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Subdifferential Regularity of Directionally Lipschitzian Functions

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Abstract. Formulas for the Clarke subdifferential are always expressed in the form of inclusion. The equality form in these formulas generally requires the functions to be directionally regular. This paper studies the directional regularity of the general class of extended-real-valued functions that are directionally Lipschitzian. Connections with the concept of subdifferential regularity are also established.

1 Introduction

A general concept of subdifferential has been introduced by Clarke [1] for any extended-real-valued function defined on a finite dimensional space X. He defined first the subdifferential of a locally Lipschitzian function f at a point $\bar{x}$ with the help of Rademacher's theorem (see also Clarke [2] for a subsequent approach using the generalized directional derivative $f^0(\bar{x};.)$ for f locally Lipschitzian over any Banach space X). This allowed him to consider a subset $S \subset X$ and for $\bar{x} \in S$ the normal cone $N_C(S;\bar{x})$ to S at $\bar{x}$ as the closed cone generated by the subdifferential at $\bar{x}$ of the distance function d_S to S. Then he defined the subdifferential of any extended-real-valued function f at a point $\bar{x}$ where f is finite with the formula

$$(\star) \qquad \partial^C f(\bar{x}) = \Big\{x^* \in X^* : (x^*, -1) \in N_C\Big(\operatorname{epi} f; \big(\bar{x}, f(\bar{x})\big)\Big)\Big\}$$

where epi f denotes the epigraph of f, that is epi $f := \{(x,r) \in X \times \mathbb{R} : f(x) \le r\}$. Note that this approach also works via [2] for any normed vector space X. An important fact to record about this subdifferential is that it enjoys a general subdifferential calculus (for compositions, sums . . .) that makes it applicable in mathematical programming, optimal control and several other mathematical fields. A prototype formula is given by

$$(\star\star) \qquad \partial^C(f+g)(\bar{x}) \subset \partial^C f(\bar{x}) + \partial^C g(\bar{x})$$

whenever f and g are locally Lipschitzian.

Rockafellar in [11] showed the route to extend the definition for functions defined over any topological vector space X. He defined a generalized directional derivative $f^{\uparrow}(\bar{x};.)$ for any extended-real-valued function f and showed that $f^{\uparrow}(\bar{x};.)$ coincides with $f^0(\bar{x};.)$ whenever f is locally Lipschitzian. With this directional derivative he defined the subdifferential of f at $\bar{x}$ as the set of all $x^* \in X^*$ such that $\langle x^*, h\rangle \le f^{\uparrow}(\bar{x};h)$ for all $h \in X$ and showed that it is equal to the set given by $(\star)$ when X is a normed space. Rockafellar also extended

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formula $(\star\star)$ under a natural qualification condition to the case where g is directionally Lipschitzian. Several crucial properties of directionally Lipschitzian functions are established in [11], [12]. The notion of directional Lipschitzness and the various directional derivatives will be recalled in Section 2.

The equality in $(\star\star)$ (as those for composition, maximum ...) requires in Clarke [1] as well as in Rockafellar [12] the functions to be H-directionally regular at $\bar{x}$ (*i.e.*, $f^{\uparrow}(\bar{x};.)$ coincides with the lower Hadamard directional derivative) and so makes clear the importance of this notion. See also Jofré and Thibault [6] for some representations of $\partial^C f(\bar{x})$ for functions f that are directionally Lipschitzian and H-directionally regular at $\bar{x}$.

Another natural dual concept of regularity in nonsmooth analysis is that of subdifferential regularity. This means that the Clarke subdifferential of f at $\bar{x}$ coincides with a prescribed subdifferential of f at $\bar{x}$. For example, the primal lower nice functions introduced by Poliquin [10] are regular in this sense with respect to the proximal subdifferential (we refer to [10] and [7] for the importance of these functions in nonsmooth analysis). In this paper we show how the Zagrodny mean value theorem allows to study connections between these concepts of regularity. Section 2 recalls some results in Rockafellar [11], [12] and the Zagrodny mean value theorem [16] and Section 3 establishes some preparatory lemmas. In Section 4 after proving the equivalence between the H-directional regularity of f at $\bar{x}$ and the subdifferential regularity with respect to the Hadamard subdifferential $\partial^H f$ (when $\partial^C f(\bar{x}) \neq \varnothing$), we characterize the H-directional regularity in terms of closedness of the Hadamard subdifferential. In Section 5 we prove a similar characterization of the subdifferential regularity with respect to the Fréchet (resp. proximal) subdifferential.

