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SYLVESTER'S PROBLEM FOR SPREADS OF CURVES

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Introduction. Spreads of curves were introduced by Grünbaum in [**1**]. A spread of curves is a continuous family of simple arcs in the real plane, every two of which intersect in exactly one point. A spread is the continuous analogue of a finite arrangement of pseudolines in the plane. Sylvester's problem for finite arrangements of pseudolines asks if every non-trivial arrangement has a simple vertex, that is a point contained in exactly two pseudolines of the arrangement. This question was answered in the affirmative by Kelly and Rottenberg [**5**]. One interesting feature of this result is that it does not depend on the pseudolines being straight lines.

Here we settle Sylvester's problem for spreads. We show that every non-trivial spread of line segments has uncountably many simple vertices. But we also give examples of non-trivial spreads with no simple vertices. Thus there is an essential difference between general spreads and spreads of line segments. We find some necessary and some sufficient conditions for a spread to have a denumerable number of simple vertices. This investigation leads to the concept of a 2-isolated point in a spread which permits Sylvester's problem for spreads to be answered affirmatively.

The author was introduced to the subject of spreads of curves by B. B. Phadke while at The Flinders University of South Australia.

1. Preliminary concepts and notation. Let C be a closed Jordan arc in the real plane and let D be its interior. Then a family L of simple arcs in the plane (homeomorphic images of $[0, 1]$) is called a *spread of curves* (or simply a *spread*) on $C \cup D$ if there is a mapping $p \mapsto l(p)$ from C onto L such that, for all $p \in C$,

(i) $l(p) \subset C \cup D$
(ii) $l(p)$ is the unique curve in L with endpoint p
(iii) the other endpoint p^* of $l(p)$ is in C
(iv) $l(p) \cap C = \{p, p^*\}$

and such that

(v) if $l_1, l_2 \in L$ and $l_1 \neq l_2$, then $l_1 \cap l_2$ is a singleton
(vi) $l : C \to L$ is a continuous map, where L is given the Hausdorff metric.

The above definition of a spread of curves is essentially the same as that used in [**1**], [**2**] and [**6**].

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Since C is homeomorphic to a circle, C may be given an orientation. For $p \in C$, let $C^+(p)$ be the open subarc of C formed by going from p to p^* in an anti-clockwise direction. Let $C^-(p) = C^+(p^*)$. So $C = \{p\} \cup C^+(p) \cup \{p^*\} \cup C^-(p)$.

Now for $p \in C$,

$$D = (l(p) \cap D) \cup p^+ \cup p^-$$

where p^σ is the component of $D \backslash l(p)$ with boundary $l(p) \cup C^\sigma(p)$ for $\sigma = \pm$. Note that $p^- = (p^*)^+$.

For every pair of points d_1, d_2 in $C \cup D$ with $d_1 \neq d_2$ there is at most one $l \in L$ with $d_1, d_2 \in l$. If such an l exists, let (d_1, d_2) be the open subarc of l from d_1 to d_2. Let $[d_1, d_2) = \{d_1\} \cup (d_1, d_2)$, $(d_1, d_2] = (d_1, d_2) \cup \{d_2\}$ and $[d_1, d_2] = (d_1, d_2) \cup \{d_1, d_2\}$.

The following elementary results are well known and are easily proved using continuity arguments.

1.1 Proposition. *The map* $(p, q) \mapsto l(p) \cap l(q)$ *is continuous on* $\{(p, q) \in C \times C: l(p) \neq l(q)\}$.

1.2 Proposition. *If* l_1 *and* l_2 *are distinct curves in* L, *then the endpoints of* l_1 *separate on* C *the endpoints of* l_2.

1.3 Proposition. *The map* $p \mapsto p^*$ *is continuous on* C.

1.4 Proposition. *If* $p \in C$ *and* $d \in p^\sigma$ *(where* $\sigma = \pm$*), then there exists a neighbourhood* V *in* C *of* p *such that* $d \in q^\sigma$ *for all* $q \in V$.

The remark that 1.2, 1.3 and 1.4 need not be true if condition (v) in the definition of a spread is deleted.

By 1.2, we may make the following definition. For $p, q \in C$ with $q \in C^+(p) \cup C^-(p)$, let (p, q) be the open subarc of C from p to q which does not contain p^* or q^*. Let $[p, q) = \{p\} \cup (p, q)$, $(p, q] = (p, q) \cup \{q\}$ and $[p, q] = (p, q) \cup \{p, q\}$.

The next two propositions are again straightforward.

1.5 Proposition. *Let* $p, q \in C$ *and* $d \in p^+ \cap q^-$. *Then there exists* $r \in (p, q)$ *such that* $d \in l(r)$.

1.6 Proposition. *Every point in* D *lies on at least one curve in* L.

A point $d \in D$ is called a *k-tuple* (*k-fold*) *point of* L where $k \in \mathbf{N} \cup \{\infty\}$ if precisely (at least) k curves of L pass through d. Let $T_k(L)$ be the set of k-tuple points of L and let $F_k(L)$ be the set of k-fold points of L. So 1.6 may be expressed as $D = F_1(L)$. A spread L is called *trivial* if $F_2(L)$ is a singleton. That is, L is trivial if all of its curves pass through a common point.

For $d \in D$, let $l(d) = \{l \in L : d \in L\}$. For $z \in C \cup D$, let

$$l^{-1}(z) = \{p \in C : z \in l(p)\}.$$

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If A is a subset of $C \cup D$, let ∂A denote the boundary of A. While if A is a subset of $l \in L$, let $\partial_l A$ denote the boundary of A in the relative topology of l.

2. Examples of spreads. Many examples of the so called natural spreads are given in [**2**]. In most cases the curves of the spread are line segments. Here we give an example of a spread which is not in general a spread of line segments.

Identify the real plane with the complex numbers $\mathbf{C}$. Let $D = \{z \in \mathbf{C} : |z| < 1\}$ and $C = \{z \in \mathbf{C} : |z| = 1\}$. Let $g : (0, \pi) \to (-1, 1)$ be continuous. Now construct the spread $L(g)$ as follows. For $\theta \in (0, \pi)$, let $l(e^{i\theta})$ be the line segment from $e^{i\theta}$ to $g(\theta)$ together with the line segment from $g(\theta)$ to $-e^{i\theta}$. Let $l(1)$ be the line segment from 1 to -1. For $\theta \in [\pi, 2\pi)$, let $l(e^{i\theta}) = l(-e^{i\theta})$. Then $L(g) = \{l(e^{i\theta}) : \theta \in [0, 2\pi]\}$ is a spread of curves on $C \cup D$.

It will be shown that $L(g)$, for suitable choices of g, has properties very much different from those of spreads of line segments.

3. The boundary of $F_2(L)$.

3.1 PROPOSITION. *We have* $T_2(L) \subset \partial F_2(L)$.

Proof. Let $d \in T_2(L)$ and let $p, q \in l^{-1}(d)$ with $q \in C^+(p)$. We may assume that $d \in s^-$ for all $s \in (p, q)$. Now either $d \in t^-$ for all $t \in (p, q^*)$ or $d \in t^+$ for all $t \in (p, q^*)$. In the former case, $(d, q^*) \subset T_1(L)$ and in the latter case, $(p, d) \subset T_1(L)$. Hence $d \in \partial F_2(L)$.

Thus the set of simple vertices of L is contained in the boundary of the set of 2-fold points. We will later show that all but a denumerable number of points of $F_2(L) \cap \partial F_2(L)$ belong to $T_2(L)$. We also remark that $F_2(L)$ is contained in the closure of $F_3(L)$ (see [**1**]), from which it follows that $\partial F_2(L) = \partial F_3(L)$. Little else can be said about $\partial F_2(L)$ for general spreads L. The set $\partial F_2(L)$ may be contained in $T_1(L)$ or contained in $F_2(L)$. The set $\partial F_2(L)$ may consist of C together with various subarcs of curves of L. The set $T_1(L) \cap \partial F_2(L)$ may be a closed Jordan arc which intersects each curve of L in exactly two points. On the other hand, $T_1(L) \cap \partial F_2(L)$ may contain a non-trivial subarc of a curve in L. For examples of spreads of the above mentioned types, refer to those constructed in 3.8.

Since $q \to l(p) \cap l(q)$ is continuous on the connected space $C^+(p)$, $F_2(L) \cap l(p)$ is a subarc of $l(p)$. Notice that $F_2(L) \cap l(p)$ is a singleton if and only if L is trivial. If $d \in \partial_l(F_2(L) \cap l)$ for some $l \in L$, then $d \in \partial F_2(L)$. It is possible for $d \in \partial F_2(L)$, but $d \notin \partial_l(F_2(L) \cap l)$ for all $l \in L$.

