J. Inst. Math. Jussieu (2020), 1–7

doi:10.1017/S1474748020000201

$\mathbb{A}_{\mathrm{inf}}$ IS INFINITE DIMENSIONAL

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(Received 19 June 2019; revised 5 April 2020; accepted 12 April 2020)

Abstract Given a perfect valuation ring R of characteristic p that is complete with respect to a rank-1 nondiscrete valuation, we show that the ring $\mathbb{A}_{\mathrm{inf}}$ of Witt vectors of R has infinite Krull dimension.

Keywords: p-adic Hodge theory; Fargues-Fontaine curves; Krull dimension; Newton polygons

2010 *Mathematics subject classification*: Primary 13F35; 13F25
Secondary 14F30

1. Introduction

Fix a prime p. Let R be a perfect valuation ring of characteristic p and denote the valuation by v. Assume v is of rank 1 and nondiscrete and that R is complete with respect to v. Let $\mathbb{A} := \mathbb{A}_{\mathrm{inf}} := W(R)$ be the ring of Witt vectors of R. This ring plays a central role in p-adic Hodge theory as it is the basic ring from which all of Fontaine's p-adic period rings are built. It is also central to the construction of the (adic) Fargues–Fontaine curve [4]. Recently, Bhatt, Morrow and Scholze constructed $\mathbb{A}_{\mathrm{inf}}$-cohomology, a cohomology theory that specializes to étale, de Rham and crystalline cohomology [3]. In these works, there is a useful analogy between $\mathbb{A}$ and a two-dimensional regular local ring. In this paper, we prove the following theorem.

Theorem 1.1. *The ring $\mathbb{A}$ has infinite Krull dimension.*

Bhatt [2, Warning 2.24] and Kedlaya [5, Remark 1.6] note that the Krull dimension of $\mathbb{A}$ is at least 3. To see this, fix a pseudouniformizer $\varpi \in R$ and let κ denote the residue field of R. Let $W(\mathfrak{m})$ be the kernel of the natural map $W(R) \to W(\kappa)$ and $[-]\colon R \to W(R)$ the Teichmüller map. Then Bhatt and Kedlaya point out that $\mathbb{A}$ contains the following explicit chain of prime ideals:

$$(0) \subset \mathfrak{p} := \bigcup_{k=0}^{\infty} [\varpi^{1/p^k}]\mathbb{A} \subset W(\mathfrak{m}) \subset (p, W(\mathfrak{m})).$$

As suggested in [5, Remark 1.6], we use Newton polygons to find an infinite chain of prime ideals between $\mathfrak{p}$ and $W(\mathfrak{m})$.

The equal characteristic analogue of Theorem 1.1 is the statement that the power series ring $R[\![X]\!]$ has infinite Krull dimension. This was first proved by Arnold [1, Theorem 1], and the structure of our argument is very similar to his. We axiomatize Arnold's argument in Section 3.

Notation. We use the convention that the symbols $<, >, \subset, \supset$ denote strict inequalities and inclusions with the exception that we allow the statement "$\infty < \infty$" to be true. Otherwise, if equality is allowed, it will be explicitly reflected in the notation using the symbols $\leqslant, \geqslant, \subseteq, \supseteq$. An inequality between two $(\mathbb{R} \cup \{\pm\infty\})$-valued functions means that the inequality holds pointwise.

2. Review of Newton polygons

As above, let R be a perfect valuation ring of characteristic p that is complete with respect to a nondiscrete valuation v of rank 1. Let $\mathfrak{m}$ be the maximal ideal of R, and fix an element $\varpi \in \mathfrak{m}$ of valuation $v(\varpi) = 1$.

Let $\mathbb{A} := W(R)$ be the ring of Witt vectors of R. Write $[-]\colon R \to \mathbb{A}$ for the Teichmüller map, which is multiplicative. Recall that every element of $\mathbb{A}$ can be written uniquely in the form $\sum_{n \geqslant 0} [x_n] p^n$ with $x_n \in R$.