2 Preliminaries

Let f be any extended-real-valued function on a real normed vector space X whose topological dual space is X^* and let $\bar{x}$ be any point where f is finite. The generalized directional derivative $f^{\uparrow}(\bar{x};.)$ is defined (see Rockafellar [11]) by

$$f^{\uparrow}(\bar{x};h) = \limsup_{\substack{(x,\alpha)\downarrow_f \bar{x} \\ t\downarrow 0}} \inf_{h'\to h} t^{-1}[f(x+th') - \alpha]$$

$$:= \sup_{H\in\mathcal{N}(h)} \Big[\limsup_{\substack{(x,\alpha)\downarrow_f \bar{x} \\ t\downarrow 0}} \Big(\inf_{h'\in H} t^{-1}[f(x+th') - \alpha]\Big)\Big],$$

where $(x,\alpha) \downarrow_f \bar{x}$ means $(x,\alpha) \in \operatorname{epi} f := \{(z,\beta) \in X \times \mathbb{R}; f(z) \leq \beta\}$ and $(x,\alpha) \longrightarrow \big(\bar{x}, f(\bar{x})\big)$ and $\mathcal{N}(h)$ denotes the filter of neighbourhoods of h.

If f is lower semicontinuous (l.s.c.) at $\bar{x}$, the definition can be expressed in the following simpler form

$$f^{\uparrow}(\bar{x};h) = \limsup_{\substack{x\to^f \bar{x} \\ t\downarrow 0}} \inf_{h'\to h} t^{-1}[f(x+th') - f(x)],$$

where $x \to^f \bar{x}$ means $x \to \bar{x}$ and $f(x) \to f(\bar{x})$.

If f is Lipschitz around $\bar{x}$, then $f^{\uparrow}(\bar{x};h)$ coincides with the Clarke directional derivative $f^0(\bar{x};.)$ defined by

$$f^0(\bar{x};h) = \limsup_{\substack{x\to\bar{x} \\ t\downarrow 0}} t^{-1}[f(x+th) - f(x)].$$

Even if f is not necessarily Lipschitz around $\bar{x}$ we will put

$$f^0(\bar{x};h) := \limsup_{\substack{(x,\alpha)\downarrow_f \bar{x}\\ t\downarrow 0}} t^{-1}[f(x+th)-\alpha].$$

The lower Hadamard directional derivative of f at $\bar{x}$ is defined (see Penot [8]) by

$$f^H(\bar{x};h) = \liminf_{\substack{h'\to h\\ t\downarrow 0}} t^{-1}[f(\bar{x}+th')-f(\bar{x})],$$

and when f is Lipschitz around $\bar{x}$ the lower Hadamard directional derivative $f^H(\bar{x};h)$ coincides with the lower Dini directional derivative $f^-(\bar{x};.)$ defined by

$$f^-(\bar{x};h) = \liminf_{t\downarrow 0} t^{-1}[f(\bar{x}+th)-f(\bar{x})].$$

The function f is said to be directionally Lipschitzian at $\bar{x}$ with respect to a vector h (see [11]) if

$$\limsup_{\substack{(x,\alpha)\downarrow_f \bar{x}\\ (t,h')\to(0^+,h)}} t^{-1}[f(x+th')-\alpha] < +\infty,$$

and this is reduced when f is l.s.c. at $\bar{x}$ to

$$\limsup_{\substack{(t,h')\to(0^+,h)\\ x\to_f \bar{x}}} t^{-1}[f(x+th')-f(x)] < +\infty.$$

If this relation holds for some h, one says that f is directionally Lipschitzian at $\bar{x}$. Observe that f is Lipschitz around $\bar{x}$ if and only if it is directionally Lipschitzian at $\bar{x}$ with respect to the vector zero (or equivalently with respect to every vector in X).

We also recall that the Clarke (resp. the Hadamard, the Fréchet) subdifferential of f at x (with $f(x)$ finite) is defined by

$$\partial^C f(x) = \{x^* \in X^* : \langle x^*, h\rangle \le f^{\uparrow}(x;h), \text{ for all } h \in X\},$$

(resp.

$$\partial^H f(x) = \{x^* \in X^* : \langle x^*, h\rangle \le f^H(x;h), \text{ for all } h \in X\},$$

$$\partial^F f(x) = \{x^* \in X^* : \liminf_{x'\to x} \frac{f(x')-f(x)-\langle x^*, x'-x\rangle}{\|x'-x\|} \ge 0\}),$$

and that the proximal subdifferential $\partial^P f(x)$ is the set of all $x^* \in X^*$ for which there exist $r, \sigma > 0$ such that for all $x' \in x + r\mathbb{B}$

$$\langle x^*, x'-x\rangle \le f(x') - f(x) + \sigma\|x'-x\|^2.$$

By convention $\partial^C f(x) = \partial^H f(x) = \partial^F f(x) = \partial^P f(x) = \varnothing$ if $f(x)$ is not finite. Note that one always has $\partial^P f(x) \subset \partial^F f(x) \subset \partial^H f(x) \subset \partial^C f(x)$.

