a semistandard parabolic subgroup of G. There is then an Iwahori decomposition $I = I_N I_M I_{\overline{N}}$. We use the Iwahori subgroup I_M of $M(L)$ to form the subgroup $\Omega_M \subset \widetilde{W}_M$; note that the canonical surjective homomorphism $\widetilde{W}_M \twoheadrightarrow \Lambda_M$ restricts to an isomorphism $\Omega_M \cong \Lambda_M$. We compose this isomorphism with the canonical homomorphism $\Lambda_M \to \mathfrak{a}_M$ to obtain a homomorphism $\Omega_M \to \mathfrak{a}_M$; for $x \in \Omega_M$ we will denote by $\nu_x \in \mathfrak{a}_M$ the image of x under this homomorphism. Note that $x \mapsto \nu_x$ is intrinsic to M and has nothing to do with P.

DEFINITION 13.1.2. For $x \in \widetilde{W}_M$, we say that $x\mathbf{a}$ is a *fundamental P-alcove* if it is a P-alcove for which $x \in \Omega_M$ or, in other words, if

$$ xI_Mx^{-1} = I_M, \quad xI_Nx^{-1} \subset I_N \quad \text{and} \quad x^{-1}I_{\overline{N}}x \subset I_{\overline{N}}. $$

Proposition 6.3.1 implies that any fundamental P-alcove is a fundamental alcove, just as the terminology suggests. An obvious question (that we have not tried to answer) is whether any fundamental alcove arises as a fundamental P-alcove for some semistandard P.

The next result gives some insight into P-alcoves, although we will make only incidental use of it. We write $\rho_N \in \mathfrak{a}^*$ for the half-sum of the elements in R_N.

PROPOSITION 13.1.3. *Write Ω_P for the set of $x \in \Omega_M$ such that $x\mathbf{a}$ is a fundamental P-alcove.*

(i) *The set Ω_P is a submonoid of Ω_M.*

(ii) *Let $x, y \in \Omega_P$. Then $IxIyI = IxyI$ and $\ell(x) + \ell(y) = \ell(xy)$, where ℓ is the usual length function on $\widetilde{W}$.*

(iii) *Let $x \in \Omega_P$. Then $\ell(x) = \langle 2\rho_N, \nu_x \rangle$.*

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Proof. (i) This is clear from the definitions.

(ii) For the first assertion, just note that

$$xIy = (xI_Nx^{-1})xy(y^{-1}I_My)(y^{-1}I_{\overline{N}}y) \subset I_NxyI_MI_{\overline{N}} \subset IxyI.$$

The second assertion follows from the first (easy, and presumably well-known).

(iii) Since both sides of the equality to be proved are additive functions on the monoid Ω_P, we may replace x by x^m for any positive integer m. Taking m to be the order of the image of x in W_M, we can reduce to the case in which x is a translation element lying in Ω_P. Such an element is of the form ϵ^μ for some cocharacter $\mu \in X_*(A)$ whose image is central in M and dominant with respect to any Borel subgroup of P containing A. It is easy to see that ν_x is simply the image of μ under the canonical inclusion of $X_*(A)$ in $\mathfrak{a}$. Thus the equality to be proved is a consequence of the equality $\ell(\epsilon^\mu) = \langle 2\rho_N, \mu\rangle$, which in turn follows from the usual formula for the length of translation elements in $\widetilde{W}$, in view of the fact that all roots of M vanish on μ. □

13.2 Levi subgroups adapted to I

Let M be a Levi subgroup of G containing A. Once again we put $I_M = M(L) \cap I$ and form $\Omega_M \subset \widetilde{W}_M$ relative to I_M. We will also make use of the homomorphism $x \mapsto \nu_x$ from Ω_M to $\mathfrak{a}_M$ that was explained in the previous subsection.