As in [4, Section 1.5.2], given $f \in \mathbb{A}$ with $f = \sum_{n \geqslant 0} [x_n] p^n$, we define the *Newton polygon* $\mathcal{N}(f)$ of f as the largest decreasing convex polygon in $\mathbb{R}^2$ lying below the set of points $\{(n, v(x_n)) \colon n \geqslant 0\}$. We shall often view $\mathcal{N}(f)$ as the graph of a function $\mathcal{N}(f)\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$. In particular, if n_f is the smallest integer such that $x_{n_f} \neq 0$, then $\mathcal{N}(f)(t) = +\infty$ for $t < n_f$ and $\mathcal{N}(f)(n_f) = v(x_{n_f})$. Furthermore, $\lim_{t \to \infty} \mathcal{N}(f)(t) = \inf_n v(x_n)$.

Following the conventions in [4, Section 1.5.2], for any integer $i \geqslant 0$ define

$$s_i(f) := \mathcal{N}(f)(i) - \mathcal{N}(f)(i+1).$$

We call $s_i(f)$ the *slope* of $\mathcal{N}(f)$ on the interval $[i, i+1]$ even though one would typically call that slope $-s_i(f)$. With this convention, the slopes form a nonnegative decreasing sequence; that is, $s_i(f) \geqslant s_{i+1}(f) \geqslant 0$ for all i. We say that n is a *node* of $\mathcal{N}(f)$ if $\mathcal{N}(f)(n) = v(x_n)$.

We recall the theory of Legendre transforms from [4, Section 1.5.1]. Given a function $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ that is not identically equal to $+\infty$, define

$$\begin{aligned} \mathcal{L}(\varphi)\colon \mathbb{R} &\to \mathbb{R} \cup \{-\infty\} \\ \lambda &\mapsto \inf_{t \in \mathbb{R}} \{\varphi(t) + \lambda t\}. \end{aligned}$$

If φ is a convex function, then one can recover φ from $\mathcal{L}(\varphi)$ via the formula

$$\varphi(t) = \sup_{\lambda \in \mathbb{R}} \{\mathcal{L}(\varphi)(\lambda) - t\lambda\}.$$

From these definitions, it is easy to see that $\mathcal{N}(f) \leqslant \mathcal{N}(g)$ if and only if $\mathcal{L}(\mathcal{N}(f)) \leqslant \mathcal{L}(\mathcal{N}(g))$.

As explained in [4, Section 1.5], for any $f, g \in \mathbb{A}$, we have

$$\mathcal{L}(\mathcal{N}(fg)) = \mathcal{L}(\mathcal{N}(f)) + \mathcal{L}(\mathcal{N}(g)). \tag{1}$$

Motivated by this, one defines a convolution product on the set of $(\mathbb{R} \cup \{+\infty\})$-valued convex functions on $\mathbb{R}$ that are not identically $+\infty$ by

$$(\varphi * \psi)(t) := \sup_{\lambda \in \mathbb{R}} \{\mathcal{L}(\varphi)(\lambda) + \mathcal{L}(\psi)(\lambda) - t\lambda\}.$$

Thus we have $\mathcal{N}(fg) = \mathcal{N}(f) * \mathcal{N}(g)$. In particular, if $\mathcal{N}(f) > 0$, then $\mathcal{N}(f^m) < \mathcal{N}(f^{m+1})$ for all $m \geqslant 1$, and for any $t \in \mathbb{R}$ we have $\lim_{m \to \infty} \mathcal{N}(f^m)(t) = +\infty$.