One says that f is *H-directionally regular* at $\bar{x}$ with respect to a vector $h \in X$ if one has $f^{\uparrow}(\bar{x}; h) = f^{H}(\bar{x}; h)$. When this holds for all $h \in X$ one says that f is H-directionally regular at $\bar{x}$.

Now, let us recall some definitions and results that will be used in all the paper. A Banach space X is called an Asplund space if every continuous convex function defined on a convex open subset U of X is Fréchet differentiable on a dense G_δ subset of U (see [9]). Recall (see J. Diestel [3]) that the dual united ball $\mathbb{B}_{X^*}$ is weak star sequentially compact, whenever the space X is Asplund or admits an equivalent norm that is Gâteaux differentiable away from the origin.

In the following definition we denote by $\operatorname{dom} f$ the effective domain of a function $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$, that is $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$.

Definition 2.1 Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be l.c.s. and $\bar{x} \in \operatorname{dom} f$ and let ∂f be any subdifferential or presubdifferential of f (in the sense of [14], [15]). We will say that ∂f is *topologically closed* at $\bar{x}$ if, for every net $(x_j, x_j^*)_{j\in J}$ in ∂f such that $x_j^* \longrightarrow^{w^*} x^*$ and $x_j \longrightarrow^{f} \bar{x}$ one has $(\bar{x}, x^*) \in \partial f$, where $\longrightarrow^{w^*}$ denotes the w^*-convergence in X^* and $(y, y^*) \in \partial f$ means that $y^* \in \partial f(y)$. When the set J is replaced by $\mathbb{N}$, we say that ∂f is sequentially closed at $\bar{x}$.

We will also say that the function f is *subdifferentially ∂-regular* at $\bar{x}$ whenever $\partial f(\bar{x})$ coincides with $\partial^C f(\bar{x})$.

We finish this section by recalling the following results (by Zagrodny [16] and Rockafellar [11]). As it appears in [14], [15] the following Zagrodny mean value theorem holds for any presubdifferential although the proof in [16] was given for the Clarke subdifferential.

***Theorem 2.1* (see [16])** *Let X be a Banach space, $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c. on X and ∂f be any subdifferential or presubdifferential of f (in the sense of [14], [15]). Let $a, b \in \operatorname{dom} f$ (with $a \neq b$). Then there exist $x_n \longrightarrow^{f} c \in [a, b[\ := \{rb + (1-r)a : r \in [0, 1[\}$ and $x_n^* \in \partial f(x_n)$ such that*

$$f(b) - f(a) \leq \lim_{n} \langle x_n^*, b - a\rangle,$$

and

$$\frac{\|b-c\|}{\|b-a\|}[f(b) - f(a)] \leq \lim_{n} \langle x_n^*, b - x_n\rangle.$$

***Proposition 2.1* (see [11])** *Let f be any extended-real-valued function on X and $\bar{x}$ be any point where f is finite. If f is directionally Lipschitzian at $\bar{x}$, then*

(i) *for every $h \in \operatorname{int} \operatorname{dom} f^{\uparrow}(\bar{x}; .)$*

$$f^{\uparrow}(\bar{x}; h) = f^{0}(\bar{x}; h) = \limsup_{\substack{(x,\alpha)\downarrow_f \bar{x} \\ (t,h')\to(0^+,h)}} t^{-1}[f(x + th') - \alpha];$$

(ii) *$\operatorname{int} \operatorname{dom} f^{\uparrow}(\bar{x}, .)$ is the set of all vectors h with respect to which f is directionally Lipschitzian and $f^{\uparrow}(\bar{x}, .)$ is continuous over $\operatorname{int} \operatorname{dom} f^{\uparrow}(\bar{x}, .)$;*

(iii) *$f^{\uparrow}(\bar{x}; h) = \liminf_{h'\to h} f^{0}(\bar{x}; h')$ for every $h \in X$.*

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3 General Results

Throughout this section we assume that X is a Banach space, $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$ is lower semicontinuous on X, $\bar{x} \in \operatorname{dom} f$ and ∂f is a subdifferential or presubdifferential of f (in the sense of [14], [15]) that satisfies $\partial f \subset \partial^C f$. Recall that $\partial^F f$ (resp. $\partial^H f$, $\partial^P f$) is a presubdifferential whenever X is an Asplund space (resp. a Banach space with a Gâteaux differentiable (away from the origin) renorm, a Hilbert space). The following two lemmas will be needed in the next section.