3.2 PROPOSITION. *If* $d \in F_2(L) \cap \partial F_2(L)$, *then there exists* $l \in L$ *such that* $d \in \partial_l(F_2(L) \cap l)$.

Proof. Let $d \in F_2(L) \cap \partial F_2(L)$. Then there exists a sequence $\{d_n\}$ in $T_1(L)$ such that $d_n \to d$. If there exists $l \in L$ such that $d \in l$ and $d_n \in l$ for infinitely many n, then $d \in \partial_l(F_2(L) \cap l)$. So assume that no such l exists. Since $d \in F_2(L)$, we may assume, by taking a subsequence of $\{d_n\}$ if necessary, that there are $p, q \in l^{-1}(d)$ with $q \in C^+(p)$ and $d_n \in p^+ \cap q^-$ for all n. For every n

there exists $p_n \in (p, q)$ such that $d_n \in l(p_n)$. We may further assume that $p_n \to r \in [p, q]$. Since $d_n \to d$ and $l(p_n) \to l(r)$, $d \in l(r)$. Suppose also that $r \neq q$ (if $r \neq p$, then a similar argument may be given).

We claim that $d \in \partial_{l(r)}(F_2(L) \cap l(r))$. If this is not true, then there exists $s \in C^+(r)$ such that $d \in s^+$. So for all n large enough, $d_n \in s^+ \cap q^-$. Hence $l^{-1}(d_n) \cap (s, q)$ is not empty for all n large enough. But $p_n \in l^{-1}(d_n)$ and $p_n \to r \notin [s, q] \subset C^+(r)$, which contradicts the fact that $d_n \in T_1(L)$ for all n.

3.3 PROPOSITION. *Suppose $p \in C$ and $d \in p^+$ are such that*

$$p^+ \cap F_2(L) \cap \partial_l(F_2(L) \cap l) = \emptyset$$

for all $l \in l(d)$. Then $d \in F_\infty(L) \cup T_1(L)$.

Proof. Suppose that $d \in T_k(L)$ where $1 < k < \infty$. Let $p_1 \in C^+(p)$ be such that $l^{-1}(d) \cap [p^*, p_1] = \{p_1\}$. Hence $d \in r^-$ for all $r \in (p^*, p_1)$ as $d \in (p^*)^-$. Since $d \in T_k(L)$, there exists a maximal sequence $\{p_n\}_{n=1}^N$ with the properties

(i) $\{p_n\}_{n=1}^N \subset C^+(p) \cap l^{-1}(d)$
(ii) $p_n \in (p, p_{n-1})$ for $n = 2, \ldots, N$
(iii) $d \in r^-$ for all $r \in (p_{n-1}, p_n)$ for $n = 1, \ldots, N$ (where $p_0 = p^*$; see fig. 1)

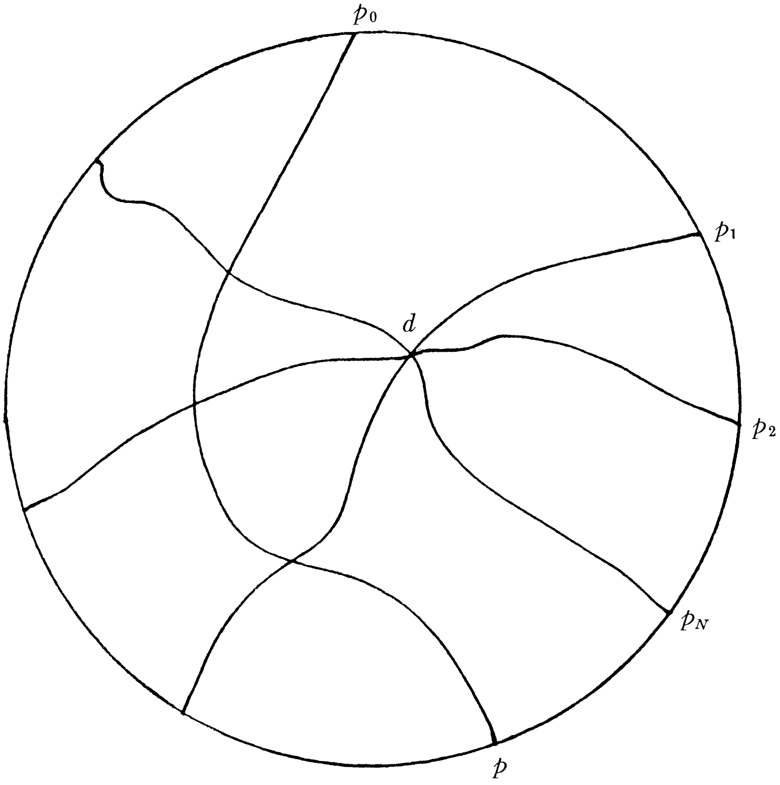

FIG. 1

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So $l(r) \cap (d, p_N) = \emptyset$ for all $r \in (p^*, p_N)$. Now if $l(r) \cap (d, p_N) = \emptyset$ for all $r \in (p_N, p)$, then

$$d \in p^+ \cap F_2(L) \cap \partial_{l(p_N)}(F_2(L) \cap l(p_N)),$$

a contradiction. Therefore there exists $r \in (p_N, p)$ such that $d \in r^-$. Hence $l^{-1}(d) \cap (p_N, p) \neq \emptyset$. There exists $p_{N+1} \in l^{-1}(d) \cap (p_N, p)$ such that $l^{-1}(d) \cap (p_N, p_{N+1}) = \emptyset$. Hence either $d \in r^-$ for all $r \in (p_N, p_{N+1})$ or $d \in r^+$ for all $r \in (p_N, p_{N+1})$.

If the latter is true, then $l(r) \cap (d, p_N) = \emptyset$ for all $r \in (p^*, p_{N+1})$. So there exists $s \in [p_{N+1}, p]$ such that

$$l(s) \cap l(p_N) = p^+ \cap F_2(L) \cap \partial_{l(p_N)}(F_2(L) \cap l(p_N)),$$

a contradiction.

Therefore $d \in r^-$ for all $r \in (p_N, p_{N+1})$. But this means that $\{p_n\}_{n=1}{}^{N+1}$ has the properties (i), (ii) and (iii). Hence no such maximal sequence exists and $d \in F_\infty(L) \cup T_1(L)$.

3.4 COROLLARY. *If $d \in D$ is such that $l \cap F_2(L)$ is open in l for all $l \in l(d)$, then $d \in F_\infty(L) \cup T_1(L)$.*

Proof. There exists $p \in C$ such that $d \in p^+$. The corollary now follows immediately from 3.3.

3.5 COROLLARY. *If $F_2(L)$ is open then $F_2(L) = F_\infty(L)$.*

Proof. Apply 3.4 to each $d \in F_2(L)$.

3.6 An example of a spread L such that $F_2(L) = F_\infty(L) = D$.

Let $g : (0, \pi) \to (-1, 1)$ be continuous, surjective and such that

$$\liminf_{x \to a} \frac{g(x) - g(a)}{x - a} = -\infty \quad \text{for all} \quad a \in (0, \pi)$$

and

$$\limsup_{x \to a} \frac{g(x) - g(a)}{x - a} = \infty \quad \text{for all} \quad a \in (0, \pi).$$

One can prove that such a function exists by using the Baire category theorem. A concrete example of such a function may be constructed from the function

$$f : (0, 50/101) \to (0, 500/909)$$

defined by $f(x) = \sum_{n=0}^{\infty} \langle 10^{2n} x \rangle / 10^n$, where $\langle y \rangle$ is the distance from y to the nearest integer. Consider the spread $L(g)$ as defined in § 2. We will show that $F_2(L(g)) = D$, whence $F_\infty(L(g)) = D$ by 3.5.

Clearly, $(-1, 1) \subset F_2(L(g))$ as g is surjective. Let $d \in 1^+$ and let $p = e^{i\theta} \in l^{-1}(d) \cap C^+(1)$ (recall that $D = F_1(L(g))$). For x in a neighbourhood of θ, let $h(x)$ be where the straight line through d and e^{ix} cuts $l(1)$. So $h(\theta) = g(\theta)$. See fig. 2.