We write $\mathcal{P}(M)$ for the set of parabolic subgroups of G having M as Levi component. For $P \in \mathcal{P}(M)$ we define $\Omega_M^{\geqslant 0}$ (respectively, $\Omega_M^{>0}$) to be the set of elements $x \in \Omega_M$ such that $\langle \alpha, \nu_x\rangle \geqslant 0$ (respectively, $\langle \alpha, \nu_x\rangle > 0$) for all $\alpha \in R_N$. It is clear that most elements of $\Omega_M^{\geqslant 0}$ lie in Ω_P; however, we are going to give a condition on M which will guarantee that every element of $\Omega_M^{\geqslant 0}$ lies in Ω_P. (Compare this with Remark 7.2.4, which shows that when $P = MN$ is standard, an element $\epsilon^\lambda w \in \Omega_M$ lies in Ω_P if and only if λ is G-dominant.)

As usual, the group $\widetilde{W}_M$ acts by affine linear transformations on both $\mathfrak{a}$ and its quotient $\mathfrak{a}/\mathfrak{a}_M$, the natural surjection $\mathfrak{a} \twoheadrightarrow \mathfrak{a}/\mathfrak{a}_M$ being $\widetilde{W}_M$-equivariant. The subgroup Ω_M then inherits an action on $\mathfrak{a}$ and $\mathfrak{a}/\mathfrak{a}_M$.

Definition 13.2.1. We say that M is *adapted to* I (respectively, *weakly adapted to* I) if there exists λ in $\mathbf{a}$ (respectively, in the closure of $\mathbf{a}$) whose image in $\mathfrak{a}/\mathfrak{a}_M$ is fixed by the action of Ω_M.

For any such λ, it is easy to see that $x\lambda = \lambda + \nu_x$ for all $x \in \Omega_M$.

Proposition 13.2.2. *If M is adapted to I, then $\Omega_M^{\geqslant 0} \subset \Omega_P$ and, consequently, for every $x \in \Omega_M$ there exists $P \in \mathcal{P}(M)$ for which $x\mathbf{a}$ is a fundamental P-alcove. Similarly, if M is weakly adapted to I, then $\Omega_M^{>0} \subset \Omega_P$.*

Proof. We begin by proving the first statement. For $\alpha \in R_N$ we must show that $x\mathbf{a} \geqslant_\alpha \mathbf{a}$, which is to say that $k(\alpha, x\mathbf{a}) \geqslant k(\alpha, \mathbf{a})$. For any $\lambda \in \mathbf{a}$ we have $k(\alpha, x\mathbf{a}) = \lceil \alpha(x\lambda)\rceil$ and $k(\alpha, \mathbf{a}) = \lceil \alpha(\lambda)\rceil$. Now pick λ as in the definition of being adapted to I. Since $x \in \Omega_M^{\geqslant 0}$, we see from the equality $x\lambda = \lambda + \nu_x$ that $\alpha(x\lambda) \geqslant \alpha(\lambda)$; it is then clear that $\lceil \alpha(x\lambda)\rceil \geqslant \lceil \alpha(\lambda)\rceil$.

Now we prove the second statement. For $\alpha \in R_N$ we now have

$$k(\alpha, \mathbf{a}) - 1 \leqslant \alpha(\lambda) < \alpha(x\lambda) \leqslant k(\alpha, x\mathbf{a}),$$

and hence $k(\alpha, \mathbf{a}) \leqslant k(\alpha, x\mathbf{a})$ as desired. □

Proposition 13.2.3. *Let M be any Levi subgroup containing A. Then there exists $w \in W$ such that ${}^wM$ is adapted to I.*

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Proof. There exist fixed points of Ω_M on $\mathfrak{a}/\mathfrak{a}_M$ lying on no affine root hyperplane for M (for example, when M is simple, one can take the barycenter of the base alcove for $\widetilde{W}_M$). We choose such a fixed point $\overline{\lambda}$ and then choose $\lambda \in \mathfrak{a}$ mapping to $\overline{\lambda}$. We are free to add any element of $\mathfrak{a}_M$ to λ, so we may assume that λ lies on no affine root hyperplane for G. If λ happens to lie in $\mathbf{a}$, then M is adapted to I. In any case, there exists a unique alcove $x'\mathbf{a}$ containing λ. The Levi subgroup is then adapted to $I' = x'Ix'^{-1}$. Taking w to be the inverse of the image of x' in W, we find that ${}^wM$ is adapted to I. □

Being adapted to I is quite a strong condition on M. It is important to realize that standard Levi subgroups are often not adapted to our standard Iwahori subgroup I, even though both notions of standard are tied to the same Borel subgroup.