There is another way of describing $\mathcal{N}(fg)$ in terms of $\mathcal{N}(f)$ and $\mathcal{N}(g)$ without explicitly using Legendre transforms. Write $f = \sum_{n \geqslant 0} [x_n] p^n$ and $g = \sum_{n \geqslant 0} [y_n] p^n$, and let n_f (respectively, n_g) be the smallest integer such that $x_n \neq 0$ (respectively, $y_n \neq 0$). Then $\mathcal{N}(fg)(t) = +\infty$ for all $t < n_f + n_g$, and $\mathcal{N}(fg)(n_f + n_g) = v(x_{n_f}) + v(y_{n_g})$. The slopes of $\mathcal{N}(fg)$ are given by interlacing the slopes of $\mathcal{N}(f)$ and $\mathcal{N}(g)$. That is, the slope sequence of $\mathcal{N}(fg)$ is given by combining the sequences $\{s_i(f) \colon i \geqslant 0\}$ and $\{s_i(g) \colon i \geqslant 0\}$ into a single decreasing sequence that incorporates all positive elements of both sequences. The relationship between this description and equation (1) is explained in [4, Section 1.5].

Lemma 2.1. *Let f be an element of $\mathbb{A}$ such that $\mathcal{N}(f) > 0$. If g is an element of $\mathbb{A}$ and $t_0 \geqslant 0$ is such that for all $t \geqslant t_0$ we have $\mathcal{N}(g)(t) \leqslant \mathcal{N}(f)(t)$, then for all m sufficiently large we have $\mathcal{N}(g) \leqslant \mathcal{N}(f^m)$.*

Proof. As noted above, since $\mathcal{N}(f) > 0$, the sequence $\{\mathcal{N}(f^m)\}_m$ converges to $+\infty$. This convergence is uniform on the compact interval $[0, t_0]$. Thus for m sufficiently large, it follows that $\mathcal{N}(g)(t) \leqslant \mathcal{N}(f^m)(t)$ for all $t \in [0, t_0]$. On the other hand, for all $t \geqslant t_0$ we have

$$\mathcal{N}(g)(t) \leqslant \mathcal{N}(f)(t) < \mathcal{N}(f^m)(t).$$

Thus $\mathcal{N}(g) \leqslant \mathcal{N}(f^m)$ for all m sufficiently large. □

Proposition 2.2. *The ideal $\mathfrak{p} := \bigcup_{k=0}^{\infty} [\varpi^{1/p^k}]\mathbb{A}$ is a prime ideal of $\mathbb{A}$.*

Proof. Note that an element f of $\mathbb{A}$ lies in $\mathfrak{p}$ if and only if $\lim_{t \to \infty} \mathcal{N}(f)(t) > 0$. If $g, g' \in \mathbb{A} \backslash \mathfrak{p}$, then $\lim_{t \to \infty} \mathcal{N}(gg')(t) = \lim_{t \to \infty} (\mathcal{N}(g) * \mathcal{N}(g'))(t) = 0$ and so $gg' \notin \mathfrak{p}$. □

3. The strategy

We define infinitely many sequences in R as follows. For all $i \geqslant 0$, define $a_{1,i} := \varpi^{1/p^i} \in R$. For $n > 1$ and $i \geqslant 0$, define $a_{n,i}$ recursively by

$$a_{n,i} := a_{n-1,i^2} \in R.$$

Thus $a_{n,i} = \varpi^{1/p^{n_i}}$, where $n_i := i^{2^{n-1}}$, and $v(a_{n,i}) = p^{-n_i}$. For each $n \geqslant 1$, define

$$h_n := \sum_{i=0}^{\infty} [a_{n,i}] p^i \in \mathbb{A}.$$

Note that $\mathcal{N}(h_n) > 0$, for any n we have $\lim_{t\to\infty} \mathcal{N}(h_n)(t) = 0$, and $\mathcal{N}(h_n)$ has a node at every integer.

Finally, we define the following subsets of $\mathbb{A}$. For $n \geqslant 1$, let

$$\mathcal{S}_n := \{g \in \mathbb{A} \colon 0 < \mathcal{N}(g) \leqslant \mathcal{N}(h_n^m) \text{ for some } m \geqslant 1\}.$$

In particular, $h_n \in \mathcal{S}_n$.