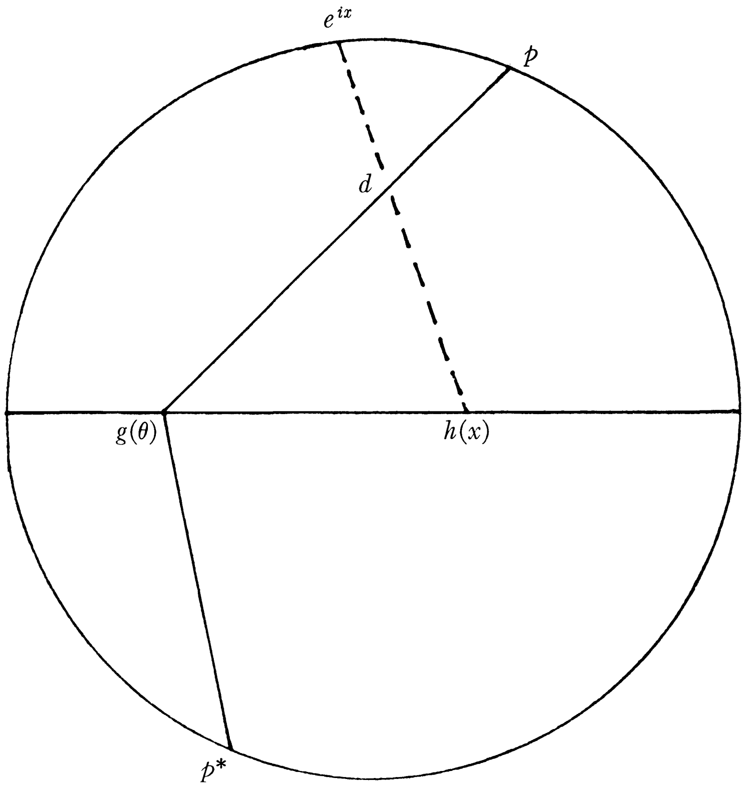

FIG. 2

Now there exists a sequence $\{\theta_n\}$ in $(0, \pi)$ such that $\theta_n \to \theta$, $\theta_n \neq \theta$ and $(g(\theta_n) - g(\theta))/(\theta_n - \theta) \to \infty$. If $l(e^{i\theta_n}) \cap (d, p) = \emptyset$ for all n, then

$$|g(\theta_n) - g(\theta)| \leqq |h(\theta_n) - h(\theta)|$$

for all n sufficiently large, which contradicts the fact that h is differentiable at θ. Hence $l(e^{i\theta_n}) \cap (d, p) \neq \emptyset$ for some n, which implies that $d \in F_2(L(g))$. If $d \in 1^-$, one uses a similar argument but with $(g(\theta_n) - g(\theta))/(\theta_n - \theta) \to -\infty$ and $h(x)$ being where the straight line through d and $-e^{ix}$ cuts $l(1)$.

Thus $F_2(L(g)) = D$ and by 3.5,

$$F_2(L(g)) = F_\infty(L(g)) = D.$$

In particular this means that $T_2(L(g)) = \emptyset$. That is, Sylvester's problem for general spreads has a negative answer.

The next proposition is an example of a statement independent of the theory of spreads of curves, but whose proof is made transparent by using the theory of spreads.

3.7 PROPOSITION. *Let I be a non-trivial subinterval of* $\mathbf{R}$. *Suppose that $h : I \to \mathbf{R}$ is continuous with range I' and that*

$$\liminf_{x\to a} \frac{h(x) - h(a)}{x - a} = -\infty \quad \textit{and} \quad \limsup_{x\to a} \frac{h(x) - h(a)}{x - a} = \infty$$

for all a in the interior of I. Then h takes every value in the interior of I' infinitely many times.

Proof. Let y be in the interior of I'. Let $m, M \in I'$ be such that $m < y < M$. Let $t_1, t_2 \in h^{-1}(\{m, M\})$ be such that $t_1 < t_2$, $h(t_1) \neq h(t_2)$ and $h((t_1, t_2)) = (m, M)$. Define $g : (0, \pi) \to (-1, 1)$ by

$$g(\theta) = \frac{2}{M - m}\left(h\left(\frac{\theta(t_2 - t_1)}{\pi} + t_1\right) - m\right) - 1.$$

Then g may be used as in 3.6 to construct the spread $L(g)$ which has the property that $F_\infty(L(g)) = D$. From $(-1, 1) \subset F_\infty(L(g))$, we deduce that g takes every value in $(-1, 1)$ infinitely many times. Hence h takes the value y infinitely many times.

We will now modify the example given in 3.6 to demonstrate that the cardinality of $T_2(L)$ may take any value not more than $\aleph_0$. An example of a spread L such that $T_2(L)$ contains a non-trivial subarc of a curve in L is given in [**2**].

3.8 THEOREM. *For all $n \in \mathbf{N}$, let $k_n \in \{0\} \cup \mathbf{N} \cup \{\aleph_0\}$. Then there exists a spread L such that* card $T_{n+1}(L) = k_n$ *for all $n \in \mathbf{N}$.*

Proof. We may assume that not all the k_n are zero, for otherwise the example of a spread given in 3.6 would have the desired properties. Let $J \subset \mathbf{N}$ and $m_j \in \mathbf{N}, j \in J$, be such that

(i) $J = \mathbf{N}$ or $J = \{1, \ldots, M\}$ for some $M \in \mathbf{N}$
(ii) card $\{j \in J : m_j = n\} = k_n$ for all $n \in \mathbf{N}$.

For all $j \in \{0\} \cup J$, let $d_j = 1 - 1/(j + 1)$ and $\theta_j = \pi d_j$. Let $s = \sup\{d_j : j \in J\}$ and $\psi = \sup\{\theta_j : j \in J\}$. For each $j \in J$ let $g_j : [\theta_{j-1}, \theta_j] \to [d_{j-1}, d_j]$ be such that

(i) g_j is continuous
(ii) $g_j(\theta_{j-1}) = d_{j-1}$, $g_j(\theta_j) = d_j$

(iii) $\displaystyle\liminf_{x\to a} \frac{g_j(x) - g_j(a)}{x - a} = -\infty$, for all $a \in (\theta_{j-1}, \theta_j)$

(iv) $\displaystyle\limsup_{x\to a} \frac{g_j(x) - g_j(a)}{x - a} = \infty$, for all $a \in (\theta_{j-1}, \theta_j)$

(v) $\displaystyle\limsup_{x\to a^-} \frac{g_j(x) - g_j(a)}{x - a} = \infty$ for $a = \theta_j$

(vi) d_j is attained exactly m_j times
(vii) $g_j(x) > d_{j-1}$ for all $x \in (\theta_{j-1}, \theta_j]$.

Such a function g_j may be easily constructed from the function $x \mapsto f(\langle x - \frac{1}{4}\rangle + \frac{1}{4})$, where $f(x) = \sum_{n=0}^{\infty} \langle 10^{2n}x\rangle/10^n$.

Let $g : (0, \pi) \to (-1, 1)$ be such that

(i) g is continuous
(ii) $g = g_j$ on $(\theta_{j-1}, \theta_j]$ for all $j \in J$
(iii) $g(\psi, \pi) = (s, 1)$

(iv) $\limsup_{x\to a} \dfrac{g(x) - g(a)}{x - a} = \infty$ for all $a \in (\psi, \pi)$

(v) $\liminf_{x\to a} \dfrac{g(x) - g(a)}{x - a} = -\infty$ for all $a \in (\psi, \pi)$.

We will show that the spread $L(g)$ has the desired properties.

Clearly, $(-1, d_0] \subset T_1(L(g))$. By 3.7 and the construction of $L(g)$, $(d_{j-1}, d_j) \subset F_\infty(L(g))$ for all $j \in J$, $(s, 1) \subset F_\infty(L(g))$ and $d_j \in T_{mj+1}(L(g))$ for all $j \in J$. Since

$$\limsup_{x\to a} \frac{g(x) - g(a)}{x - a} = \infty \quad \text{for all} \quad a \in (0, \pi)$$

then $1^+ \subset F_2(L(g))$ by the reasoning used in 3.6. Let $z \in 1^+$. Then $l \cap 1^+ \subset F_2(L(g))$ for all $l \in l(z)$. Hence $z \in F_\infty(L(g))$ by 3.3. Now let $z \in 1^-$. If $z \in l(e^{i\theta_j})$ for some $j \in J$, then $z \in T_1(L(g))$. If $z \notin l(e^{i\theta_j})$ for all $j \in J$, then

$$\liminf_{x\to a} \frac{g(x) - g(a)}{x - a} = -\infty \quad \text{for every} \quad a \in (0, \pi) \quad \text{such that} \quad z \in l(e^{ia}).$$

By reasoning similar to that used in 3.6, $l \cap (-1)^+ \subset F_2(L(g))$ for all $l \in l(z)$. Hence $z \in F_\infty(L(g))$ by 3.3. Putting the above together yields that

$$\operatorname{card} T_{n+1}(L(g)) = \operatorname{card} \{j \in J : m_j = n\} = k_n$$

for all $n \in \mathbf{N}$.

Two spreads, L_1 and L_2, are called *combinatorially isomorphic* if there exists a bijection $\eta : L_1 \to L_2$ such that l_1, l_2, l_3 in L_1 are concurrent if and only if $\eta(l_1), \eta(l_2), \eta(l_3)$ in L_2 are concurrent. Clearly card $T_k(L_1) =$ card $T_k(L_2)$ for all $k \in \mathbf{N} \cup \{\infty\}$ if L_1 and L_2 are combinatorially isomorphic. So by 3.8 there are uncountably many pairwise non-combinatorially isomorphic spreads.