Corollary 13.2.4. *For every $[b] \in B(G)$ there exists a semistandard representative $x \in \widetilde{W}$ of $[b]$ such that $x\mathbf{a}$ is a fundamental alcove and hence $IxI \subset [b]$.*

Proof. This follows from the previous two propositions and Definition 7.2.3. □

13.3 Superset method

Let $b \in G(L)$. The *superset* $\widetilde{W}(b)$ associated to b is the set of $x \in \widetilde{W}$ such that IxI is contained in $Iy^{-1}IbIyI$ for some $y \in \widetilde{W}$. The reason for the name 'superset' is that the set of $x \in \widetilde{W}$ such that $X_x(b) \neq \emptyset$ is contained in $\widetilde{W}(b)$. Indeed, if $X_x(b) \neq \emptyset$, then there exists $g \in G(L)$ such that $g^{-1}b\sigma(g) \in IxI$. There also exists $y \in \widetilde{W}$ such that $g \in IyI$, and then

$$IxI = Ig^{-1}b\sigma(g)I \subset Iy^{-1}IbIyI.$$

Proposition 13.3.1. *Suppose that $x_0\mathbf{a}$ is a fundamental alcove, and let b_0 be any element of Ix_0I. Then*

$$\{x \in \widetilde{W} : X_x(b_0) \neq \emptyset\} = \widetilde{W}(b_0).$$

Proof. We already know the inclusion $\subset$. To establish $\supset$, we consider $x \in \widetilde{W}(b_0)$ and choose $y \in \widetilde{W}$ such that $IxI \subset Iy^{-1}Ib_0IyI$. Then IxI meets $y^{-1}Ib_0Iy$, and since (by our hypothesis on x_0) every element of Ib_0I has the form $i^{-1}b_0\sigma(i)$ for suitable $i \in I$, there is some element in IxI of the form $\dot{y}^{-1}i^{-1}b_0\sigma(i)\dot{y}$, where $\dot{y}$ is a representative of y in the F-points of the normalizer of A in G. Since $\dot{y} = \sigma(\dot{y})$, this shows that IxI meets $[b_0]$, as desired. □

Corollary 13.3.2. *For every $[b] \in B(G)$ there is a semistandard representative $b_0 \in [b]$ for which the superset method applies, yielding*

$$\{x \in \widetilde{W} : X_x(b_0) \neq \emptyset\} = \widetilde{W}(b_0).$$

Proof. Combine Corollary 13.2.4 with Proposition 13.3.1. □

14. Examples

14.1 To illustrate our results and conjectures (Conjecture 9.4.2 and Conjecture 9.5.1(a)), in this section we present two examples for the group GSp_4 (i.e. for Dynkin type C_2). In the first example, shown in Figure 3, we take $b = 1$; in the second example, shown in Figure 4, b is one of the generators of the subgroup $\Omega \subset \widetilde{W}$ of all length-zero elements (the picture is independent of the choice of generator; in fact, it depends only on the parity of the image of b under an isomorphism $\Omega \cong \mathbb{Z}$).