Lemma 3.1 *If f is directionally Lipschitzian at $\bar{x}$ with respect to $\bar{h} \in X$, then there exists $\beta \in \mathbb{R}$ such that for every sequence $(x_n, x_n^*)_{n\in\mathbb{N}}$ in ∂f, with $x_n \longrightarrow^f \bar{x}$, there are $n_0 \in \mathbb{N}$ and a neigbourhood H of $\bar{h}$ such that for all $n \geq n_0$ one has*

$$\langle x_n^*, h\rangle \leq \beta \quad \textit{for all } h \in H.$$

Proof As f is directionally Lipschitzian at $\bar{x}$ with respect to $\bar{h}$, there exist $\beta \in \mathbb{R}$, $\delta > 0$ such that

$$(1) \qquad t^{-1}[f(x+th) - f(x)] \leq \beta, \quad \text{for all } t \in\,]0,\delta[\,, h \in \bar{h} + \delta\mathbb{B}, x \in U(f,\bar{x},\delta),$$

where $U(f,\bar{x},\delta) := \{x \in X : x \in \bar{x} + \delta\mathbb{B} \text{ and } |f(x) - f(\bar{x})| \leq \delta\}$ and $\mathbb{B}$ denotes the closed united ball of X. Let $(x_n, x_n^*)_{n\in\mathbb{N}}$ in ∂f, with $x_n \longrightarrow^f \bar{x}$. Then there exist $n_0 \in \mathbb{N}$ such that $x_n \in U(f,\bar{x},\delta)$, for all $n \geq n_0$. Fix any $h_0 \in \bar{h} + \frac{\delta}{2}\mathbb{B}$ and any $n \geq n_0$. Then (by (1))

$$t^{-1}[f(x+th) - f(x)] \leq \beta, \quad \text{for all } t \in\,]0,\delta[\,, h \in h_0 + \frac{\delta}{2}\mathbb{B}, x \in U\left(f, x_n, \frac{\delta}{2}\right),$$

which ensures that

$$f^{\uparrow}(x_n, h_0) \leq \beta, \quad \text{for all } n \geq n_0 \text{ and all } h_0 \in \bar{h} + \frac{\delta}{2}\mathbb{B}.$$

Thus, as $x_n^* \in \partial f(x_n) \subset \partial^C f(x_n)$, one has

$$\langle x_n^*, h\rangle \leq f^{\uparrow}(x_n, h) \leq \beta, \quad \text{for all } n \geq n_0 \text{ and all } h \in \bar{h} + \frac{\delta}{2}\mathbb{B},$$

which completes the proof. ∎

Lemma 3.2 *If f is directionally Lipschitzian at $\bar{x}$ and $\partial^C f(\bar{x}) \neq \varnothing$, then for each $h \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x}, .)$ there exist a sequence $u_n \longrightarrow^f \bar{x}$ and a bounded sequence $(u_n^*)_{n\in\mathbb{N}}$ in X^* such that*

(i) $u_n^* \in \partial f(u_n)$ *for all* $n \in \mathbb{N}$;
(ii) $f^{\uparrow}(\bar{x}, h) \leq \limsup_n \langle u_n^*, h\rangle$.

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Proof Let $\bar{h} \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x}, .)$. As $\partial^C f(\bar{x}) \neq \varnothing$, one has (see [11])

$$f^{\uparrow}(\bar{x}; \bar{h}) := \sup\{\langle x^*, \bar{h}\rangle; x^* \in \partial^C f(\bar{x})\} > -\infty, \tag{2}$$

and by (i) in Proposition 2.1 one also has

$$f^{\uparrow}(\bar{x}; \bar{h}) = \limsup_{\substack{x \to^f \bar{x} \\ t \downarrow 0}} t^{-1}[f(x + t\bar{h}) - f(x)].$$

Let us consider sequences $x_n \longrightarrow^f \bar{x}$ and $t_n \longrightarrow 0^+$ such that

$$f^{\uparrow}(\bar{x}; \bar{h}) = \lim_n t_n^{-1}[f(x_n + t_n\bar{h}) - f(x_n)]. \tag{3}$$

For each $n \in \mathbb{N}$, we put $a_n := x_n$ and $b_n := x_n + t_n\bar{h}$. Note that, for n large enough, $f(x_n)$ is finite and that $f(b_n)$ is also finite because f is directionally Lipschitzian at $\bar{x}$ with respect to $\bar{h}$. Then (by Theorem 2.1), there exist a sequence $c_{k,n} \longrightarrow c_n \in [a_n, b_n[$ and a sequence $(x^*_{k,n})_{k\in\mathbb{N}}$ in X^* such that $x^*_{k,n} \in \partial f(c_{k,n})$ for all $k \in \mathbb{N}$ and

$$t_n^{-1}[f(x_n + t_n\bar{h}) - f(x_n)] \leq \lim_{k\to+\infty} \langle x^*_{k,n}, \bar{h}\rangle. \tag{4}$$

Thus for each $n \in \mathbb{N}$, there exists $s(n) \in \mathbb{N}$ such that

$$\lim_k \langle x^*_{k,n}, \bar{h}\rangle \leq \langle x^*_{s(n),n}, \bar{h}\rangle + \frac{1}{n+1}, \tag{5}$$

and

$$\|c_{s(n),n} - c_n\| \leq \frac{1}{n+1}.$$

Put $u^*_n := x^*_{s(n),n}$ and $u_n := c_{s(n),n}$. Then $u^*_n \in \partial f(u_n)$ for each $n \in \mathbb{N}$, which ensures (i).

