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Symmetric and antisymmetric tensor products for the function-theoretic operator theorist

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Abstract. We study symmetric and antisymmetric tensor products of Hilbert-space operators, focusing on norms and spectra for some well-known classes favored by function-theoretic operator theorists. We pose many open questions that should interest the field.

1 Introduction

Tensor products and their symmetrization have appeared in the literature since the mid-nineteenth century, such as in Riemann's foundational work on differential geometry [26, 27]. Tensors describe many-body quantum systems [24] and symmetric tensors underpin the foundations of general relativity [3]. In a separate yet overlapping vein, multilinear algebra [16] and representation theory [11] utilize symmetric tensor product spaces.

Decomposing a symmetric tensor into a minimal linear combination of tensor powers of the same vector arises in mobile communications, machine learning, factor analysis of k-way arrays, biomedical engineering, psychometrics, and chemometrics (see [4, 6, 9, 30, 33] and the references therein). We refer the reader to [5] for a study of this decomposition problem. Symmetric tensors also arise in statistics [23].

In quantum mechanics, many-body systems are represented in terms of tensor products of wave functions. In the simplest case, where the systems do not interact, the Hamiltonian of the many-body system corresponds to a symmetric tensor product of operators [20, Chapter 4, Section 9]. Recently, there has been an endeavor within the physics community to study self-adjoint extensions of symmetric tensor products of operators [18, 19, 22]. Furthermore, the symmetric part of a quantum geometric tensor can be exploited as a tool to detect quantum phase transitions in $\mathcal{PT}$-symmetric quantum mechanics [34].

Unfortunately, there is little literature about symmetric tensor products of non-normal operators. The purpose of this paper is to introduce the basic ideas to

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the function-theoretic operator theory community. We study some fundamental operator-theoretic questions in this area, such as finding the norm and spectrum of symmetric tensor products of operators. We work through some examples with familiar operators, such as the unilateral shift, its adjoint, and diagonal operators. Given the ramifications of symmetric tensor products in a broad spectrum of fields, we hope that initiating this study will shed new light on classical problems and lead to new directions of study for function-theoretic operator theorists.

The layout of this paper is as follows. Section 2 introduces symmetric and antisymmetric tensor power spaces, the domains for the operators in Section 3. In Section 4, we collect results on operator-theoretic properties of symmetric tensor products of bounded operators. The materials in Sections 2–4 are known, but perhaps difficult for the function-theoretic operator theorist to locate in one place. More novel material occupies Sections 5–9, although it is possible that some of the contents of Section 5 have appeared before. Section 5 is devoted to the norms of symmetric tensor powers of operators, while Section 6 focuses on the spectrum. We study symmetric tensor products of diagonal operators in Section 7, the forward and backward shift operators in Section 8, and the symmetric tensor product of shifts and diagonal operators in Section 9. We conclude in Section 10 with a host of open questions that should appeal to researchers in function-theoretic operator theory.

2 Symmetric and antisymmetric tensor power spaces

Symmetric and antisymmetric tensor powers are familiar in mathematical physics, but less so in function-theoretic operator theory. We summarize the basics, with abbreviated explanations or without proof (see [1, Section I.5] or [32, Section 3.8] for the details).

Let $\mathcal{H}$ be a complex Hilbert space, in which the inner product $\langle\cdot,\cdot\rangle$ is linear in the first argument and conjugate linear in the second. We assume that $\mathcal{H}$ has a countable orthonormal basis. A superscript $^-$ denotes the closure with respect to the norm of $\mathcal{H}$.

Let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$. For $\mathbf{u}_1,\mathbf{u}_2\ldots,\mathbf{u}_n\in\mathcal{H}$, the *simple tensor* $\mathbf{u}_1\otimes\mathbf{u}_2\otimes\cdots\otimes\mathbf{u}_n:\mathcal{H}^n\to\mathbb{C}$ acts as follows:

$$(\mathbf{u}_1\otimes\mathbf{u}_2\otimes\cdots\otimes\mathbf{u}_n)(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)=\langle\mathbf{u}_1,\mathbf{v}_1\rangle\langle\mathbf{u}_2,\mathbf{v}_2\rangle\cdots\langle\mathbf{u}_n,\mathbf{v}_n\rangle.$$

A simple tensor is a conjugate-multilinear function of its arguments. The map taking an n-tuple in $\mathcal{H}^n$ to the corresponding simple tensor is linear in each argument.

Let $\mathcal{H}^{\widehat{\otimes}n}$ denote the $\mathbb{C}$-vector space spanned by the simple tensors. There is a unique inner product on $\mathcal{H}^{\widehat{\otimes}n}$ such that

$$\langle\mathbf{u}_1\otimes\mathbf{u}_2\otimes\cdots\otimes\mathbf{u}_n,\,\mathbf{v}_1\otimes\mathbf{v}_2\otimes\cdots\otimes\mathbf{v}_n\rangle:=\langle\mathbf{u}_1,\mathbf{v}_1\rangle\langle\mathbf{u}_2,\mathbf{v}_2\rangle\cdots\langle\mathbf{u}_n,\mathbf{v}_n\rangle \tag{2.1}$$

for all $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n,\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\in\mathcal{H}$ [32, Proposition 3.8.2]. Moreover,

$$\|\mathbf{u}_1\otimes\mathbf{u}_2\otimes\cdots\otimes\mathbf{u}_n\|=\|\mathbf{u}_1\|\|\mathbf{u}_2\|\cdots\|\mathbf{u}_n\|.$$

Definition 2.2 (Tensor powers of Hilbert spaces) Let $\mathcal{H}^{\otimes 0}:=\mathbb{C}$. For $n=1,2,\ldots$, let $\mathcal{H}^{\otimes n}$ denote the completion of $\mathcal{H}^{\widehat{\otimes}n}$ with respect to the inner product (2.1).

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For $n = 2$, we may write $\mathcal{H} \otimes \mathcal{H}$ instead of $\mathcal{H}^{\otimes 2}$. If $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is an orthonormal basis for $\mathcal{H}$, then $\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_n}$ for $(i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$ is an orthonormal basis for $\mathcal{H}^{\otimes n}$. Here, $\mathbb{N} := \{1, 2, 3, \ldots\}$ denotes the set of natural numbers.

Let Σ_n be the group of permutations of $[n] := \{1, 2, \ldots, n\}$. For all $\pi \in \Sigma_n$ and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \in \mathcal{H}$, define

$$\widehat{\pi}(\mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_n) := \mathbf{u}_{\pi(1)} \otimes \mathbf{u}_{\pi(2)} \otimes \cdots \otimes \mathbf{u}_{\pi(n)}.$$

The density of the span of the simple tensors ensures that $\widehat{\pi}$ extends to a bounded linear map on $\mathcal{H}^{\otimes n}$.