Proposition 3.1. *The sets $\mathcal{S}_n$ satisfy the following three properties:*

(1) *for all $n \geqslant 1$ we have $\mathcal{S}_{n+1} \subset \mathcal{S}_n$;*

(2) *each $\mathcal{S}_n$ is multiplicatively closed;*

(3) *for any $g \in \mathcal{S}_{n+1}$ and $f \in \mathbb{A}$, we have that $g + fh_n \in \mathcal{S}_{n+1}$.*

We prove this proposition in Section 4.

Theorem 3.2. *The ring $\mathbb{A}$ has infinite Krull dimension.*

Proof. We follow Arnold's proof of [1, Theorem 1]. We prove that for any $n \geqslant 1$, there exists a chain of prime ideals of $\mathbb{A}$, say $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$, such that $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$.

For $n = 1$, let $\mathfrak{p}_1 = \mathfrak{p}$. To see that $\mathfrak{p} \cap \mathcal{S}_1 = \emptyset$, note that if $f \in \mathfrak{p}$, then $f \in [\varpi^{1/p^k}]\mathbb{A}$ for some $k \geqslant 0$, and so $\mathcal{N}(f) \geqslant 1/p^k$. On the other hand, if $f \in \mathcal{S}_1$, then for some $m \geqslant 1$ we have that $\lim_{t\to\infty} \mathcal{N}(f)(t) \leqslant \lim_{t\to\infty} \mathcal{N}(h_1^m)(t) = 0$.

Fix $n \geqslant 1$ and suppose for induction that there is a chain $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals of $\mathbb{A}$ such that $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$. Consider the ideal $\mathfrak{a}_n := \mathfrak{p}_n + h_n\mathbb{A}$. Note that $\mathfrak{a}_n \neq \mathfrak{p}_n$ since $h_n \in \mathcal{S}_n$ and $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$. We claim that $\mathfrak{a}_n \cap \mathcal{S}_{n+1} = \emptyset$. Indeed, given $g \in \mathcal{S}_{n+1}$, we have that $g + h_n f \in \mathcal{S}_{n+1}$ for all $f \in \mathbb{A}$ by property (3) of the sets $\mathcal{S}_n$. By property (1), it follows that $g + h_n f \in \mathcal{S}_n$ for all $f \in \mathbb{A}$. If $g \in \mathfrak{a}_n$, then there is some $f \in \mathbb{A}$ such that $g + h_n f \in \mathfrak{p}_n$. But $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$, so it follows that $g \notin \mathfrak{a}_n$.

Since $\mathcal{S}_{n+1}$ is multiplicatively closed by property (2), there is a prime ideal $\mathfrak{p}_{n+1}$ of $\mathbb{A}$ such that $\mathfrak{p}_n \subset \mathfrak{a}_n \subseteq \mathfrak{p}_{n+1}$ and $\mathfrak{p}_{n+1} \cap \mathcal{S}_{n+1} = \emptyset$. By induction on n, it follows that $\mathbb{A}$ has infinite Krull dimension. □

Remark 3.3. (a) *Arnold has used an argument as above to show that the ring $R[\![X]\!]$ has infinite Krull dimension [1, Theorem 1]. In fact given any ring A, if one can exhibit elements h_n of A and sets $\mathcal{S}_n$ satisfying the properties in Proposition 3.1 together with a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \cap \mathcal{S}_1 = \emptyset$, then the above argument shows that A has infinite Krull dimension.*

(b) *There is a rigorous way to view the power series ring $R[\![X]\!]$ as an equal characteristic version of $\mathbb{A}$ (see [4, Section 1.3]). Our definitions make sense in this more general setting, and our arguments give another proof that $R[\![X]\!]$ has infinite Krull dimension.*

4. The proof of Proposition 3.1

In this section we prove Proposition 3.1. Recall that v is the valuation on R and $s_i(h_n) := v(a_{n,i-1}/a_{n,i})$ is the ith slope of $\mathcal{N}(h_n)$.