As f is directionally Lipschitzian at $\bar{x}$ with respect to $\bar{h}$ (see (ii) in Proposition 2.1), then by Lemma 3.1 above, there exist $\beta \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that for all h around $\bar{h}$ and all $n \geq n_0$

$$\langle u^*_n, h\rangle \leq \beta.$$

By (2), there exist $n_1 \in \mathbb{N}$ and $\sigma \in \mathbb{R}$ such that for all $n \geq n_1$

$$\sigma \leq t_n^{-1}[f(u_n + t_n\bar{h}) - f(u_n)] \leq \langle u^*_n, \bar{h}\rangle + \frac{1}{n+1}, \quad \text{(by (4) and (5))}$$

and hence $\sigma - \frac{1}{n_1+1} \leq \langle u^*_n, \bar{h}\rangle$ for all $n \geq n_1$. Put $N := \max\{n_0, n_1\}$ and $\sigma_1 := \sigma - \frac{1}{n_1+1}$. Choose $\delta > 0$ such that $\langle u^*_n, h\rangle \leq \beta$ for all $h \in \bar{h} + \delta\mathbb{B}$ and all $n \geq N$. Then, for all $n \geq N$ and all $b \in \mathbb{B}$, one has

$$\begin{aligned}\langle u^*_n, b\rangle &\leq \frac{1}{\delta}[\langle u^*_n, \bar{h} + \delta b\rangle - \langle u^*_n, \bar{h}\rangle] \\ &\leq \frac{1}{\delta}[\beta - \sigma_1].\end{aligned}$$

So for all $n \geq N$,

$$\|u^*_n\| \leq M := \frac{\beta - \sigma_1}{\delta},$$

which ensures that the sequence $(u^*_n)_{n\in\mathbb{N}}$ is bounded. Furthermore, (ii) is ensured by (3), (4) and (5), and hence the proof is finished. ■

4 H-Regularity.

In this section we are going to prove (under some general assumptions) that the H-directional regularity is equivalent to the closedness of the Hadamard subdifferential. We begin by showing that the H-directional regularity of f at $\bar{x}$ coincides with the H-subdifferential regularity of f at $\bar{x}$ whenever the Clarke subdifferential of f at $\bar{x}$ is nonempty.

Proposition 4.1 *Let X be a real normed vector space, $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c. and $\bar{x} \in \operatorname{dom} f$ with $\partial^C f(\bar{x}) \neq \varnothing$. Then the following assertions are equivalent*

(i) f is H-directionally regular at $\bar{x}$;
(ii) f is subdifferentially ∂^H-regular at $\bar{x}$.

Proof The implication (i)⇒(ii) is obvious. So, we will prove the reverse implication (ii)⇒(i). Fix any $h \in X$. Suppose that $\partial^H f(\bar{x}) = \partial^C f(\bar{x})$. As the inequality $f^H(\bar{x}; h) \leq f^{\uparrow}(\bar{x}; h)$ always holds, one has

$$\begin{aligned} f^{\uparrow}(\bar{x}; h) &= \sup\{\langle x^*, h\rangle : x^* \in \partial^C f(\bar{x})\} \\ &= \sup\{\langle x^*, h\rangle : x^* \in \partial^H f(\bar{x})\} \\ &\leq f^H(\bar{x}; h) \\ &\leq f^{\uparrow}(\bar{x}; h). \end{aligned}$$

This ensures that f is H-directionally regular at $\bar{x}$ and proves the reverse implication. ■

As a direct application of this proposition one obtains that the primal lower nice functions introduced by Poliquin [10] (see [10] and [7] for the importance of these functions) are H-directionally regular at all points of the domains of the subdifferentials. Indeed it is shown in [7], [10] that all subdifferentials coincide for these functions.

Consider now the following lemma which has its own interest. It will allow us to prove the next propositions 4.2 and 4.3.