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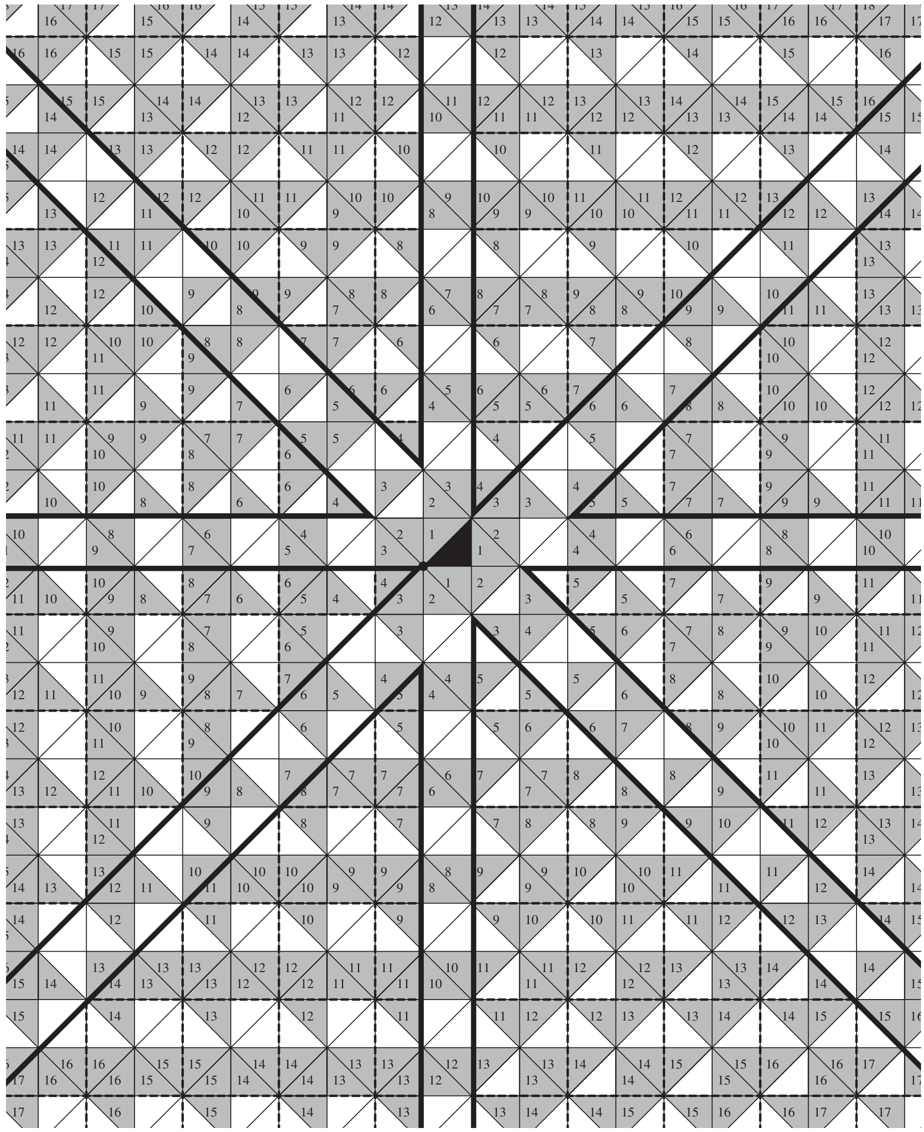

Figure 3. Dimensions of affine Deligne–Lusztig varieties for Dynkin type C_2 with $b = 1$.

In both cases, we identify the coset $W_a b \subset \widetilde{W}$ with the set of alcoves in the standard apartment. In Figures 3 and 4, the origin is marked by a dot, and the base alcove is shown in black. Gray alcoves correspond to non-empty affine Deligne–Lusztig varieties (and the number given is the dimension), while white alcoves correspond to empty ones.

The thick black lines indicate the shrunken Weyl chambers. The dashed lines indicate the W-cosets $\varepsilon^\mu W$ inside the shrunken Weyl chambers. Recall the maps η_1 and η_2 from § 9.5: viewing each dashed square as a copy of the finite Weyl group, η_1 maps an element to the position it has inside the dashed square it lies in (i.e. to the corresponding element of W). On the other hand, the map η_2 is constant on each finite Weyl chamber; that is, it maps an alcove to the finite Weyl chamber it lies in, considered as an element of W. As the conjecture predicts, inside a shrunken Weyl chamber all dashed squares look the same (independently of b!).