Proposition 2.3 *Let $\pi, \tau \in \Sigma_n$. (a) $\widehat{\pi\tau} = \widehat{\pi}\widehat{\tau}$. (b) The map $\widehat{\pi}$ on $\mathcal{H}^{\otimes n}$ is unitary.*

Proof (a) Since the span of the simple tensors is dense in $\mathcal{H}^{\otimes n}$, it suffices to observe that

$$\begin{aligned}\widehat{\pi\tau}(\mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_n) &= \mathbf{u}_{(\pi\tau)(1)} \otimes \mathbf{u}_{(\pi\tau)(2)} \otimes \cdots \otimes \mathbf{u}_{(\pi\tau)(n)} \\ &= \widehat{\pi}(\mathbf{u}_{\tau(1)} \otimes \mathbf{u}_{\tau(2)} \otimes \cdots \otimes \mathbf{u}_{\tau(n)}) \\ &= \widehat{\pi}(\widehat{\tau}(\mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_n))\end{aligned}$$

for any $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \in \mathcal{H}$.

(b) For any $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathcal{H}$, (2.1) ensures that

$$\begin{aligned}&\langle \widehat{\pi}(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n), \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_n \rangle \\ &\quad= \langle \mathbf{v}_{\pi(1)} \otimes \mathbf{v}_{\pi(2)} \otimes \cdots \otimes \mathbf{v}_{\pi(n)}, \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_n \rangle \\ &\quad= \prod_{i=1}^{n} \langle \mathbf{v}_{\pi(i)}, \mathbf{u}_i \rangle = \prod_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u}_{\pi^{-1}(j)} \rangle \\ &\quad= \langle \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n, \mathbf{u}_{\pi^{-1}(1)} \otimes \mathbf{u}_{\pi^{-1}(2)} \otimes \cdots \otimes \mathbf{u}_{\pi^{-1}(n)} \rangle \\ &\quad= \langle \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n, \widehat{\pi^{-1}}(\mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_n) \rangle.\end{aligned}$$

Therefore, $\widehat{\pi}^* = \widehat{\pi^{-1}}$ and hence $\widehat{\pi}^{-1} = \widehat{\pi}^*$ by (a). ■

We now define certain subspaces of $\mathcal{H}^{\otimes n}$ that respect the action of the operators $\widehat{\pi}$.

Definition 2.4 (Symmetric and antisymmetric tensor powers of Hilbert spaces). Let $\operatorname{sgn} \pi$ denote the sign of a permutation $\pi \in \Sigma_n$.

(a) Let $\mathcal{H}^{\odot 1} := \mathcal{H}$ and $\mathcal{H}^{\odot n} := \{\mathbf{v} \in \mathcal{H}^{\otimes n} : \widehat{\pi}(\mathbf{v}) = \mathbf{v} \text{ for all } \pi \in \Sigma_n\}$ for $n \geqslant 2$.
(b) Let $\mathcal{H}^{\wedge 1} := \{\mathbf{0}\}$ and $\mathcal{H}^{\wedge n} := \{\mathbf{v} \in \mathcal{H}^{\otimes n} : \widehat{\pi}(\mathbf{v}) = (-1)^{\operatorname{sgn}\pi}\mathbf{v} \text{ for all } \pi \in \Sigma_n\}$ for $n \geqslant 2$.

We may write $\mathcal{H} \odot \mathcal{H}$ and $\mathcal{H} \wedge \mathcal{H}$ instead of $\mathcal{H}^{\odot 2}$ and $\mathcal{H}^{\wedge 2}$, respectively. In this case, there is only one nonidentity $\pi \in \Sigma_2$.

Example 2.5 Let $H^2(\mathbb{D})$ denote the Hardy space on the unit disk $\mathbb{D}$. The monomials $1, z, z^2, \ldots$ are an orthonormal basis for $H^2(\mathbb{D})$, so the simple tensors $z^i \otimes z^j$ for $i, j = 0, 1, \ldots$ are an orthonormal basis for $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$. The unitary map $z^i \otimes z^j \mapsto z^i w^j$ identifies $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$ with $H^2(\mathbb{D}^2)$, the Hardy space on the bidisk $\mathbb{D}^2$ [10]. Thus, we identify $H^2(\mathbb{D}) \odot H^2(\mathbb{D})$ and $H^2(\mathbb{D}) \wedge H^2(\mathbb{D})$ with

$$H^2_{\text{sym}}(\mathbb{D}^2) := \{f(z, w) \in H^2(\mathbb{D}^2) : f(z, w) = f(w, z) \text{ for all } z, w \in \mathbb{D}\} \tag{2.6}$$

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and

$$H^2_{\mathrm{asym}}(\mathbb{D}^2) := \{f(z,w) \in H^2(\mathbb{D}^2) : f(z,w) = -f(w,z) \text{ for all } z,w \in \mathbb{D}\}, \tag{2.7}$$

respectively. We freely use these identifications in what follows.

Definition 2.8 (Symmetrization and antisymmetrization operators) Define $\mathrm{A}_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ and $\mathrm{S}_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ by

$$\mathrm{S}_n := \frac{1}{n!}\sum_{\pi \in \Sigma_n} \widehat{\pi} \qquad \text{and} \qquad \mathrm{A}_n := \frac{1}{n!}\sum_{\pi\in\Sigma_n} \operatorname{sgn}(\pi)\widehat{\pi}.$$

Proposition 2.9 (a) S_n *is the orthogonal projection from* $\mathcal{H}^{\otimes n}$ *onto* $\mathcal{H}^{\odot n}$.
(b) A_n *is the orthogonal projection from* $\mathcal{H}^{\otimes n}$ *onto* $\mathcal{H}^{\wedge n}$.
In particular, $\mathcal{H}^{\odot n}$ *and* $\mathcal{H}^{\wedge n}$ *are closed subspaces of* $\mathcal{H}^{\otimes n}$.

Proof (a) Use Proposition 2.3 and the fact that $\widehat{\pi}\mathrm{S}_n = \mathrm{S}_n$ for all $\pi \in \Sigma_n$ to show that $\mathrm{S}_n^2 = \mathrm{S}_n = \mathrm{S}_n^*$ and $\operatorname{ran}\mathrm{S}_n = \mathcal{H}^{\odot n}$. The proof of (b) is similar. ∎

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathcal{H}$ and define the simple *symmetric* and *antisymmetric tensors*

$$\mathbf{v}_1 \odot \mathbf{v}_2 \odot \cdots \odot \mathbf{v}_n := \mathrm{S}_n(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n) \quad \text{and}$$
$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n := \mathrm{A}_n(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n).$$

A factor of $1/\sqrt{n!}$ is included in some sources [32, (3.8.33)] and $\vee$ is sometimes used instead of $\odot$. Note that $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n = \mathbf{0}$ if $\mathbf{v}_i = \mathbf{v}_j$ for some $i \neq j$.