Lemma 4.1 *Let X be a real normed vector space, $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c. and $\bar{x} \in \operatorname{dom} f$. Suppose that f is directionally Lipschitzian at $\bar{x}$. Then, for all $h \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x}; .)$ one has*

$$\limsup_{x \to_f \bar{x}} f^{\uparrow}(x; h) \leq f^{\uparrow}(\bar{x}; h).$$

Proof Fix any $h \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x}; .)$. As f is directionally Lipschitzian at $\bar{x}$, one has (by (i) in Proposition 2.1)

$$f^{\uparrow}(\bar{x}; h) = \limsup_{\substack{x \to_f \bar{x} \\ t \downarrow 0}} t^{-1}[f(x + th) - f(x)],$$

and there exists $\delta > 0$ such that f is directionally Lipschitzian at each $x \in U(f, \bar{x}, \delta)$. Let $\gamma > f^{\uparrow}(\bar{x}; h)$. By definition of upper limit, there exists $0 < \delta' < \delta$ such that

$$t^{-1}[f(x + th) - f(x)] < \gamma,$$

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for all $x \in U(f, \bar{x}, \delta')$ and all $t \in \left]0, \delta'\right[$. Fix any $x_0 \in U(f, \bar{x}, \frac{\delta'}{2})$. Then we have

$$t^{-1}[f(x+th) - f(x)] < \gamma,$$

for all $x \in U(f, x_0, \frac{\delta'}{2})$ and all $t \in \left]0, \delta'\right[$, and hence as f is directionally Lipschitzian at each point $x \in U(f, \bar{x}, \delta')$ one has

$$f^{\uparrow}(x_0; h) = \limsup_{\substack{x \to^f x_0 \\ t \downarrow 0}} t^{-1}[f(x+th) - f(x)] \leq \gamma.$$

Thus, since this inequality holds for all $x_0 \in U(f, \bar{x}, \frac{\delta'}{2})$ and all $\gamma > f^{\uparrow}(\bar{x}; h)$, then taking upper limits as $x_0 \longrightarrow^f \bar{x}$ gives

$$\limsup_{x_0 \to^f \bar{x}} f^{\uparrow}(x_0; h) \leq f^{\uparrow}(\bar{x}; h),$$

which completes the proof. ■

Proposition 4.2 *Let X be a real normed vector space, $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c. and $\bar{x} \in \operatorname{dom} f$. If f is directionally Lipschitzian at $\bar{x}$, then $\partial^C f$ is topologically closed at $\bar{x}$.*

Proof This follows immediately from Lemma 4.1 and the equality

$$\partial^C f(\bar{x}) = \{x^* : \langle x^*, h\rangle \leq f^{\uparrow}(\bar{x}; h) \text{ for all } h \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x}; .)\},$$

because f is directionally Lipschitzian at $\bar{x}$. ■

Proposition 4.3 *Let X be a Banach space, $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c. and $\bar{x} \in \operatorname{dom} f$. Suppose that f is directionally Lipschitzian at $\bar{x}$. Then the following assertions are equivalent*

(i) f *is H-directionally regular at $\bar{x}$ with respect to any vector $h \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x}; .)$;*
(ii) $\limsup_{x \to^f \bar{x}} f^H(x; h) \leq f^H(\bar{x}; h)$, *for all $h \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x}; .)$.*

Proof The implication (i)⇒(ii) is immediate from Lemma 4.1 and the inequality $f^H(x; h) \leq f^{\uparrow}(x; h)$. Let us prove the reverse implication. Assume (ii) and fix any $\bar{h} \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x}; .)$. As we always have (see [4], [5])

$$f^{\uparrow}(\bar{x}; \bar{h}) \leq \sup_{\epsilon > 0}\Bigl(\limsup_{x \to^f \bar{x}}\bigl(\inf_{h \in \bar{h} + \epsilon\mathbb{B}} f^H(x; h)\bigr)\Bigr),$$

then

$$\begin{aligned} f^{\uparrow}(\bar{x}; \bar{h}) &\leq \limsup_{x \to^f \bar{x}} f^H(x; \bar{h}) \\ &\leq f^H(\bar{x}; \bar{h}) \quad \text{(by (ii))} \\ &\leq f^{\uparrow}(\bar{x}; \bar{h}). \end{aligned}$$

This ensures the H-directional regularity of f at $\bar{x}$ with respect to $\bar{h}$ and hence the proof is complete. ■

Now, we are ready to prove the first of our main theorems.

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Theorem 4.1 *Assume that X is a Banach space admitting an equivalent norm that is Gâteaux differentiable away from the origin. Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c. on X and directionally Lipschitzian at $\bar{x} \in \operatorname{dom} f$ with $\partial^C f(\bar{x}) \neq \emptyset$. Then the following assertions are equivalent*

(i) f *is H-directionally regular at $\bar{x}$;*
(ii) $\partial^H f$ *is topologically closed at $\bar{x}$ and* $\operatorname{dom} f^{\uparrow}(\bar{x};.) = \operatorname{dom} f^H(\bar{x};.)$*;*
(iii) $\partial^H f$ *is sequentially closed at $\bar{x}$ and* $\operatorname{dom} f^{\uparrow}(\bar{x};.) = \operatorname{dom} f^H(\bar{x};.)$.