The next three lemmas are directed towards proving that $F_3(L) \cap \partial F_2(L)$ is denumerable for every spread L. The latter two are very important and are the key to the subsequent theory.

3.9 LEMMA. *Let $B \subset C \times C \times C$ have the following properties:*

(i) $(b_1, b_2, b_3) \in B \Rightarrow b_2 \in C^+(b_1), b_3 \in C^+(b_2)$

(ii) *If $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$ are in B, then at least one of the following occurs*

(1) $b = c$
(2) $(b_1, b_2) \cap (c_1, c_2) = \emptyset$
(3) $(b_2, b_3) \cap (c_2, c_3) = \emptyset$:
Then B is denumerable.

Proof. Fix a homeomorphism between C and the unit circle. If C_1 is a subarc of C, let $\mu(C_1)$ be the length of the corresponding subarc of the unit circle. For $n \in \mathbf{N}$, let

$$B_n = \{(b_1, b_2, b_3) \in B : \mu(b_1, b_2) > 2\pi/n, \mu(b_2, b_3) > 2\pi/n\}.$$

Then $B = \bigcup_{n \in \mathbf{N}} B_n$. To complete the proof it suffices to show that B_n is finite for all $n \in \mathbf{N}$. Now by Ramsey's theorem, there exists a number N_n such that if B_n contains at least N_n elements, then either

(i)′ B_n contains a set Y of $n + 1$ elements, every two of which satisfy (2) or

(ii)′ B_n contains a set Z of $n + 1$ elements, every two of which satisfy (3). If (i)′ occurs, then $\sum_{b \in Y} \mu(b_1, b_2) > ((n + 1)/n)2\pi$, a contradiction. If (ii)′ occurs then $\sum_{b \in Z} \mu(b_2, b_3) > ((n + 1)/n)2\pi$, a contradiction. Hence B_n is finite as required.

3.10 LEMMA. *Let $p \in C$, $A \subset D$, $\tau_1 : A \to C^+(p)$, $\tau_2 : A \to C^+(p)$ and $\rho : A \to C^+(p)$ satisfy the following conditions for all $a \in A$:*
(i) $\tau_1(a) \in (p, \rho(a))$, $\tau_2(a) \in (\rho(a), p^*)$
(ii) $\tau_1(a), \rho(a), \tau_2(a) \in l^{-1}(a)$
(iii) $a \notin t^+$ for all $t \in [\tau_1(a), \tau_2(a)]$
(iv) $a_1 \notin \tau_2(a_2)^+ \cap \tau_1(a_2)^-$ *for all* $a_1, a_2 \in A$.
Then A is denumerable.

Proof. We will show that if $a, b \in A$ then at least one of the following occurs
(vi) $a = b$
(vii) $(\tau_1(a), \rho(a)) \cap (\tau_1(b), \rho(b)) = \emptyset$
(viii) $(\rho(a), \tau_2(a)) \cap (\rho(b), \tau_2(b)) = \emptyset$.
By applying 3.9 to $B = \{(\tau_1(a), \rho(a), \tau_2(a)) : a \in A\}$ it would then follow that A is denumerable as the map $a \mapsto (\tau_1(a), \rho(a), \tau_2(a))$, $a \in A$, is injective (by (i) and (ii)). To prove the above it suffices to consider the following cases
(1) : $\tau_1(b) = \tau_1(a)$ but $b \neq a$
(2) : $\tau_1(b) \in (\tau_1(a), \rho(a))$
(3) : $\tau_1(b) \in [\rho(a), p^*)$.

Case (1). By symmetry we may assume that $b \in (\tau_1(a), a)$ (fig. 3). If $\rho(b) \in (\tau_1(a), \tau_2(a))$, then $a \in \rho(b)^+$, which contradicts (iii). Hence $\rho(b) \in [\tau_2(a), p^*)$ and therefore $(\rho(a), \tau_2(a)) \cap (\rho(b), \tau_2(b)) = \emptyset$.

Case (2). Since $a \notin \tau_1(b)^+$ by (iii), one of the following two subcases occurs
(2a) : $a \in l(\tau_1(b))$
(2b) : $a \in \tau_1(b)^-$.

Case (2a). See fig. 4.

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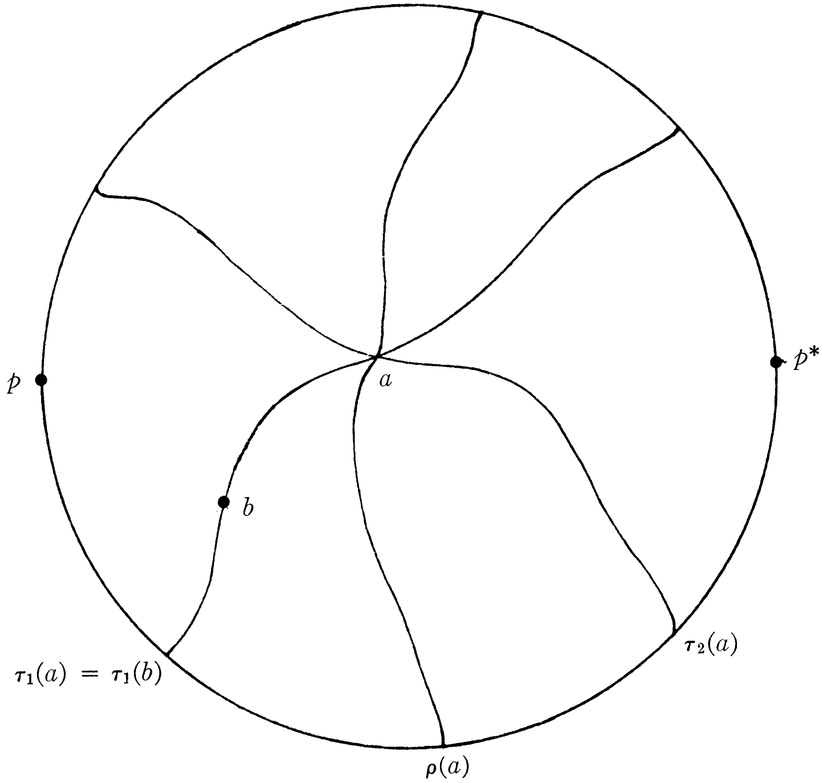

FIG. 3

By (iv), $b \in (\tau_1(b), a)$. If $\rho(b) \in (\tau_1(b), \tau_2(a))$, then $a \in \rho(b)^+$, which contradicts (iii). Hence $\rho(b) \in [\tau_2(a), p^*)$ and therefore $(\rho(a), \tau_2(a)) \cap (\rho(b), \tau_2(b)) = \emptyset$.

Case (2b). Let $\{c\} = l(\tau_1(b)) \cap (\rho(a), a)$ (fig. 5).
We distinguish the two subcases
$(2b\alpha) : b \in (\tau_1(b), c]$
$(2b\beta) : b \in (c, \tau_1(b)^*)$.

Case $(2b\alpha)$. If $\tau_2(b) \in (\rho(a), p^*)$, then $a \in \tau_2(b)^+ \cap \tau_1(b)^-$, which contradicts (iv). Therefore $\tau_2(b) \in (\tau_1(b), \rho(a)]$ and hence

$$(\rho(a), \tau_2(a)) \cap (\rho(b), \tau_2(b)) = \emptyset.$$

Case $(2b\beta)$. If $\tau_2(b) \in (\rho(a), p^*)$, then $\rho(a) \in (\tau_1(b), \tau_2(b))$ and $b \in \rho(a)^+$, which contradicts (iii). Therefore $\tau_2(b) \in (\tau_1(b), \rho(a)]$ and hence

$$(\rho(a), \tau_2(a)) \cap (\rho(b), \tau_2(b)) = \emptyset.$$

Case (3). Clearly, $(\tau_1(a), \rho(a)) \cap (\tau_1(b), \rho(b)) = \emptyset$.