Proposition 2.10 *Let* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots$ *be an orthonormal basis for* $\mathcal{H}$.

(a) $\mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_n}$ *for* $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n$ *form an orthogonal basis for* $\mathcal{H}^{\odot n}$.
(b) $\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_n}$ *for* $1 < i_1 < i_2 < \cdots < i_n$ *form an orthogonal basis for* $\mathcal{H}^{\wedge n}$.

We say "orthogonal" instead of "orthonormal" because the vectors described in the previous proposition need not be unit vectors. Let m_ℓ denote the number of occurrences of ℓ in $(i_1, i_2, \ldots, i_n) \in [d]^n$. Then there are $m_1!m_2!\cdots m_d!$ permutations of $\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_n}$ that give rise to the same simple tensor. Thus,

$$\|\mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_n}\| = \left(\frac{m_1!m_2!\cdots m_r!}{n!}\right)^{1/2}.$$

If $\dim \mathcal{H} = d$ is finite, then (using the notation for binomial coefficients)

$$\dim \mathcal{H}^{\odot n} = \binom{d+n-1}{n} \quad \text{and} \quad \dim \mathcal{H}^{\wedge n} = \begin{cases} \binom{d}{n}, & \text{if } n \leqslant d,\\ 0, & \text{if } n > d.\end{cases}$$

The case $n = 2$ is special since $\dim \mathcal{H}^{\otimes 2} = d^2 = \binom{d+1}{2} + \binom{d}{2} = \dim \mathcal{H}^{\odot 2} + \dim \mathcal{H}^{\wedge 2}$, which suggests Proposition 2.11. The simple symmetric and antisymmetric tensors are

$$\mathbf{u} \odot \mathbf{v} = \tfrac{1}{2}(\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u}) \in \mathcal{H}^{\odot 2} \quad \text{and} \quad \mathbf{u} \wedge \mathbf{v} = \tfrac{1}{2}(\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u}) \in \mathcal{H}^{\wedge 2}$$

for $\mathbf{u}, \mathbf{v} \in \mathcal{H}$. If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots$ is an orthonormal basis for $\mathcal{H}$, then

(a) $\sqrt{2}(\mathbf{e}_i \odot \mathbf{e}_j)$ for $i < j$ and $\mathbf{e}_i \odot \mathbf{e}_i$ for $i \geqslant 1$ form an orthonormal basis for $\mathcal{H}^{\odot 2}$, and
(b) $\sqrt{2}(\mathbf{e}_i \wedge \mathbf{e}_j)$ for $i < j$ form an orthonormal basis for $\mathcal{H}^{\wedge 2}$.

Proposition 2.11 *$\mathcal{H}^{\otimes 2} = \mathcal{H}^{\odot 2} \oplus \mathcal{H}^{\wedge 2}$ is an orthogonal decomposition.*

Proof Let π be the nonidentity permutation in Σ_2. If $\mathbf{x} \in \mathcal{H}^{\otimes 2}$, then

$$\mathbf{x} = \underbrace{\tfrac{1}{2}(\mathbf{x} + \widehat{\pi}(\mathbf{x}))}_{S_2(\mathbf{x}) \in \mathcal{H}^{\odot 2}} + \underbrace{\tfrac{1}{2}(\mathbf{x} - \widehat{\pi}(\mathbf{x}))}_{A_2(\mathbf{x}) \in \mathcal{H}^{\wedge 2}}$$

and hence $\mathcal{H}^{\otimes 2} = \mathcal{H}^{\odot 2} + \mathcal{H}^{\wedge 2}$. Since $\widehat{\pi}$ is unitary (Proposition 2.3) and involutive (since $\widehat{\pi}^2 = I$), it is self-adjoint. Let $\mathbf{u} \in \mathcal{H}^{\odot 2}$ and $\mathbf{v} \in \mathcal{H}^{\wedge 2}$, then $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \widehat{\pi}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \widehat{\pi}\mathbf{v} \rangle = \langle \mathbf{u}, -\mathbf{v} \rangle = -\langle \mathbf{u}, \mathbf{v} \rangle$, so $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. Thus, $\mathcal{H}^{\odot 2} \cap \mathcal{H}^{\wedge 2} = \{\mathbf{0}\}$, and $\mathcal{H}^{\odot 2} \perp \mathcal{H}^{\wedge 2}$. ∎

Example 2.12 Recall from Example 2.5 the identification of $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$ with $H^2(\mathbb{D}^2)$. The orthogonal decomposition of Proposition 2.11 becomes

$$H^2(\mathbb{D}^2) = H^2_{\text{sym}}(\mathbb{D}^2) \oplus H^2_{\text{asym}}(\mathbb{D}^2), \tag{2.13}$$

where the direct summands are defined by (2.6) and (2.7), respectively. In this context, $z^i w^i$ and $(z^i w^j + z^j w^i)/\sqrt{2}$ for $0 \leqslant i < j$ form an orthonormal basis for $H^2_{\text{sym}}(\mathbb{D}^2)$ and $(z^i w^j - z^j w^i)/\sqrt{2}$ for $i < j$ form an orthonormal basis for $H^2_{\text{asym}}(\mathbb{D}^2)$.

Lemma 2.14 *If $\sum_{i \leqslant j} |a_{ij}|^2 < \infty$, then $\sum_{i \leqslant j} a_{ij} \mathbf{e}_i \odot \mathbf{e}_j \in \mathcal{H} \odot \mathcal{H}$.*

Proof Proposition (2.10) ensures that

$$\Big\| \sum_{i \leqslant j} a_{ij} \mathbf{e}_i \odot \mathbf{e}_j \Big\|^2 = \Big\| \sum_{i<j} \frac{a_{ij}}{\sqrt{2}} \sqrt{2} \mathbf{e}_i \odot \mathbf{e}_j + \sum_{i=1}^{\infty} a_{ii} \mathbf{e}_i \odot \mathbf{e}_i \Big\|^2 \leqslant \sum_{i \leqslant j} |a_{ij}|^2 < \infty. \qquad ∎$$

Lemma 2.15 *For $\mathbf{u}, \mathbf{v} \in \mathcal{H}$, we have $\frac{1}{\sqrt{2}} \|\mathbf{u}\| \|\mathbf{v}\| \leqslant \|\mathbf{u} \odot \mathbf{v}\| \leqslant \|\mathbf{u}\| \|\mathbf{v}\|$; both inequalities are sharp. In particular, the symmetric tensor product of two nonzero vectors is nonzero.*