Proposition 4.1. *Fix $n, m \geqslant 1$. For $t > 2m^2$ we have that*

$$\mathcal{N}(h_{n+1}^m)(t) < \mathcal{N}(h_n)(t).$$

Proof. Let $\ell = km + r \in \mathbb{Z}$ with $k > 2m$ and $0 \leqslant r < m$. We have

$$\mathcal{N}(h_n)(\ell) = v(a_{n,\ell}) = v(a_{n,km+r})$$

and

$$\mathcal{N}(h_{n+1}^m)(\ell) = mv(a_{n+1,k}) - s_{k+1}(h_{n+1})r \leqslant mv(a_{n+1,k}) = mv(a_{n,k^2}).$$

To see that $mv(a_{k^2}) < v(a_{n,km+r})$, recall that $v(a_{n,i}) = p^{-i^{2^{n-1}}}$. Thus we must show that

$$m < p^{k^{2^n} - (km+r)^{2^{n-1}}}.$$

Since $r < m$, it suffices to show that $m < p^{k^{2^n} - ((k+1)m)^{2^{n-1}}}$. One checks this quickly using that $k > 2m$ and therefore $k^2 - (km + m) > m$. □

Corollary 4.2. *For all $n \geqslant 1$ we have $\mathcal{S}_{n+1} \subset \mathcal{S}_n$.*

Proof. If $g \in \mathcal{S}_{n+1}$ then for some $m \geqslant 1$ we have $0 < \mathcal{N}(g) \leqslant \mathcal{N}(h_{n+1}^m)$. By Proposition 4.1 and Lemma 2.1, it follows that for m' sufficiently large (depending on m and n), we have $\mathcal{N}(h_{n+1}^m) < \mathcal{N}(h_n^{m'})$, so $g \in \mathcal{S}_n$. To see that the inclusion is strict, note that Proposition 4.1 also implies that $h_n \notin \mathcal{S}_{n+1}$, but $h_n \in \mathcal{S}_n$. □

Proposition 4.3. *Let h be an element of $\mathbb{A}$ such that $\mathcal{N}(h) > 0$. Then for any $f \in \mathbb{A}$, $\mathcal{N}(fh) \geqslant \mathcal{N}(h)$.*

Proof. The Newton polygon $\mathcal{N}(fh)$ starts at $n_f + n_h$. Note that the slopes of $\mathcal{N}(fh)$ are all positive and form a monotone sequence converging to zero. Therefore all slopes $s_i(h)$ of h eventually occur as slopes of $\mathcal{N}(hf)$. It follows that for any $l \geqslant n_f + n_h$, $\mathcal{N}(fh)(l) \geqslant \sum_{i \geqslant l}^{\infty} s_i(h) = \mathcal{N}(h)(l)$. □

Proposition 4.4. *For each $n \geqslant 1$, the set $\mathcal{S}_n$ is multiplicatively closed.*

Proof. Let $f, g \in \mathcal{S}_n$. Then by Proposition 4.3, we have that $\mathcal{N}(fg) \geqslant \mathcal{N}(g) > 0$.

For m sufficiently large, we have $0 < \mathcal{N}(f), \mathcal{N}(g) \leqslant \mathcal{N}(h_n^m)$. Thus for any $\lambda, t \in \mathbb{R}$ we have

$$\mathcal{N}(f)(t) + \lambda t \leqslant \mathcal{N}(h_n^m)(t) + \lambda t.$$

Taking the infimum over $t \in \mathbb{R}$, it follows that $\mathcal{L}(\mathcal{N}(f))(\lambda) \leqslant \mathcal{L}(\mathcal{N}(h_n^m))(\lambda)$ for all $\lambda \in \mathbb{R}$. Similarly, $\mathcal{L}(\mathcal{N}(g)) \leqslant \mathcal{L}(\mathcal{N}(h_n^m))$. Therefore

$$\mathcal{L}(\mathcal{N}(fg)) = \mathcal{L}(\mathcal{N}(f)) + \mathcal{L}(\mathcal{N}(g)) \leqslant 2\mathcal{L}(\mathcal{N}(h_n^m)) = \mathcal{L}(\mathcal{N}(h_n^{2m})).$$