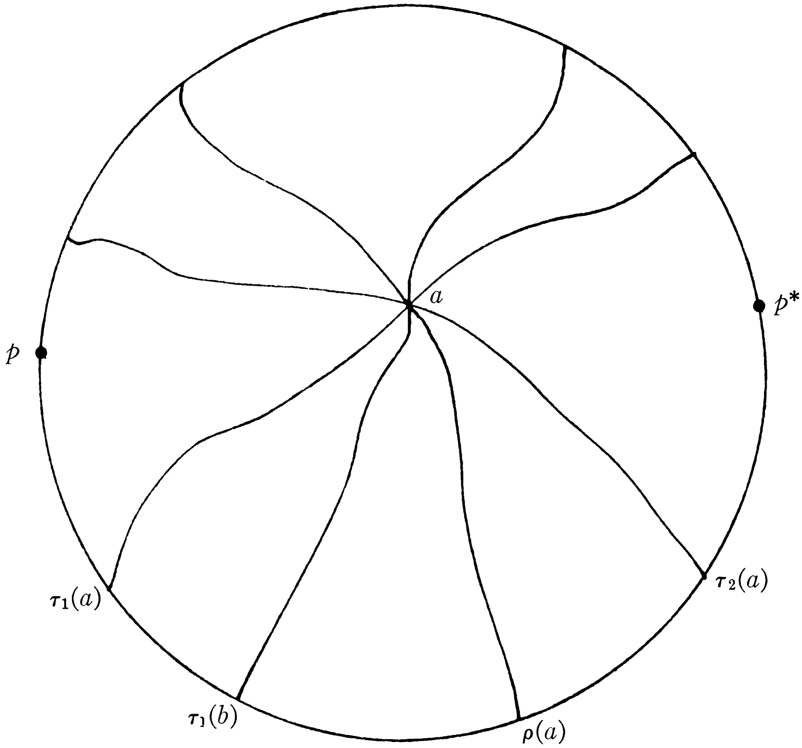

FIG. 4

If condition (iii) in the statement of 3.10 is replaced by
(iii)′ $a \notin t^-$ for all $t \in [\tau_1(a), \tau_2(a)]$,
then A would obviously be denumerable as before. For reference we state this fact in the next lemma.

3.11 LEMMA. *Let $p \in C$, $A \subset D$, $\tau_1 : A \to C^+(p)$, $\tau_2 : A \to C^+(p)$ and $\rho : A \to C^+(p)$ satisfy conditions* (i), (ii) *and* (iv) *of* 3.10 *and* (iii)′ *for all $a \in A$. Then A is denumerable.*

3.12 THEOREM. *For every spread L, $F_3(L) \cap \partial F_2(L)$ is denumerable.*

Proof. Let $A = F_3(L) \cap \partial F_2(L)$. Then for every $a \in A$ there exists by 3.2 $\tau_1(a) \in l^{-1}(a)$ such that $a \notin t^+$ for all $t \in C^+(\tau_1(a))$. Since $A \subset F_3(L)$, for every $a \in A$ there exist $\rho(a), \tau_2(a) \in l^{-1}(a)$ such that $\tau_2(a) \in C^+(\tau_1(a))$ and $\rho(a) \in (\tau_1(a), \tau_2(a))$ (fig. 6).

Let $\{p_n\}$ be a countable dense subset of C. Then $A = \bigcup_n A_n$ where

$$A_n = \{a \in A : \tau_1(a), \rho(a), \tau_2(a) \in C^+(p_n)\}.$$

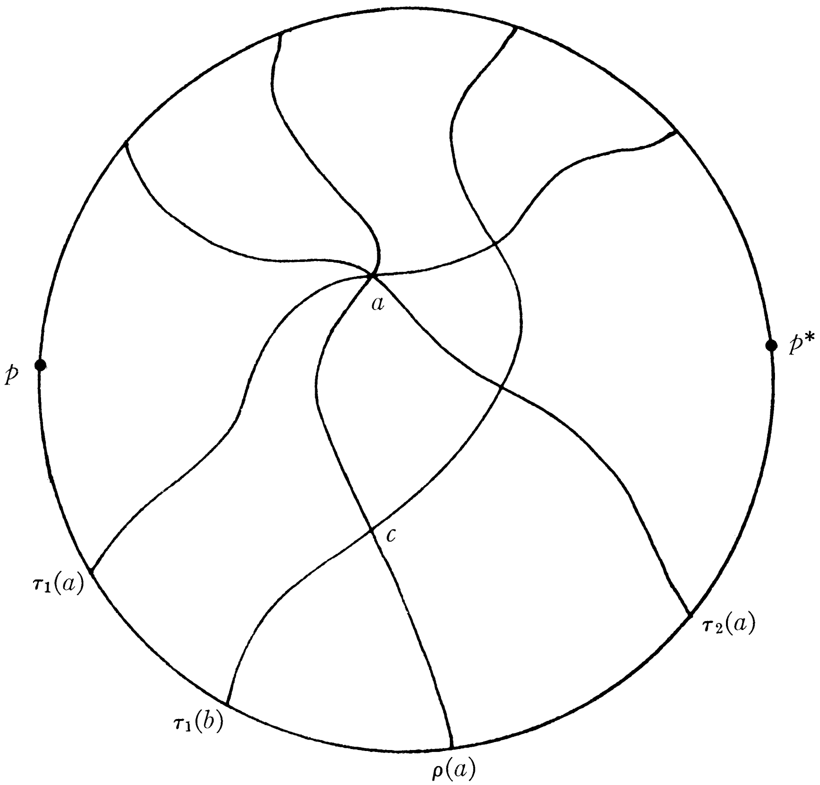

FIG. 5

So to prove that A is denumerable, it suffices to show that each A_n is denumerable. Clearly, p_n, A_n, $\tau_i : A_n \to C^+(p_n)$ $(i = 1, 2)$ and $\rho : A_n \to C^+(p_n)$ satisfy (i), (ii) and (iii) of 3.10. We will now check condition (iv).

Let $a_1, a_2 \in A_n$. If $a_1 \in \tau_2(a_2)^+ \cap \tau_1(a_2)^-$ then either $\tau_2(a_2) \in C^+(\tau_1(a_1))$ and $a_1 \in \tau_2(a_2)^+$ or $\tau_1(a_2)^* \in C^+(\tau_1(a_1))$ and $a_1 \in \tau_1(a_2)^{*+}$, which contradicts the above. Hence (iv) is satisfied. Therefore A_n is denumerable by 3.10.

3.13 COROLLARY. *For every spread L, $T_2(L)$ is denumerable if and only if $F_2(L) \cap \partial F_2(L)$ is denumerable.*

Proof. This follows immediately from 3.1 and 3.12.

4. Isolated curves in $l(d)$. In this section we analyse the properties of spreads L such that $T_2(L)$ is denumerable. This will lead to a sufficient condition for a spread L to have uncountably many simple vertices. An application of this result will show that every non-trivial spread of line segments has uncountably many simple vertices. Another consequence will be the recognition of the importance of isolated curves in $l(d)$. A new definition of a k-tuple

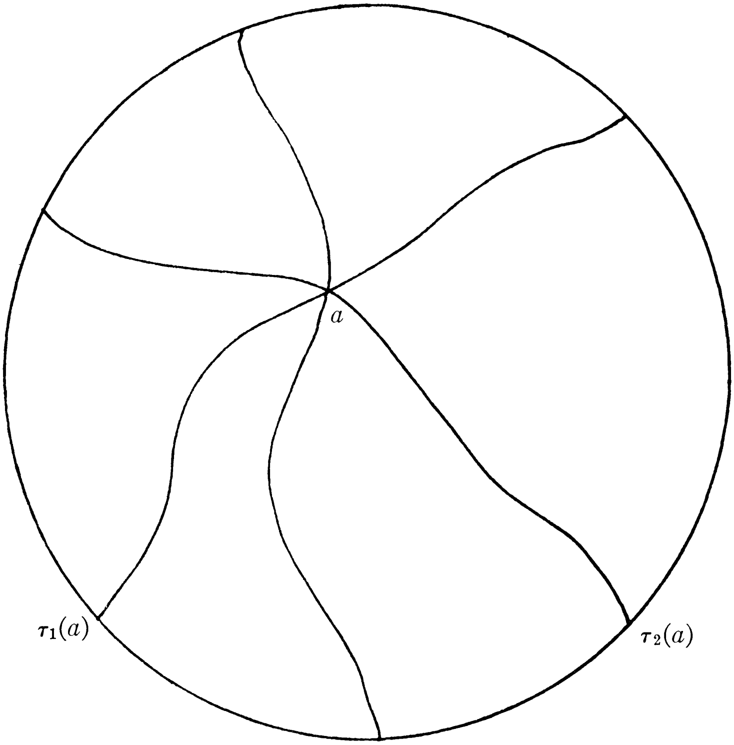

FIG. 6

point will ensue in Section 5 which allows an affirmative answer to Sylvester's problem.

The following lemma about isolated curves in $l(d)$ is the first step towards describing the topology of $l(d)$ for all but a denumerable number of points d on a curve in a spread L with $T_2(L)$ denumerable. The lemma will also play a part in 5.1.