Proof The Cauchy–Schwarz inequality provides the upper inequality since

$$\begin{aligned} \|\mathbf{u} \odot \mathbf{v}\|^2 &= \tfrac{1}{4} \|\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u}\|^2 = \tfrac{1}{4} \langle \mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u}, \ \mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u} \rangle \\ &= \tfrac{1}{4} (\|\mathbf{u} \otimes \mathbf{v}\|^2 + \|\mathbf{v} \otimes \mathbf{u}\|^2 + |\langle \mathbf{u} \otimes \mathbf{v}, \mathbf{v} \otimes \mathbf{u} \rangle|^2) \\ &= \tfrac{1}{4} (2\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 + 2|\langle \mathbf{u}, \mathbf{v} \rangle|^2) \\ &\leqslant \tfrac{1}{4} (2\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|^2 \|\mathbf{v}\|^2) \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2. \end{aligned} \tag{2.16}$$

In (2.16), $|\langle \mathbf{u}, \mathbf{v} \rangle|^2$ is nonnegative, so we obtain the lower inequality. The upper inequality is sharp if $\mathbf{u} = \mathbf{v}$ and the lower inequality is sharp if $\mathbf{u} \perp \mathbf{v}$. ∎

3 Symmetric and antisymmetric tensor products of operators

For $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$, define $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ on simple tensors by

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_n)(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_n) = A_1\mathbf{v}_1 \otimes A_2\mathbf{v}_2 \otimes \cdots \otimes A_n\mathbf{v}_n.$$

This extends by linearity to the linear span $\mathcal{H}^{\widehat{\otimes} n}$ of the simple tensors. The density of $\mathcal{H}^{\widehat{\otimes} n}$ in $\mathcal{H}^{\otimes n}$ ensures that $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ has a unique bounded extension to $\mathcal{H}^{\otimes n}$, also denoted $A_1 \otimes A_2 \otimes \cdots \otimes A_n$, which satisfies [32, (3.8.17)]:

$$\|A_1 \otimes A_2 \otimes \cdots \otimes A_n\| = \|A_1\|\|A_2\| \cdots \|A_n\|. \tag{3.1}$$

We may write $A^{\otimes n}$ instead of $A \otimes A \otimes \cdots \otimes A$ (n times).

Proposition 3.2 *Let $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{H}^{\odot n}$ and $\mathcal{H}^{\wedge n}$ are invariant under*

$$\mathrm{S}_n(A_1, A_2, \ldots, A_n) = \frac{1}{n!} \sum_{\pi \in \Sigma_n} (A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)}) \in \mathcal{B}(\mathcal{H}^{\otimes n}).$$

Proof Let $T = \mathrm{S}_n(A_1, A_2, \ldots, A_n)$. For $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathcal{H}$,

$$\begin{aligned} T(\mathbf{v}_1 \odot \mathbf{v}_2 \odot \cdots \odot \mathbf{v}_n) &= \frac{1}{n!} \sum_{\pi \in \Sigma_n} (A_{\pi(1)} \otimes A_{\pi(2)} \cdots \otimes A_{\pi(n)})(\mathbf{v}_1 \odot \mathbf{v}_2 \odot \cdots \odot \mathbf{v}_n) \\ &= \frac{1}{n!} \sum_{\pi \in \Sigma_n} (A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)}) \left(\frac{1}{n!} \sum_{\tau \in \Sigma_n} \mathbf{v}_{\tau(1)} \otimes \mathbf{v}_{\tau(2)} \otimes \cdots \otimes \mathbf{v}_{\tau(n)} \right) \\ &= \frac{1}{(n!)^2} \sum_{\pi \in \Sigma_n} \sum_{\tau \in \Sigma_n} (A_{\pi(1)}\mathbf{v}_{\tau(1)} \otimes A_{\pi(2)}\mathbf{v}_{\tau(2)} \otimes \cdots \otimes A_{\pi(n)}\mathbf{v}_{\tau(n)}) \\ &= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} A_{\sigma(1)}\mathbf{v}_1 \odot A_{\sigma(2)}\mathbf{v}_2 \odot \cdots \odot A_{\sigma(n)}\mathbf{v}_n, \end{aligned}$$

a sum of elements in $\mathcal{H}^{\odot n}$. The density of the simple symmetric tensors in $\mathcal{H}^{\odot n}$ ensures that $T\mathcal{H}^{\odot n} \subseteq \mathcal{H}^{\odot n}$. The proof that $T\mathcal{H}^{\wedge n} \subseteq \mathcal{H}^{\wedge n}$ is similar. ■

The proposition above suggests the following definition.

Definition 3.3 (Symmetric tensor products of operators) Let $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$. Then $A_1 \odot A_2 \odot \cdots \odot A_n$ and $A_1 \wedge A_2 \wedge \cdots \wedge A_n$ are the restrictions of

$$\mathrm{S}_n(A_1, A_2, \ldots, A_n) = \frac{1}{n!} \sum_{\pi \in \Sigma_n} (A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)})$$

to $\mathcal{H}^{\odot n}$ and $\mathcal{H}^{\wedge n}$, respectively. We may write $A^{\odot n}$ and $A^{\wedge n}$ instead of $A \odot A \odot \cdots \odot A$ (n times) and $A \wedge A \wedge \cdots \wedge A$ (n times), respectively.

Symmetric tensor products are permutation invariant:

$$A_1 \odot A_2 \odot \cdots \odot A_n = A_{\pi(1)} \odot A_{\pi(2)} \odot \cdots \odot A_{\pi(n)} \quad \text{for all } \pi \in \Sigma_n.$$

If $A, B, C \in \mathcal{B}(\mathcal{H})$, then the domain of $A \odot B$ is $\mathcal{H} \odot \mathcal{H}$, which is not equal to $\mathcal{H}$, the domain of C. Thus, $(A \odot B) \odot C$ is not well defined. Note that $I \odot I \odot \cdots \odot I = I$.

Proposition 3.4 (a) *For all $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$, we have $\|A_1 \odot A_2 \odot \cdots \odot A_n\| \leqslant \|A_1\|\|A_2\| \cdots \|A_n\|$.* (b) *For all $A \in \mathcal{B}(\mathcal{H})$, we have $\|A^{\odot n}\| = \|A\|^n$.*

Proof (a) Since $A_1 \odot A_2 \odot \cdots \odot A_n$ is the restriction of $\frac{1}{n!} \sum_{\pi \in \Sigma_n} (A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)})$ to $\mathcal{H}^{\odot n}$, its norm is at most

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$$\left\|\frac{1}{n!}\sum_{\pi\in\Sigma_n}(A_{\pi(1)}\otimes\cdots\otimes A_{\pi(n)})\right\| \leqslant \frac{1}{n!}\sum_{\pi\in\Sigma_n}\|(A_{\pi(1)}\otimes A_{\pi(2)}\otimes\cdots\otimes A_{\pi(n)})\|$$
$$= \frac{1}{n!}\sum_{\pi\in\Sigma_n}\|A_{\pi(1)}\|\|A_{\pi(2)}\|\cdots\|A_{\pi(n)}\|$$
$$= \frac{1}{n!}\sum_{\pi\in\Sigma_n}\|A_1\|\|A_2\|\ldots\|A_n\|$$
$$= \|A_1\|\|A_2\|\ldots\|A_n\|.$$