Proof Let us prove first the implication "(i)⇒(ii)". The assumption (i) evidently implies $\operatorname{dom} f^{\uparrow}(\bar{x};.) = \operatorname{dom} f^H(\bar{x};.)$ and $\partial^H f(\bar{x}) = \partial^C f(\bar{x})$. So Proposition 4.2 ensures that $\partial^H f$ is topologically closed at $\bar{x}$ and this finishes the proof of the first implication. Since the implication "(ii)⇒(iii)" is obvious, it remains to show the third one "(iii)⇒(i)". Fix any vector $\bar{h} \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x};.)$. We apply Lemma 3.2 with the Hadamard subdifferential $\partial^H f$ and we obtain a sequence $x_n \longrightarrow^f \bar{x}$ and a bounded sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^* such that

(a) $x_n^* \in \partial^H f(x_n)$ for all $n \in \mathbb{N}$;
(b) $f^{\uparrow}(\bar{x};\bar{h}) \leq \limsup_{n\to+\infty}\langle x_n^*, \bar{h}\rangle$.

As the space X admits an equivalent norm that is Gâteaux differentiable away from the origin, the closed united ball of X^* is w^*-sequentially compact and hence we may suppose that the sequence $(x_n^*)_{n\in\mathbb{N}}$ converges with respect to the weak star topology to some $x^* \in X^*$. Therefore, by the sequential closedness of $\partial^H f$ we have $x^* \in \partial^H f(\bar{x})$, and hence by (b) and Proposition 2.1

$$f^0(\bar{x};\bar{h}) = f^{\uparrow}(\bar{x};\bar{h}) \leq \langle x^*, \bar{h}\rangle. \tag{6}$$

Note that the analysis above ensures that $\partial^H f(\bar{x}) \neq \emptyset$. Fix now $\bar{h} \in \operatorname{dom} f^{\uparrow}(\bar{x};.)$, consider the lower semicontinuous convex function φ defined by

$$\varphi(h) := \sup\{\langle x^*, h\rangle : x^* \in \partial^H f(\bar{x})\},$$

and note that

$$\varphi(h) \leq f^H(\bar{x};h) \leq f^{\uparrow}(\bar{x};h) \quad \text{for all } h \in X. \tag{7}$$

Fix $v \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x};.)$ and put $h_t := \bar{h} + t(v - \bar{h})$ for $t \in [0,1]$. By (7) we have $h_t \in \operatorname{dom}\varphi$ for all $t \in [0,1]$ and hence the function $t \mapsto \varphi(h_t)$ is continuous on $[0,1]$ (see Theorem 10.2 in [13]). Moreover for each $t \in]0,1]$ we have $h_t \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x};.)$ and (6) ensures that there exists $x_t^* \in \partial^H f(\bar{x})$ with

$$f^0(\bar{x};h_t) = f^{\uparrow}(\bar{x};h_t) \leq \langle x_t^*, h_t\rangle \leq \varphi(h_t).$$

Therefore by Proposition 2.1

$$\begin{aligned} f^{\uparrow}(\bar{x};\bar{h}) &= \liminf_{h\to\bar{h}} f^0(\bar{x};h) \\ &\leq \liminf_{t\to 0^+} f^0(\bar{x};h_t) \\ &\leq \lim_{t\to 0^+} \varphi(h_t) \\ &= \varphi(\bar{h}) \\ &\leq f^H(\bar{x};\bar{h}). \end{aligned}$$

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So $f^{\uparrow}(\bar{x};\bar{h}) = f^{H}(\bar{x};\bar{h})$ for all $\bar{h} \in \operatorname{dom} f^{\uparrow}(\bar{x};.)$. As $\operatorname{dom} f^{\uparrow}(\bar{x};.) = \operatorname{dom} f^{H}(\bar{x};.)$ by assumption, we get $f^{\uparrow}(\bar{x};\bar{h}) = f^{H}(\bar{x};\bar{h})$ for all $\bar{h} \in X$ and the proof is complete. ■

We close this section with the following result, which gives the equivalence between the various notions of H-regularity given above and also the closedness of the Hadamard subdifferential, when the function is assumed be locally Lipschitzian. Its proof follows directly from Propositions 4.1 and 4.3 and Theorem 4.1.

Corollary 4.1 *Assume that X is a Banach space admitting an equivalent norm that is Gâteaux differentiable away from the origin. Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitzian function around $\bar{x} \in \operatorname{dom} f$. Then the following assertions are equivalent*

(i) $f^{H}(.;h)$ *is upper semicontinuous at $\bar{x}$ for all $h \in X$;*
(ii) f *is H-directionally regular at $\bar{x}$;*
(iii) f *is subdifferentially ∂^{H}-regular at $\bar{x}$;*
(iv) $\partial^{H} f$ *is topologically closed at $\bar{x}$;*
(v) $\partial^{H} f$ *is sequentially closed at $\bar{x}$.*

Remark 4.1 Note that Theorem 4.1 and its Corollary 4.1 still holds for any Banach space X for which ∂^{H} is a presubdifferential.