4.1 LEMMA. *Let* $p \in C$, $A \subset D$, $\tau_1 : A \to C^+(p)$, $\tau_2 : A \to C^+(p)$ *and* $\rho : A \to C^+(p)$ *have the following properties for all* $a \in A$*:*

(i) $T_2(L) \cap l(\rho(a)) = \emptyset$

(ii) $\tau_1(a) \in (p, \rho(a))$, $\tau_2(a) \in (\rho(a), p^*)$

(iii) $l^{-1}(a) \cap [\tau_1(a), \tau_2(a)] = \{\tau_1(a), \rho(a), \tau_2(a)\}$

(iv) $a_1 \notin \tau_2(a_2)^+ \cap \tau_1(a_2)^-$ *for all* $a_1, a_2 \in A$

(v) $a_1 \notin \tau_2(a_2)^- \cap \tau_1(a_2)^+$ *for all* $a_1, a_2 \in A$.

Then A is denumerable.

Proof. For $i \in \{1, 2\}$ and $\sigma \in \{+, -\}$, let

$$A_i^\sigma = \{a \in A : a \in t^\sigma \text{ for all } t \in (\rho(a), \tau_i(a))\}.$$

Then by (iii), A is the disjoint union of $A^{++} = A_1^+ \cap A_2^+$, $A^{+-} = A_1^+ \cap A_2^-$, $A^{-+} = A_1^- \cap A_2^+$ and $A^{--} = A_1^- \cap A_2^-$. It suffices to show that all the $A^{\sigma\mu}$ are denumerable.

$A^{\sigma\sigma}$ is denumerable ($\sigma = \pm$). For all $a \in A^{\sigma\sigma}$ and for all $t \in [\tau_1(a), \tau_2(a)]$, we have $a \notin t^{-\sigma}$. So A^{++} and A^{--} are denumerable by 3.10 and 3.11.

A^{+-} is denumerable. Let $a \in A^{+-}$. Then $l(t) \cap [\rho(a), a] = \emptyset$ for all $t \in (\tau_1(a), \tau_2(a))$ with $t \neq \rho(a)$. Hence there exists a unique

$$\lambda(a) \in (\rho(a), a] \cap F_2(L) \cap \partial_{l(\rho(a))}(F_2(L) \cap l(\rho(a))).$$

By (i), $\lambda(a) \in F_3(L)$. So $\lambda(a) \in F_3(L) \cap \partial F_2(L)$ which is denumerable by 3.12. To prove that A^{+-} is denumerable, it suffices to show that $\lambda : A^{+-} \to F_3(L) \cap \partial F_2(L)$ is injective. Let $a, b \in A^{+-}$ be such that $\lambda(a) = \lambda(b)$.

If $\rho(a) \neq \rho(b)$ then $\lambda(a) \in l(t)$ for all $t \in (\rho(a), \rho(b))$, and hence there exists $t \in (\tau_1(a), \tau_2(a))$, $t \neq \rho(a)$, such that $\lambda(a) \in l(t) \cap (\rho(a), a]$, a contradiction.

Hence $\rho(a) = \rho(b)$. We may assume by symmetry that $a \in (\rho(a), b]$. Then

$$b \in (\tau_2(a)^+ \cap \tau_1(a)^-) \cup \{a\}.$$

So by (iv), $a = b$ as required.

A^{-+} is denumerable. Define $\tau_i^* : A \to C^+(p^*)$, $i = 1, 2$, and $\rho^* : A \to C^+(p^*)$ by $\tau_i^*(a) = \tau_i(a)^*$ and $\rho^*(a) = \rho(a)^*$ for $i = 1, 2$ and $a \in A$. Then p^*, A, τ_1^*, τ_2^* and ρ^* have the following properties for all $a \in A$

(i)* $T_2(L) \cap l(\rho^*(a)) = \emptyset$

(ii)* $\tau_1^*(a) \in (p^*, \rho^*(a))$, $\tau_2^*(a) \in (\rho^*(a), p)$

(iii)* $l^{-1}(a) \cap [\tau_1^*(a), \tau_2^*(a)] = \{\tau_1^*(a), \rho^*(a), \tau_2^*(a)\}$

(iv) * $a_1 \notin \tau_2^*(a_2)^- \cap \tau_1^*(a_2)^+$ for all $a_1, a_2 \in A$

(v)* $a_1 \notin \tau_2^*(a_2)^+ \cap \tau_1^*(a_2)^-$ for all $a_1, a_2 \in A$.

For $i \in \{1, 2\}$ and $\sigma \in \{+, -\}$, let

$$A_i^{*\sigma} = \{a \in A : a \in t^\sigma \text{ for all } t \in (\rho^*(a), \tau_i^*(a))\}.$$

Then $A^{*+-} = A_1^{*+} \cap A_2^{*-}$ is denumerable by what has already been proved. Since $A^{-+} = A^{*+-}$, A^{-+} is denumerable.

4.2 THEOREM. *For every $p \in C$ there exists a denumerable set $A(p) \subset l(p)$ such that for every $d \in l(p)$ one of the following occurs:*

(i) $d \in A(p)$

(ii) *$l(d)$ contains no isolated curves other than (possibly) $l(p)$*

(iii) *every l in $l(d)$ such that $l \neq l(p)$ and l is isolated in $l(d)$ satisfies $T_2(L) \cap l \neq \emptyset$.*

Proof. Let $A(p)$ be the set of all $a \in l(p) \cap D$ for which there exists $l \in l(a)$, $l \neq l(p)$, such that l is isolated in $l(a)$ and $T_2(L) \cap l = \emptyset$. Clearly, it suffices to show that $A(p)$ is denumerable.

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For each $a \in A(p)$, select $\rho(a) \in l^{-1}(a) \cap C^+(p)$ such that $l(\rho(a))$ is isolated in $l(a)$ and $T_2(L) \cap l(\rho(a)) = \emptyset$. Then select $\tau_1(a) \in [p, \rho(a))$ and $\tau_2(a) \in (\rho(a), p^*]$ such that

$$l^{-1}(a) \cap [\tau_1(a), \rho(a)] = \{\tau_1(a), \rho(a)\}$$

and

$$l^{-1}(a) \cap [\rho(a), \tau_2(a)] = \{\rho(a), \tau_2(a)\}.$$

Since $a \notin T_2(L)$, $\tau_2(a) \neq \tau_1(a)^*$. Let $a_1, a_2 \in A(p)$. Then

$$a_1 \notin \tau_2(a_2)^+ \cap \tau_1(a_2)^- \subset p^-$$

and

$$a_1 \notin \tau_2(a_2)^- \cap \tau_1(a_2)^+ \subset p^+.$$

Now let $\{p_n\}$ be a countable dense subset of C. Then $A(p) = \bigcup_n A_n$ where

$$A_n = \{a \in A(p) : \tau_1(a), \rho(a), \tau_2(a) \in C^+(p_n)\}.$$

Hence $A(p)$ is denumerable as each A_n is denumerable by 4.1.

4.3 COROLLARY. *Let L be a spread such that $T_2(L)$ is denumerable. Then for every $p \in C$ there exists a denumerable set $A(p) \subset l(p)$ such that for every $d \in l(p) \cap F_2(L)$ one of the following occurs:*
(i) $d \in A(p)$
(ii) *$l^{-1}(d)$ is a perfect set*
(iii) *$l^{-1}(d) \cap C^+(p)$ is a perfect set.*

Proof. The set of points $a \in l(p)$ for which there exists $l \in l(a)$ such that $l \neq l(p)$ and $l \cap T_2(L) \neq \emptyset$ is denumerable. The result is now a direct consequence of 4.2.

Note that a perfect subset of C is uncountable. If a curve l in the spread L contains no points which lie on uncountably many curves of L, then $T_2(L)$ is uncountable by 4.3. This extends the main result of T. Zamfirescu in [**8**]. T. Zamfirescu has given in [**7**] a sufficient condition for a spread L to have simple vertices. (Note that none of the results of [**7**] and [**8**] have been correctly stated in MR *40* #2035 and MR *50* #8297 respectively.)

In [**4**], C. Ivan has studied spreads L which have a curve l such that the interior of $l \cap F_2(L)$ (in the relative topology of l) is contained in $T_{2k}(L)$ for some $k \in \mathbf{N}$. By 4.3, all such spreads have uncountably many simple vertices if they are not trivial.

By using 4.3 we can now amplify the conclusion of 3.7 about certain continuous but nowhere differentiable functions.

4.4 THEOREM. *Let h, I and I' be as in 3.7. Then the set of values in I' which h takes a denumerable number of times is itself denumerable.*

Proof. It suffices to show that whenever $m, M \in I'$ and $m < M$ then the set of values in (m, M) which h takes a denumerable number of times is itself denumerable. Define the function g as in the proof of 3.7. It suffices to prove that the set of values in $(-1, 1)$ which g takes a denumerable number of times is itself denumerable. Since $T_2(L(g)) = \emptyset$ (see 3.6), this follows by applying 4.3 to the spread $L(g)$ with $p = 1$.

The next two lemmas will be used to deal with the distinction made in 4.2 between $l(p)$ and the other curves of $l(d)$, $d \in l(p)$.