(b) First, we have $\|A^{\odot n}\| \leqslant \|A\|^n$ from (a). Then

$$\|A\|^n = \sup_{\substack{\mathbf{v}\in\mathcal{H}\\ \|\mathbf{v}\|=1}}\|A\mathbf{v}\|^n = \sup_{\substack{\mathbf{v}\in\mathcal{H}\\ \|\mathbf{v}\|=1}}\|A^{\otimes n}(\mathbf{v}\otimes\mathbf{v}\otimes\cdots\otimes\mathbf{v})\| \leqslant \|A^{\odot n}\|. \qquad \blacksquare$$

Example 3.5 If $A, B \in \mathcal{B}(\mathcal{H})$, then Propositions 2.11 and 3.2 ensure that

$$\frac{1}{2}(A\otimes B + B\otimes A) = \begin{bmatrix} A\odot B & 0 \\ 0 & A\wedge B\end{bmatrix} : \begin{bmatrix}\mathcal{H}\odot\mathcal{H} \\ \mathcal{H}\wedge\mathcal{H}\end{bmatrix} \to \begin{bmatrix}\mathcal{H}\odot\mathcal{H} \\ \mathcal{H}\wedge\mathcal{H}\end{bmatrix}. \tag{3.6}$$

Example 3.7 Let $\mathcal{H} = H^2(\mathbb{D})$ and let $T_g : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the Toeplitz operator with symbol $g \in L^\infty(\mathbb{D})$. Then (3.1) ensures that $T_g \otimes T_g : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$, the linear extension of the map $z^i w^j \mapsto T_g(z^i)T_g(w^j)$, has norm $\|g\|_\infty^2$. Proposition 3.4 says that $T_g \odot T_g$, the restriction of $T_g \otimes T_g$ to $H^2_{\mathrm{sym}}(\mathbb{T}^2)$, also has norm $\|g\|_\infty^2$.

4 Basic properties

In this section, we collect some results on the operator-theoretic properties of symmetric tensor products of bounded Hilbert-space operators.

Lemma 4.1 $(A\odot B)(C\odot D) = \frac{1}{2}(AC\odot BD + AD\odot BC)$ *for* $A, B, C, D \in \mathcal{B}(\mathcal{H})$.

Proof Restrict $\frac{1}{2}(A\otimes B + B\otimes A)\frac{1}{2}(C\otimes D + D\otimes C) = \frac{1}{4}(AC\otimes BD + BD\otimes AC + AD\otimes BC + BC\otimes AD)$ to $\mathcal{H}\odot\mathcal{H}$ and obtain the desired formula. $\blacksquare$

Example 4.2 Equip $\mathbb{C}^2$ with the standard basis $\mathbf{e}_1, \mathbf{e}_2$ and consider

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} \quad\text{and}\quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}. \tag{4.3}$$

With respect to the orthonormal basis $\mathbf{e}_1\otimes\mathbf{e}_1, \mathbf{e}_1\otimes\mathbf{e}_2, \mathbf{e}_2\otimes\mathbf{e}_1, \mathbf{e}_2\otimes\mathbf{e}_2$ of $\mathcal{H}\otimes\mathcal{H}$, we see that $\frac{1}{2}(A\otimes B + B\otimes A)$ has the matrix representation

$$\frac{1}{2}\begin{bmatrix} 2a_{11}b_{11} & a_{11}b_{12}+b_{11}a_{12} & a_{12}b_{11}+b_{12}a_{11} & 2a_{12}b_{12} \\ a_{11}b_{21}+b_{11}a_{21} & a_{11}b_{22}+b_{11}a_{22} & a_{12}b_{21}+b_{12}a_{21} & a_{12}b_{22}+b_{12}a_{22} \\ a_{21}b_{11}+b_{21}a_{11} & a_{21}b_{12}+b_{21}a_{12} & a_{22}b_{11}+b_{22}a_{11} & a_{22}b_{12}+b_{22}a_{12} \\ 2a_{21}b_{21} & a_{21}b_{22}+b_{21}a_{22} & a_{22}b_{21}+b_{22}a_{21} & 2a_{22}b_{22}\end{bmatrix}.$$

With respect to the orthonormal basis $\mathbf{e}_1\odot\mathbf{e}_1, \sqrt{2}(\mathbf{e}_1\odot\mathbf{e}_2), \mathbf{e}_2\odot\mathbf{e}_2$ of $\mathcal{H}\odot\mathcal{H}$, the symmetric tensor product $A\odot B$ has the matrix representation

$$\begin{bmatrix} a_{11}b_{11} & \frac{a_{11}b_{12}+b_{11}a_{12}}{\sqrt{2}} & a_{12}b_{12} \\ \frac{a_{11}b_{21}+b_{11}a_{21}}{\sqrt{2}} & \frac{a_{11}b_{22}+b_{11}a_{22}+a_{12}b_{21}+b_{12}a_{21}}{2} & \frac{a_{12}b_{22}+b_{12}a_{22}}{\sqrt{2}} \\ a_{21}b_{21} & \frac{a_{21}b_{22}+b_{21}a_{22}}{\sqrt{2}} & a_{22}b_{22}\end{bmatrix}. \tag{4.4}$$

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Proposition 4.5 *If $A_1, A_2, \dots, A_n \in \mathcal{B}(\mathcal{H})$ have a common invariant subspace $\mathcal{V} \subseteq \mathcal{H}$, then $\odot^n \mathcal{V}$ is invariant for $A_1 \odot A_2 \odot \cdots \odot A_n$.*

Proof This follows from Proposition 3.2. ∎

Proposition 4.6 *Let $A_1, A_2, \dots, A_n \in \mathcal{B}(\mathcal{H})$. Then $(A_1 \odot A_2 \odot \cdots \odot A_n)^* = A_1^* \odot A_2^* \odot \cdots \odot A_n^*$ and $(A_1 \wedge A_2 \wedge \cdots \wedge A_n)^* = A_1^* \wedge A_2^* \wedge \cdots \wedge A_n^*$.*

Proof Recall that $\mathcal{H}^{\odot n}$ and $\mathcal{H}^{\wedge n}$ are invariant under $S_n(A_1, A_2, \dots, A_n)$. Since $(A_1 \otimes A_2 \otimes \cdots \otimes A_n)^* = A_1^* \otimes A_2^* \otimes \cdots \otimes A_n^*$, the result follows. ∎