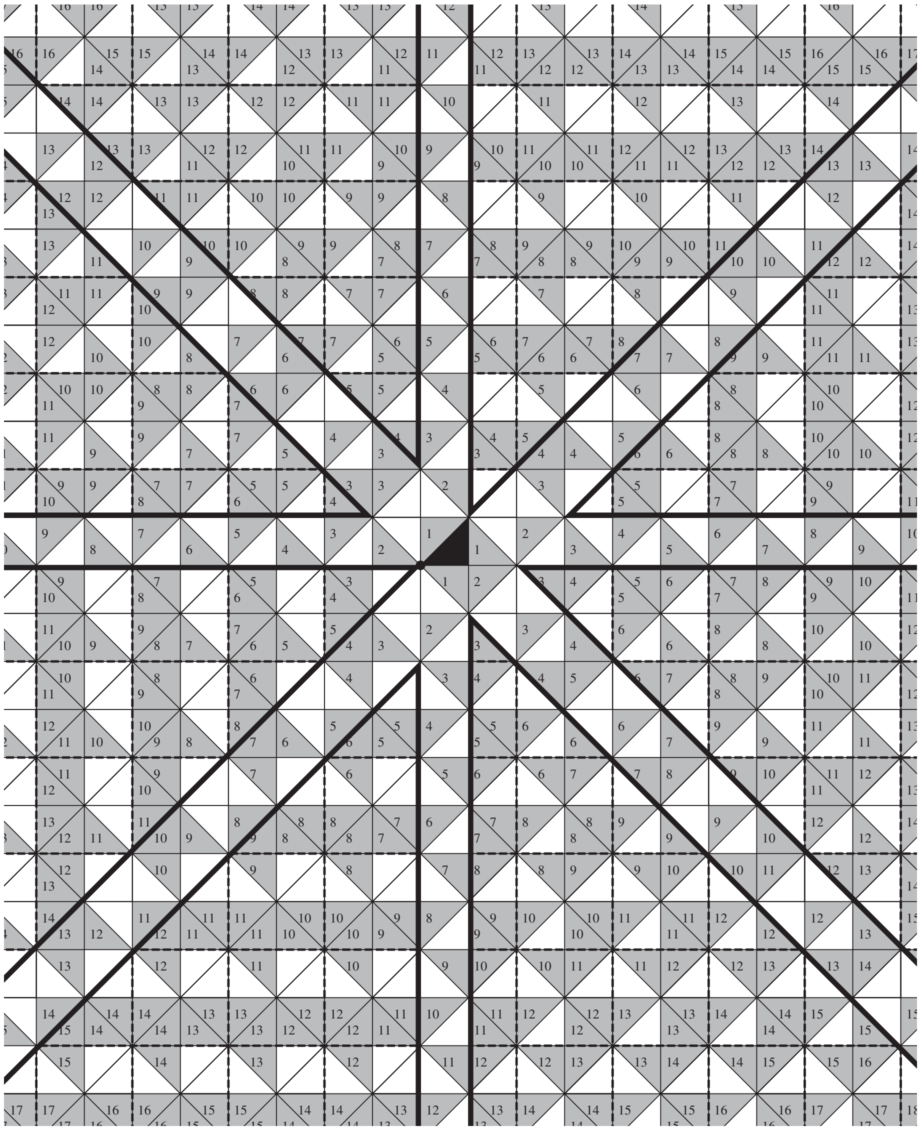

FIGURE 4. Dimensions of affine Deligne–Lusztig varieties for Dynkin type C_2 with b 'supersingular'.

For further examples, we refer to [GHKR06], and also to the version of that paper on the arXiv.org e-print server (arXiv:math/0504443v1).

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