4.5 LEMMA. *Let $p \in C$ and $q \in C^+(p)$. For every $s \in (p, q)$ there is a unique $\lambda(s) \in l(s) \cap D$ such that $l^{-1}(\lambda(s)) \cap [p, q^*] \neq \emptyset$ and $\lambda(s) \in r^+ \cup l(r)$ for all $r \in [p, q^*]$. Moreover, the map $\lambda : (p, q) \to D$ is continuous.*

Proof. The existence of $\lambda(s)$ for every $s \in (p, q)$ follows from the continuity of the map $r \mapsto l(r) \cap l(s)$ on the compact set $[p, q^*]$. If λ is not continuous, then there are $s \in (p, q)$, a sequence $\{s_n\}$ in (p, q) and a sequence $\{r_n\}$ in $[p, q^*]$ such that $s_n \to s$, $r_n \to r \in [p, q^*]$, $\lambda(s_n) \in l(r_n)$, $\lambda(s_n) \to z \in C \cup D$ and $z \neq \lambda(s)$. Since $\lambda(s_n) \in l(r_n) \to l(r)$, $z \in l(r)$. Since $\lambda(s_n) \in l(s_n) \to l(s)$, $z \in l(s)$. So by the definition of λ, $z \in (\lambda(s), s^*)$. Now there exists $v \in l^{-1}(\lambda(s)) \cap [p, q^*]$. Hence $z \in v^-$ and $\lambda(s_n) \in v^-$ for all n large enough, which contradicts the definition of λ. Therefore λ is continuous.

4.6 LEMMA. *Let $p \in C$ and $q \in C^+(p)$. Let λ be defined as in 4.5. Let*

$$E = \{d \in \lambda(p, q) : \text{there exist } \tau_1(d), \tau_2(d) \in l^{-1}(d) \cap [p, q^*] \text{ with } \tau_2(d) \in (\tau_1(d), p]\},$$

$$A_1 = \{d \in \lambda(p, q) : \text{there exist } \rho(d) \in \lambda^{-1}(d) \text{ and } \tau_1(d) \in (p, \rho(d)) \text{ such that } l^{-1}(d) \cap [\tau_1(d), q] = \{\tau_1(d), \rho(d)\} \text{ and } l(\rho(d)) \cap T_2(L) = \emptyset\},$$

$$A_2 = \{d \in \lambda(p, q) : \text{there exist } \rho(d) \in \lambda^{-1}(d) \text{ and } \tau_2(d) \in (\rho(d), q) \text{ such that } l^{-1}(d) \cap [p, \tau_2(d)] = \{\rho(d), \tau_2(d)\} \text{ and } l(\rho(d)) \cap T_2(L) = \emptyset\}.$$

Then E, A_1 and A_2 are denumerable.

Proof. E is denumerable. Let d_1 and d_2 be distinct elements of E. Then

$$l(r) \cap \tau_1(d_1)^+ \cap \tau_2(d_1)^+ = \emptyset, \; l(r) \cap (\tau_2(d_1), d_1) = \emptyset$$

and

$$l(r) \cap (d_1, \tau_1(d_1)^*) = \emptyset$$

for all $r \in (\tau_1(d_1), \tau_2(d_1))$. Hence $l(r) \cap \lambda(p, q) \subset \{d_1\}$ for all $r \in (\tau_1(d_1), \tau_2(d_1))$. Therefore

$$(\tau_1(d_1), \tau_2(d_1)) \cap (\tau_1(d_2), \tau_2(d_2)) = \emptyset,$$

whenever d_1 and d_2 are distinct elements of E. Thus E is denumerable.

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A_1 is denumerable. If $d \in A_1$ then $d \in t^-$ for all $t \in (\rho(d), q]$ and either

(1) $d \in t^+$ for all $t \in (\tau_1(d), \rho(d))$

or

(2) $d \in t^-$ for all $t \in (\tau_1(d), \rho(d))$.

Let $A_1{}^i = \{d \in A_1 : \text{(i) holds}\}$ for $i = 1, 2$. If $d \in A_1{}^1$ then there exists a unique

$$\eta(d) \in (\rho(d), d] \cap F_2(L) \cap \partial_{l(\rho(d))}(F_2(L) \cap l(\rho(d))).$$

Since $l(\rho(d)) \cap T_2(L) = \emptyset$, $\eta(d) \in F_3(L)$. So $\eta(A_1{}^1)$ is denumerable by 3.12. We claim that $\eta : A_1{}^1 \to D$ is injective.

Let d_1 and d_2 be distinct elements of $A_1{}^1$ such that $\eta(d_1) = \eta(d_2)$. Since $\lambda(\rho(d)) = d$ for all $d \in A_1{}^1$, $\rho(d_1) \neq \rho(d_2)$. We may assume that $\rho(d_2) \in (\rho(d_1), q)$. Then $d_1 \in \rho(d_2)^-$, which implies that $\eta(d_1) \in (d_1, \rho(d_1)^*)$ as $\{\eta(d_1)\} = l(\rho(d_1)) \cap l(\rho(d_2))$. But this contradicts $\eta(d_1) \in (\rho(d_1), d_1]$.

So $A_1{}^1$ is denumerable. To prove that A_1 is denumerable, it suffices to show that $A_1{}^2$ is denumerable. Let d_1 and d_2 be distinct elements of $A_1{}^2$. Suppose that $\tau_1(d_2) \in (\tau_1(d_1), \rho(d_1))$. Then $d_1 \in \tau_1(d_2)^-$ (fig. 7).

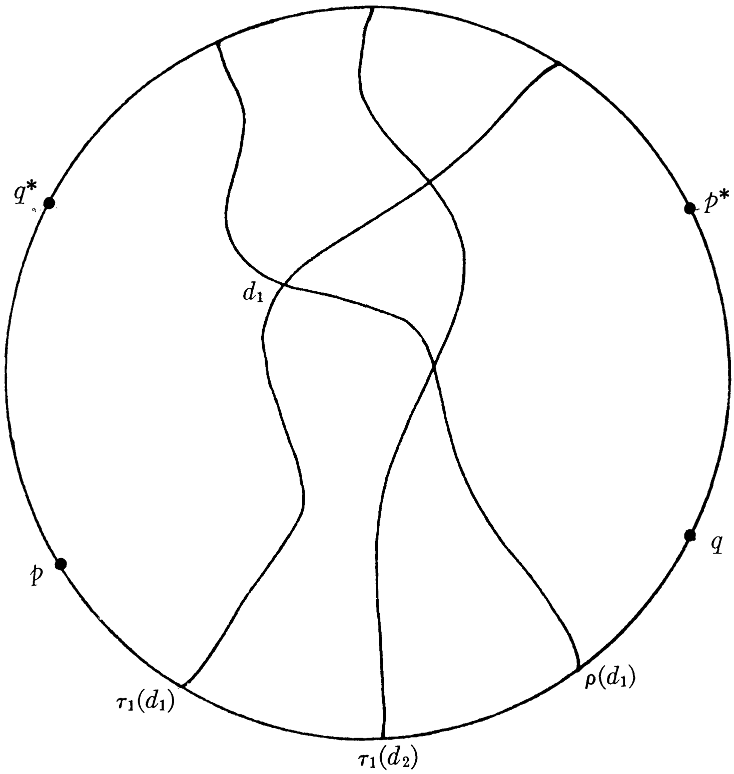

FIG. 7

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Now $d_2 \in \rho(d_1)^- \cup l(\rho(d_1))$ and hence $d_2 \in (\tau_1(d_2), z]$ where $\{z\} = l(\tau_1(d_2)) \cap l(\rho(d_1))$. Since $d_2 = \lambda(\rho(d_2))$, there exists $r \in [p, q^*] \cap l^{-1}(d_2)$. But then $d_1 \in r^-$ which contradicts $d_1 = \lambda(\rho(d_1))$. Therefore

$$(\tau_1(d_1), \rho(d_1)) \cap (\tau_1(d_2), \rho(d_2)) = \emptyset$$

whenever d_1 and d_2 are distinct elements of A_1^2. Hence A_1^2 is denumerable.

A_2 is denumerable. The proof is similar to that given to show that A_1 is denumerable.

4.7 THEOREM. *If L is a non-trivial spread such that $l^{-1}(d) \supset [p, q]$ for some $d \in D$, $p \in C$ and $q \in C^+(p)$, then $T_2(L)$ is uncountable.*

Proof. There exists either $r \in (p, q^*)$ such that $l(r) \cap p^+ \cap q^- \neq \emptyset$ or $r \in (p, q^*)$ such that $l(r) \cap p^- \cap q^+ \neq \emptyset$. Without loss of generality, assume the former is true. Let E and λ be as in 4.6. Since λ is not constant and continuous on (p, q), $\lambda(p, q)$ is uncountable. Since E is denumerable, $T_2(L)$ which contains $\lambda(p, q)\backslash E$ is uncountable.