Remark 4.7 Observe that $(A \odot A^*)^* = A^* \odot A = A \odot A^*$; that is, $A \odot A^*$ is self-adjoint. For example, for the 2×2 matrix A in (4.3), the formula (4.4) gives

$$A \odot A^* = \begin{bmatrix} |a_{11}|^2 & \frac{a_{11}\overline{a_{21}} + \overline{a_{11}} a_{12}}{\sqrt{2}} & a_{12}\overline{a_{21}} \\ \frac{a_{11}\overline{a_{12}} + \overline{a_{11}} a_{21}}{\sqrt{2}} & \frac{a_{11}\overline{a_{22}} + \overline{a_{11}} a_{22} + |a_{12}|^2 + |a_{21}|^2}{2} & \frac{a_{12}\overline{a_{22}} + \overline{a_{21}} a_{22}}{\sqrt{2}} \\ a_{21}\overline{a_{12}} & \frac{a_{21}\overline{a_{22}} + \overline{a_{12}} a_{22}}{\sqrt{2}} & |a_{22}|^2 \end{bmatrix}.$$

Theorem 4.8 *Let $A_1, A_2, \dots, A_n \in \mathcal{B}(\mathcal{H})$.*

(a) *If $A_1, A_2, \dots, A_n$ are self-adjoint, then $A_1 \odot A_2 \odot \cdots \odot A_n$ is self-adjoint.*
(b) *If $A_1, A_2, \dots, A_n$ are normal and commute, then $A_1 \odot A_2 \odot \cdots \odot A_n$ is normal.*
(c) *If $U \in \mathcal{B}(\mathcal{H})$ is unitary, then $U \odot U \odot \cdots \odot U$ is unitary.*

Proof (a) This follows from Proposition 4.6.

(b) The Fuglede–Putnam theorem [25] ensures that $A_i A_j^* = A_j^* A_i$ for $1 \leqslant i, j \leqslant n$. Proposition 4.6 and a computation establish the normality of $A_1 \odot A_2 \odot \cdots \odot A_n$.

(c) By Proposition 4.6, $(U \odot U \odot \cdots \odot U)^*(U \odot U \odot \cdots \odot U) = I \odot I \odot \cdots \odot I$ and similarly $(U \odot U \odot \cdots \odot U)(U \odot U \odot \cdots \odot U)^* = I \odot I \odot \cdots \odot I$. ∎

Example 4.9 The normal matrices $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ do not commute, but $A \odot B = \begin{bmatrix} 1 & -\frac{1-i}{\sqrt{2}} & -i \\ \frac{1+i}{\sqrt{2}} & 1 & -\frac{1-i}{\sqrt{2}} \\ i & \frac{1+i}{\sqrt{2}} & 1 \end{bmatrix}$ is not normal. Thus, the commutativity hypothesis is necessary in (b).

Example 4.10 If $A, B \in \mathcal{B}(\mathcal{H})$ are self-adjoint and noncommuting, then $A \odot B$ is self-adjoint, and hence normal. Thus, the converse of (b) is false.

Example 4.11 The matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are unitary, but $A \odot B = \begin{bmatrix} 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix}$ is not.

Proposition 4.12 *For orthogonal projections P, Q, where $P, Q \neq 0, I$ and $PQ = QP = 0$, the map $2P \odot Q$ is an orthogonal projection. Furthermore, $2P \odot Q \neq I \odot I, 0$.*

Proof Since P and Q are self-adjoint, $2S_2(P, Q)$ is self-adjoint. Since $PQ = QP = 0$, we have $(2S_2(P, Q))^2 = 2S_2(P, Q)$. Thus, $2S_2(P, Q)$ is an orthogonal projection, so $2S_2(P, Q)|_{\mathcal{H} \odot \mathcal{H}} = 2P \odot Q$ is an orthogonal projection. To show $2P \odot Q \neq I \odot I, 0$

observe if $\mathbf{x}, \mathbf{y} \in \mathcal{H}$ are nonzero, $P\mathbf{x} = \mathbf{x}$, and $Q\mathbf{y} = \mathbf{y}$, then $(2P \odot Q)(\mathbf{x} \odot \mathbf{y}) = \mathbf{x} \odot \mathbf{y}$ and $(2P \odot Q)(\mathbf{x} \odot \mathbf{x}) = 0$. ∎

One can define tensor powers of bounded conjugate-linear operators. An analogue of Proposition 3.2 shows that $\mathcal{H}^{\odot n}$ is invariant under $\mathrm{S}_n(C_1, C_2, \ldots, C_n)$ for any bounded conjugate-linear operators $C_1, C_2, \ldots, C_n$. We say that C is a *conjugation* on $\mathcal{H}$ if C is conjugate linear, isometric, and involutive. We say that $T \in B(\mathcal{H})$ is *C-symmetric* if $T = CT^*C$ [13–15]. If C is a conjugation, let $C^{\odot n}$ denote the restriction of $C^{\otimes n}$ to $\mathcal{H}^{\odot n}$.

Proposition 4.13 *Let C be a conjugation on $\mathcal{H}$, and let $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ be C-symmetric. (a) $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ is $C^{\otimes n}$-symmetric. (b) $A_1 \odot A_2 \odot \cdots \odot A_n$ is $C^{\odot n}$-symmetric.*

Proof (a) Since $(A_1 \otimes A_2 \otimes \cdots \otimes A_n)^* = A_1^* \otimes A_2^* \otimes \cdots \otimes A_n^*$, the result follows.

(b) Since $C^{\otimes n}(\mathcal{H}^{\odot n}) \subseteq \mathcal{H}^{\odot n}$, it follows that $C^{\odot n}$ is a well-defined conjugation on $\mathcal{H}^{\odot n}$. The desired result follows from part (a) and Proposition 4.6. ∎

5 Norms and spectral radius

In this section, we provide various bounds for the norm of symmetric tensor products of operators, as well as a spectral-radius formula for symmetric tensor powers. It may be that (a) and (b) of Theorem 5.1 are already known, although we did not encounter them before.

Theorem 5.1 *Let $A, B \in \mathcal{B}(\mathcal{H})$.*

(a) $\frac{1}{\sqrt{2}} \sup_{\mathbf{x} \in \mathcal{H}, \|\mathbf{x}\|=1} \|A\mathbf{x}\| \|B\mathbf{x}\| \leqslant \|A \odot B\|$, *and this is sharp.*
(b) *If $A, B \neq 0$, then $A \odot B \neq 0$.*
(c) $\rho(A^{\odot n}) = \rho(A)^n$, *in which $\rho(A) := \sup\{|\lambda| \in \sigma(A)\}$ is the spectral radius*

Proof (a) If $\|\mathbf{x}\| = 1$, then $\mathbf{x} \otimes \mathbf{x} \in \mathcal{H} \odot \mathcal{H}$ has norm one, so Lemma 2.15 ensures that

$$\frac{\|A\mathbf{x}\| \|B\mathbf{x}\|}{\sqrt{2}} \leqslant \|A\mathbf{x} \odot B\mathbf{x}\| = \left\| \frac{(A \otimes B + B \otimes A)(\mathbf{x} \otimes \mathbf{x})}{2} \right\| \leqslant \|A \odot B\|.$$

Equality is attained for $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, and $\mathbf{x} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Indeed, (4.4) ensures that

$$A \odot B = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and hence} \quad \|A \odot B\| = \frac{1}{\sqrt{2}},$$

while $\mathbf{x}$ is of unit norm and $\|A\mathbf{x}\| = \|B\mathbf{x}\| = 1$, so $\frac{\|A\mathbf{x}\| \|B\mathbf{x}\|}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \|A \odot B\|$.