5 Fréchet and Proximal Regularity

As in the section above and Theorem 4.1, we use the general results established in Section 3, to prove the equivalence between the topological closedness of the Fréchet subdifferential (resp. proximal subdifferential) and the Fréchet (resp. proximal) subdifferential regularity of f whenever it is directionally Lipschitzian.

Theorem 5.1 *Let X be an Asplund space, $f\colon X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be l.s.c. on X with $\bar{x} \in \operatorname{dom} f$ and let $s(\bar{x};.)$ be the support function of $\partial^{F} f(\bar{x})$, that is*

$$s(\bar{x};h) := \sup\{\langle y^*, h\rangle; y^* \in \partial^{F} f(\bar{x})\}.$$

Suppose that f is directionally Lipschitzian at $\bar{x}$. Then f is subdifferentially ∂^{F}-regular at $\bar{x}$ if and only if $\partial^{F} f$ is topologically closed at $\bar{x}$ and $\operatorname{dom} f^{\uparrow}(\bar{x};.) = \operatorname{dom} s(\bar{x};.)$.

Proof The implication "$\Rightarrow$" follows from Proposition 4.2. So we will prove the reverse implication "$\Leftarrow$". Fix any $\bar{h} \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x};.)$. As in the proof of Theorem 4.1 we apply Lemma 3.2 to obtain an element $x^* \in \partial^{F} f(\bar{x})$ that satisfies $f^{\uparrow}(\bar{x};\bar{h}) \leq \langle x^*, \bar{h}\rangle$, and hence $\partial^{F} f(\bar{x}) \neq \varnothing$. Thus for each $\bar{h} \in \operatorname{int}\operatorname{dom} f^{\uparrow}(\bar{x};.)$ there exists $x^* \in \partial^{F} f(\bar{x})$ such that

$$f^{\uparrow}(\bar{x};\bar{h}) \leq \langle x^*, \bar{h}\rangle \leq s(\bar{x};\bar{h}) = \sup\{\langle y^*, \bar{h}\rangle; y^* \in \partial^{F} f(\bar{x})\}. \tag{8}$$

Put $D := \operatorname{dom} f^{\uparrow}(\bar{x};.)$. It is obvious that the function $s(\bar{x};.)$ is convex and lower semicontinuous. Fix $v \in \operatorname{int} D$ and $h \in D$ and put $h_t := h + t(v - h)$ for each $t \in [0,1]$. Observe

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that $h_t \in \operatorname{int} D$ for each $t \in]0,1]$ and that the function $t \mapsto s(\bar{x}; h_t)$ is continuous on $[0,1]$ (see Theorem 10.2 in [13]). So, we have (see (iii) in Proposition 2.1)

$$\begin{aligned} f^{\uparrow}(\bar{x}; h) &= \liminf_{h' \to h} f^0(\bar{x}; h') \\ &\leq \liminf_{t \to 0^+} f^0(\bar{x}; h_t) \\ &\leq \lim_{t \to 0^+} s(\bar{x}; h_t) \quad \text{(by (8))} \\ &= s(\bar{x}; h). \end{aligned}$$

Therefore $f^{\uparrow}(\bar{x}; h) \leq s(\bar{x}; h)$ for all $h \in \operatorname{dom} f^{\uparrow}(\bar{x}; .)$. As $\operatorname{dom} f^{\uparrow}(\bar{x}; .) = \operatorname{dom} s(\bar{x}; .)$, the inequality above holds for each $h \in X$, which ensures that the support function of $\partial^C f(\bar{x})$ is not greater than the support function of $\partial^F f(\bar{x})$. Since $\partial^F f(\bar{x})$ is weak-star closed (because of the topological closedness of $\partial^F f$ at $\bar{x}$) we get that $\partial^C f(\bar{x}) \subset \partial^F f(\bar{x})$ and hence $\partial^C f(\bar{x}) = \partial^F f(\bar{x})$ (the reverse inclusion being always true). So the proof is complete. ∎

We conclude the paper with the theorem below, which concerns the proximal regularity when X is assumed to be a Hilbert space. We omit the proof since it follows the arguments in Theorem 5.1.

Theorem 5.2 *Let X be a Hilbert space, $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c. on X with $\bar{x} \in \operatorname{dom} f$ and let $s(\bar{x}; .)$ be the support function of $\partial^P f(\bar{x})$. Suppose that f is directionally Lipschitzian at $\bar{x}$. Then f is subdifferentially ∂^P-regular at $\bar{x}$ if and only if $\partial^P f$ is topologically closed at $\bar{x}$ and* $\operatorname{dom} f^{\uparrow}(\bar{x}; .) = \operatorname{dom} s(\bar{x}; .)$.

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