In fact the above proof shows that $\{s \in (p, q) : l(s) \cap T_2(L) = \emptyset\}$ is denumerable.

4.8 THEOREM. *Let L be a non-trivial spread, let $p \in C$ and $q \in C^+(p)$. Suppose that $r \in (p, q^*)$ is such that the curve $l(r)$ has a parameterization $\varphi : [a, b] \rightarrow l(r)$ for which the map $f : [p, q] \rightarrow [a, b]$ defined by $s \mapsto \varphi^{-1}(l(r) \cap l(s))$ is of bounded variation, i.e. such that*

$$\left\{ \sum_{i=1}^{n} |f(s_i) - f(s_{i+1})| : s_{i+1} \in (s_i, q] \text{ for } i = 1, \ldots, n;\ s_1 \in [p, q] \right\}$$

is bounded. Then $T_2(L)$ is uncountable.

Proof. Let $X = \cup \{l(s) \cap l(r) : s \in [p, q]\}$. By 4.7, we may assume that X is a non-trivial subarc of $l(r)$. Since f is of bounded variation, there are an uncountable number of points d in X such that $l^{-1}(d) \cap [p, q]$ is finite. Hence there are an uncountable number of points d in X such that $l^{-1}(d)$ and $l^{-1}(d) \cap C^+(r)$ are not perfect sets. Therefore $T_2(L)$ is uncountable by 4.3.

The next theorem is a special case of 4.8 and shows in particular that every non-trivial spread of line segments has uncountably many simple vertices.

4.9 THEOREM. *Let L be a non-trivial spread such that $l(s)$ is a line segment for every $s \in [r, q]$ where $r \in C$ and $q \in C^+(r)$. Then $T_2(L)$ is uncountable.*

Proof. Coordinatize the plane so that $l(r)$ is contained in the set of points with coordinates $(t, 0)$, $t \in \mathbf{R}$. Let $p \in (r, q)$. For every $s \in [p, q]$, let $\alpha(s) \in (0, \pi)$ be the angle between $l(s)$ and $l(r)$. Then $\alpha : [p, q] \rightarrow (0, \pi)$ is a continuous injection. Moreover, for every $s \in [p, q]$ there exists a unique $\delta(s) \in \mathbf{R}$ such that $l(s)$ is contained in

$$\{(x, y) : x \sin \alpha(s) - y \cos \alpha(s) - \delta(s) = 0\}.$$

The map $\delta : [p, q] \to \mathbf{R}$ is continuous. Hence the point of intersection of $l(s)$ with $l(r)$ has coordinates

$$(\delta(s)/\sin \alpha(s), 0).$$

For $s \in [p, q]$, let $f(s) = \delta(s)/\sin \alpha(s)$. To complete the proof it suffices to show that $f : [p, q] \to \mathbf{R}$ is of bounded variation. We will use the argument of the proof of Theorem 1 of [**3**].

For distinct s_1 and s_2 in $[p, q]$, the point of intersection of $l(s_1)$ with $l(s_2)$ has coordinates

$$\left(\frac{\delta(s_2)\cos\alpha(s_1) - \delta(s_1)\cos\alpha(s_2)}{\sin\,(\alpha(s_2) - \alpha(s_1))}, \frac{\delta(s_2)\sin\alpha(s_1) - \delta(s_1)\sin\alpha(s_2)}{\sin\,(\alpha(s_2) - \alpha(s_1))}\right)$$

and belongs to the bounded set D. By computing the sum of the squares of the coordinates of these intersection points, we deduce that there exists a positive number K such that

$$\left(\frac{\delta(s_2) - \delta(s_1)}{\sin\,(\alpha(s_2) - \alpha(s_1))}\right)^2 + \frac{2\delta(s_1)\delta(s_2)}{1 + \cos\,(\alpha(s_2) - \alpha(s_1))} \leqq K$$

whenever s_1 and s_2 are distinct elements of $[p, q]$. Now

$$\left\{\frac{2\delta(s_1)\delta(s_2)}{1 + \cos\,(\alpha(s_2) - \alpha(s_1))} : s_1, s_2 \in [p, q]\right\}$$

is bounded. So there exists a positive number K' such that

$$\left|\frac{\delta(s_2) - \delta(s_1)}{\sin\,(\alpha(s_2) - \alpha(s_1))}\right| \leqq K'$$

whenever s_1 and s_2 are distinct elements of $[p, q]$. From this it follows easily that there exists a positive number K'' such that

$$|\delta(s_2) - \delta(s_1)| \leqq K''|\alpha(s_2) - \alpha(s_1)|$$

for all $s_1, s_2 \in [p, q]$. Hence $\delta : [p, q] \to \mathbf{R}$ is of bounded variation. Clearly $s \to 1/\sin \alpha(s)$ is also of bounded variation on $[p, q]$. Hence f is of bounded variation on $[p, q]$ as it is the product of two functions of bounded variation on $[p, q]$.

5. The concept of a k-isolated point. The preceding results have demonstrated that isolated curves in $l(d)$ are useful to work with. Furthermore, the study of spreads L such that $T_2(L)$ is denumerable leads to their consideration. The following definition encapsules the process of distinguishing between points d of D by the number of isolated curves in $l(d)$.

A point $d \in D$ is called a *k-isolated* point of the spread L if either
(1) $k = \aleph_0$ and $l(d)$ contains $\aleph_0$ isolated curves; or
(2) $k \in \mathbf{N}$ and $d \in T_k(L)$; or
(3) $k \in \mathbf{N}$ and $l(d)$ consists of $k - 1$ isolated curves and a perfect set.

Let $T_k^i(L)$ be the set of k-isolated points in D. Then

$$D = \cup \{T_k^i(L) : k \in \mathbf{N} \cup \{\aleph_0\}\} \text{ and } T_k(L) \subset T_k^i(L).$$

We will now show that every non-trivial spread L has an uncountable number of 2-isolated points. Thus, from the point of view of Sylvester's problem, the concept of a 2-isolated point in a spread of curves is the appropriate continuous analogue of a simple vertex in a finite arrangement of pseudolines.

5.1 THEOREM. *If L is a non-trivial spread, then $T_2^i(L)$ is uncountable.*

Proof. We may assume that $T_2(L)$ is denumerable. Since L is non-trivial, let p, q, λ, E, A_1 and A_2 be as in 4.6 and such that $G = \lambda(p, q) \cap p^+ \cap q^-$ is uncountable (λ is continuous). Now let $G' = G \backslash (E \cup A_1 \cup A_2)$. Then G' is uncountable as E, A_1 and A_2 are denumerable by 4.6. If $d \in G'$, then $l^{-1}(d) \cap [p, q^*]$ is a singleton belonging to (p, q^*) and $l^{-1}(d) \cap (p, q) \neq \emptyset$. Let G'' be the set of all those d in G' for which $l^{-1}(d) \cap (p, q)$ has an isolated point. Hence $G' \backslash G'' \subset T_2^i(L)$. So to complete the proof it suffices to show that G'' is denumerable.

Let $A = \{d \in \lambda(p, q) :$ there exist $\rho(d) \in l^{-1}(d) \cap (p, q)$, $\tau_1(d) \in (p, \rho(d))$ and $\tau_2(d) \in (\rho(d), q)$ such that $l^{-1}(d) \cap [\tau_1(d), \tau_2(d)] = \{\tau_1(d), \rho(d), \tau_2(d)\}$ and $l(\rho(d)) \cap T_2(L) = \emptyset\}$. Note that the definition of λ implies that $\lambda(\rho(d)) = d$ for all $d \in A$. We aim to show that A is denumerable by using 4.1. It remains to check (iv) and (v) of the statement of 4.1. For this purpose, let $a_1, a_2 \in A$. Since $a_2 \in \lambda(p, q)$, there exists $r_2 \in l^{-1}(a_2) \cap [p, q^*]$. Hence

$$a_1 \notin \tau_2(a_2)^+ \cap \tau_1(a_2)^-$$

because $a_1 \in \lambda(p, q)$ implies that $a_1 \notin r_2^-$. Let $r_1 \in l^{-1}(a_1) \cap [p, q^*]$. Then $a_2 \notin r_1^-$ implies that

$$a_1 \notin \tau_2(a_2)^- \cap \tau_1(a_2)^+.$$

Thus we may apply 4.1 to deduce that A is denumerable.

Now let $A' = \{a \in \lambda(p, q) :$ there exists $\rho(a) \in \lambda^{-1}(a)$ such that $l(\rho(a)) \cap T_2(L) \neq \emptyset\}$. Since $T_2(L)$ is denumerable, A' is denumerable. Finally, $G'' \subset A \cup A'$ and hence G'' is denumerable as required.

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