(b) Let $A, B \neq 0$. If there is a unit vector $\mathbf{x}$ such that $A\mathbf{x} \neq \mathbf{0}$ and $B\mathbf{x} \neq \mathbf{0}$, then (a) ensures that $0 < \frac{1}{\sqrt{2}} \|A\mathbf{x}\| \|B\mathbf{x}\| \leqslant \|A \odot B\|$. So suppose that $A\mathbf{x} = \mathbf{0}$ or $B\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathcal{H}$. Pick $\mathbf{u}$ such that $A\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v}$ such that $B\mathbf{v} \neq \mathbf{0}$. Then $B\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$; moreover, $\mathbf{u} \neq -\mathbf{v}$. Let $\mathbf{x} = \frac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}$, then (a) leads to the contradiction

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$$0 < \frac{1}{\sqrt{2}} \frac{\|A\mathbf{u}\|\|B\mathbf{v}\|}{\|\mathbf{u}+\mathbf{v}\|^2} = \frac{1}{\sqrt{2}}\|A\mathbf{x}\|\|B\mathbf{x}\| \leqslant \|A \odot B\|.$$

In both cases, $A \odot B \neq 0$ since it has positive norm.

(c) Since $(A^{\odot n})^k = (A^k)^{\odot n}$ for each $k \in \mathbb{N}$, Proposition 3.4 ensures that $\|(A^{\odot n})^k\| = \|(A^k)^{\odot n}\| = \|A^k\|^n$. Gelfand's formula [7, Proposition 3.8, Chapter 5] yields

$$\rho(A^{\odot n}) = \inf_{k\in\mathbb{N}} \|(A^{\odot n})^k\|^{\frac{1}{k}} = \inf_{k\in\mathbb{N}} \|A^k\|^{\frac{n}{k}} = \rho(A)^n. \qquad \blacksquare$$

In contrast to symmetric tensor products, the antisymmetric products of nonzero operators may be 0. If P is a rank-one orthogonal projection, then $P^{\wedge n} = 0$ for $n \geqslant 2$.

Theorem 5.2 (a) *If $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ and the A_i have orthogonal ranges, then $\|A_1 \odot A_2 \odot \cdots \odot A_n\| \leqslant \frac{1}{\sqrt{n!}}\|A_1\|\|A_2\|\cdots\|A_n\|$. For $n = 2$, the inequality is sharp.*

(b) *If $(\ker B)^\perp \subseteq \ker A$ and $\operatorname{ran} B \subseteq (\operatorname{ran} A)^\perp$, then $\frac{1}{2}\|A\|\|B\| \leqslant \|A \odot B\| \leqslant \frac{1}{\sqrt{2}}\|A\|\|B\|$. The inequalities are sharp.*

Proof (a) Recall that the set of finite sums $\sum_{i=1}^k \mathbf{v}_i^1 \otimes \mathbf{v}_i^2 \otimes \cdots \otimes \mathbf{v}_i^n$ of simple tensors are dense in $\mathcal{H}^{\otimes n}$. Take the supremum over such vectors and observe that

$$\begin{aligned}
&\Big\| \sum_{\pi\in\Sigma_n} A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)} \Big\| \\
&\quad = \sup \frac{\|(\sum_{\pi\in\Sigma_n} A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)})(\sum_{i=1}^k \mathbf{v}_i^1 \otimes \mathbf{v}_i^2 \otimes \cdots \otimes \mathbf{v}_i^n)\|}{\|\sum_{i=1}^k \mathbf{v}_i^1 \otimes \mathbf{v}_i^2 \otimes \cdots \otimes \mathbf{v}_i^n\|} \\
&\quad = \sup \frac{\|(\sum_{\pi\in\Sigma_n} \sum_{i=1}^k (A_{\pi(1)}\mathbf{v}_i^1 \otimes A_{\pi(2)}\mathbf{v}_i^2 \otimes \cdots \otimes A_{\pi(n)}\mathbf{v}_i^n)\|}{\|\sum_{i=1}^k \mathbf{v}_i^1 \otimes \mathbf{v}_i^2 \otimes \cdots \otimes \mathbf{v}_i^n\|} \\
&\quad = \sup \frac{(\sum_{\pi\in\Sigma_n} \|\sum_{i=1}^k A_{\pi(1)}\mathbf{v}_i^1 \otimes A_{\pi(2)}\mathbf{v}_i^2 \otimes \cdots \otimes A_{\pi(n)}\mathbf{v}_i^n\|^2)^{1/2}}{\|\sum_{i=1}^k \mathbf{v}_i^1 \otimes \mathbf{v}_i^2 \otimes \cdots \otimes \mathbf{v}_i^n\|} \\
&\quad = \sup_{\mathbf{x}\in\mathcal{H}^{\otimes n}} \frac{\left(\sum_{\pi\in\Sigma_n} \|(A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)})\mathbf{x}\|^2\right)^{1/2}}{\|\mathbf{x}\|} \\
&\quad \leqslant \sqrt{n!}\|A_1\|\|A_2\|\cdots\|A_n\|.
\end{aligned}$$

The prepenultimate equality above follows because the A_i have orthogonal ranges; the final inequality is due to (3.1). For $n = 2$, the matrices from the proof of Theorem 5.1(a) have orthogonal ranges and demonstrate that the inequality is sharp.