Hence, we have that $\mathcal{L}(\mathcal{N}(fg))(\lambda) - t\lambda \leqslant \mathcal{L}(\mathcal{N}(h_n^{2m}))(\lambda) - t\lambda$ for all $t, \lambda \in \mathbb{R}$. It follows that

$$\mathcal{N}(fg)(t) = \sup_{\lambda}\{\mathcal{L}(\mathcal{N}(fg))(\lambda) - t\lambda\} \leqslant \sup_{\lambda}\{\mathcal{L}(\mathcal{N}(h_n^{2m}))(\lambda) - t\lambda\} = \mathcal{N}(h_n^{2m})(t)$$

for all $t \in \mathbb{R}$. Therefore $fg \in \mathcal{S}_n$. □

Let $f, g \in \mathbb{A}$, and write $f = \sum_{n=0}^{\infty}[x_n]p^n$ and $g = \sum_{n=0}^{\infty}[y_n]p^n$. In order to prove property (3) from Proposition 3.1 we need to understand the Newton polygon of $f+g$ in terms of those of f and g. For that, we show a property of Witt vector addition in Lemma 4.5 below. First, recall the translation between Teichmüller expansions and Witt coordinates:

$$\sum_{n=0}^{\infty}[x_n]p^n = (x_0, x_1^p, x_2^{p^2}, \ldots, x_n^{p^n}, \ldots).$$

Recall also that addition of Witt vectors is governed by the polynomials

$$S_n(X_0, \ldots, X_n; Y_0, \ldots, Y_n),$$

which are defined recursively by

$$S_0(X_0; Y_0) \coloneqq X_0 + Y_0$$

and

$$\sum_{k=0}^{n} p^k S_k(X_0, \ldots, X_k; Y_0, \ldots, Y_k)^{p^{n-k}} = \sum_{k=0}^{n} p^k (X_k^{p^{n-k}} + Y_k^{p^{n-k}}).$$

Thus

$$\begin{aligned} f + g &= (S_0(x_0; y_0), \ldots, S_n(x_0, \ldots, x_n^{p^n}; y_0, \ldots, y_n^{p^n}), \ldots) \\ &= \sum_{n=0}^{\infty}[S_n(x_0, \ldots, x_n^{p^n}; y_0, \ldots, y_n^{p^n})^{p^{-n}}]p^n. \end{aligned}$$

Lemma 4.5. *For all $n \geqslant 0$ we have that*

$$S_n(x_0, \ldots, x_n^{p^n}; y_0, \ldots, y_n^{p^n}) = x_n^{p^n} + y_n^{p^n} + \Sigma_n,$$

where Σ_n is a sum of terms of the form $\prod_{k=0}^{n-1} x_k^{p^k i_k} y_k^{p^k j_k}$ such that $\sum_{k=0}^{n-1} p^k(i_k + j_k) = p^n$.

Proof. Note that if the lemma holds for some n, then S_n^p is a sum of terms of the form $\prod_{k=0}^{n} x_k^{p^k i_k} y_k^{p^k j_k}$ such that $\sum_{k=0}^{n} p^k(i_k + j_k) = p^{n+1}$. The lemma then follows from the definition of S_n and induction on n. □

Proposition 4.6. *Let $f = \sum_{n=0}^{\infty}[x_n]p^n$, $g = \sum_{n=0}^{\infty}[y_n]p^n \in \mathbb{A}$. Assume that $\mathcal{N}(g)$ is strictly decreasing. Suppose there exists a $t_0 \geqslant 0$ such that for all $t \geqslant t_0$ we have $\mathcal{N}(g)(t) < \mathcal{N}(f)(t)$. Then there exists $t_1 \geqslant t_0$ such that for all $t \geqslant t_1$, we have that $\mathcal{N}(g + f)(t) \leqslant \mathcal{N}(g)(t)$.*

Proof. We first show the desired inequality when $t \geqslant t_0$ is a node of $\mathcal{N}(g)$; these exist since g is strictly decreasing. Let $n \geqslant t_0$ be a node of $\mathcal{N}(g)$. Since $\mathcal{N}(g)$ is strictly decreasing, we have that

$$v(y_n) = \mathcal{N}(g)(n) < v(y_m)$$

for all $m < n$. Since $n \geqslant t_0$ and $\mathcal{N}(f)$ is decreasing, for all $m \leqslant n$ we have that

$$v(y_n) = \mathcal{N}(g)(n) < \mathcal{N}(f)(n) \leqslant v(x_m).$$

Thus $v(y_n^{p^n}) < v(x_n^{p^n})$ and for any $i_0, j_0, \ldots, i_{n-1}, j_{n-1}$ such that $\sum_{k=0}^{n-1} p^k(i_k + j_k) = p^n$, it follows that

$$v\left(\prod_{k=0}^{n-1} y_k^{p^k i_k} x_k^{p^k j_k}\right) > p^n v(y_n) = v(y_n^{p^n}).$$

By Lemma 4.5, it follows that

$$v(S_n(y_0, \ldots, y_n^{p^n}; x_0, \ldots, x_n^{p^n})^{p^{-n}}) = v(y_n).$$

Therefore

$$\mathcal{N}(g + f)(n) \leqslant v(S_n(y_0, \ldots, y_n^{p^n}; x_0, \ldots, x_n^{p^n})^{p^{-n}}) = v(y_n) = \mathcal{N}(g)(n),$$

and the inequality holds at all nodes of $\mathcal{N}(g)$ beyond t_0.

Let $t_1 \geqslant t_0$ be the first node of $\mathcal{N}(g)$. Given $t \geqslant t_1$, let n_1 and n_2 be two consecutive nodes such that $n_1 \leqslant t \leqslant n_2$. On this segment, $\mathcal{N}(g)$ is the straight line connecting $(n_1, v(y_{n_1}))$ and $(n_2, v(y_{n_2}))$. Since $\mathcal{N}(g + f)$ is a convex function lying below $\mathcal{N}(g)$ at the two end points n_1 and n_2, it follows that $\mathcal{N}(g + f)(t) \leqslant \mathcal{N}(g)(t)$, as desired. □

Corollary 4.7. *If $g \in \mathcal{S}_{n+1}$ and $f \in \mathbb{A}$, then $g + fh_n \in \mathcal{S}_{n+1}$.*

Proof. Since $g \in \mathcal{S}_{n+1}$, it follows that $\mathcal{N}(g)$ is strictly decreasing and there exists $m \geqslant 0$ such that $\mathcal{N}(g) \leqslant \mathcal{N}(h_{n+1}^m)$. By Propositions 4.1 and 4.3, for all $t > 2m^2$, we have

$$\mathcal{N}(g)(t) \leqslant \mathcal{N}(h_{n+1}^m)(t) < \mathcal{N}(h_n)(t) \leqslant \mathcal{N}(fh_n)(t).$$

By Proposition 4.6, it follows that for all t sufficiently large,

$$\mathcal{N}(g + fh_n)(t) \leqslant \mathcal{N}(g)(t) \leqslant \mathcal{N}(h_{n+1}^m)(t).$$

By Lemma 2.1, it follows that $g + fh_n \in \mathcal{S}_{n+1}$. □

Acknowledgments. We thank Kevin Buzzard for helpful comments on an earlier draft. We are grateful to the anonymous referee for helpful comments and suggestions. The first author gratefully acknowledges support from the National Science Foundation through award DMS-1604148.

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