(b) Part (a) ensures that the desired upper inequality holds and is sharp. It suffices to examine the lower inequality. For $\mathbf{u} \in (\ker A)^\perp$ and $\mathbf{v} \in \ker A$,

$$\begin{aligned}
\|(A \otimes B + B \otimes A)(\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u})\| &= \|A\mathbf{u} \otimes B\mathbf{v} + B\mathbf{v} \otimes A\mathbf{u}\| \\
&= (\|A\mathbf{u} \otimes B\mathbf{v}\|^2 + \|B\mathbf{v} \otimes A\mathbf{u}\|^2)^{1/2}
\end{aligned}$$

since $(A\mathbf{u} \otimes B\mathbf{v}) \perp (B\mathbf{v} \otimes A\mathbf{u})$. Then

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$$
\begin{aligned}
\|A \odot B\| &\geqslant \sup_{\substack{\mathbf{u}\in\ker A^{\perp}\\ \mathbf{v}\in\ker A}} \frac{\|(A\odot B)(\mathbf{u}\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u})\|}{\|\mathbf{u}\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}\|}\\
&= \frac{1}{2}\sup_{\substack{\mathbf{u}\in\ker A^{\perp}\\ \mathbf{v}\in\ker A}} \frac{(\|A\mathbf{u}\otimes B\mathbf{v}\|^2+\|B\mathbf{v}\otimes A\mathbf{u}\|^2)^{1/2}}{\sqrt{2}\|\mathbf{v}\|\|\mathbf{u}\|}\\
&= \frac{1}{2}\sup_{\substack{\mathbf{u}\in\ker A^{\perp}\\ \mathbf{v}\in\ker A}} \frac{\|A\mathbf{u}\|\|B\mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1}{2}\|A\|\|B\|
\end{aligned}
$$

since $\|A\| = \sup_{\mathbf{u}\in(\ker A)^{\perp}} \frac{\|A\mathbf{u}\|}{\|\mathbf{u}\|}$ and $(\ker B)^{\perp} \subseteq \ker A$.

To see that the lower inequality is sharp, let $A = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$and $B = I - A$, so $(\ker B)^{\perp} \subseteq \ker A$ and $\operatorname{ran} B \subseteq (\operatorname{ran} A)^{\perp}$. Then Proposition 4.12 yields $\|A \odot B\| = \frac{1}{2}$. ∎

6 Spectrum

Here, we present results on the spectrum of symmetric products of Hilbert-space operators (the finite-dimensional case is simpler; see [1, p. 18]). We find a complete description in some special cases. In what follows, $\sigma(A)$, $\sigma_{\mathrm{p}}(A)$, and $\sigma_{\mathrm{ap}}(A)$ denote the spectrum, point spectrum, and approximate point spectrum of A, respectively [12, Definition 2.4.5]. For $X, Y \subseteq \mathbb{C}$, let $X + Y := \{x + y : x \in X,\ y \in Y\}$ and $XY := \{xy : x \in X,\ y \in Y\}$.

Theorem 6.1 (Brown–Pearcy [2]) *$\sigma(A \otimes B) = \sigma(A)\sigma(B)$ for all $A, B \in \mathcal{B}(\mathcal{H})$.*

Proposition 6.2 *Let $A, B \in \mathcal{B}(\mathcal{H})$.*

(a) $\sigma\big(\frac{1}{2}(A\otimes B + B\otimes A)\big) = \sigma(A\odot B) \cup \sigma(A \wedge B)$.

(b) $\sigma_{\mathrm{p}}\big(\frac{1}{2}(A\otimes B + B\otimes A)\big) = \sigma_{\mathrm{p}}(A\odot B) \cup \sigma_{\mathrm{p}}(A \wedge B)$.

Proof This follows from the direct-sum decomposition (3.6). ∎

Theorem 6.3 *Let $A \in \mathcal{B}(\mathcal{H})$.*

(a) $\sigma(A\odot I) = \frac{1}{2}(\sigma(A) + \sigma(A))$.

(b) $\sigma(A\odot A) = \sigma(A)\sigma(A)$.

Proof (a) First, observe that

$$
\begin{aligned}
\sigma(A\odot I) &\subseteq \sigma\big(\tfrac{1}{2}(A\otimes I + I\otimes A)\big) && \text{(Lemma 6.2)}\\
&= \tfrac{1}{2}(\sigma(A)+\sigma(A)) && \text{(by [29, Theorem 2.1])}.
\end{aligned}
$$

Let $\lambda, \mu \in \sigma_{\mathrm{ap}}(A)$. There are sequences $\{\mathbf{u}_i\}_{i=1}^{\infty}$ and $\{\mathbf{v}_i\}_{i=1}^{\infty}$ of unit vectors such that $\|(A-\lambda I)\mathbf{u}_i\| \to 0$ and $\|(A-\mu I)\mathbf{v}_i\| \to 0$. Then Lemma 2.15 ensures that

$$
\begin{aligned}
&\left\|\left(A\odot I - \left(\frac{\lambda}{2}+\frac{\mu}{2}\right)(I\odot I)\right)\left(\frac{\mathbf{u}_i\odot\mathbf{v}_i}{\|\mathbf{u}_i\odot\mathbf{v}_i\|}\right)\right\|\\
&\quad\leqslant \sqrt{2}\left\|\left(A\odot I - \left(\frac{\lambda}{2}+\frac{\mu}{2}\right)(I\odot I)\right)(\mathbf{u}_i\odot\mathbf{v}_i)\right\|\\
&\quad= \frac{1}{2\sqrt{2}}\left\|\big(A\otimes I + I\otimes A - (\lambda+\mu)(I\otimes I)\big)(\mathbf{u}_i\otimes\mathbf{v}_i + \mathbf{v}_i\otimes\mathbf{u}_i)\right\|
\end{aligned}
$$

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$$= \frac{1}{2\sqrt{2}} \big\| A\mathbf{u}_i \otimes \mathbf{v}_i + \mathbf{u}_i \otimes A\mathbf{v}_i - \lambda \mathbf{u}_i \otimes \mathbf{v}_i - \mathbf{u}_i \otimes \mu \mathbf{v}_i$$
$$+ A\mathbf{v}_i \otimes \mathbf{u}_i + \mathbf{v}_i \otimes A\mathbf{u}_i - \mathbf{v}_i \otimes \lambda \mathbf{u}_i - \mu \mathbf{v}_i \otimes \mathbf{u}_i \big\|$$
$$= \frac{1}{2\sqrt{2}} \big\| (A - \lambda I)\mathbf{u}_i \otimes \mathbf{v}_i + \mathbf{v}_i \otimes (A - \lambda I)\mathbf{u}_i$$
$$+ \mathbf{u}_i \otimes (A - \mu I)\mathbf{v}_i + (A - \mu I)(\mathbf{v}_i \otimes \mathbf{u}_i) \big\|$$
$$\leqslant \frac{1}{2\sqrt{2}} \big(2\|(A - \lambda I)\mathbf{u}_i\| \|\mathbf{v}_i\| + 2\|(A - \mu I)\mathbf{v}_i\| \|\mathbf{u}_i\| \big)$$
$$= \frac{1}{\sqrt{2}} \|(A - \lambda I)\mathbf{u}_i\| + \frac{1}{\sqrt{2}} \|(A - \mu I)\mathbf{v}_i\| \to 0.$$

Thus, $\frac{1}{2}(\lambda + \mu) \in \sigma_{\text{ap}}(A \odot I)$ and hence

$$\sigma_{\text{ap}}(A \odot I) \supseteq \tfrac{1}{2}(\sigma_{\text{ap}}(A) + \sigma_{\text{ap}}(A)). \tag{6.4}$$

Recall that $\Omega(A) = \sigma(A) \backslash \sigma_{\text{ap}}(A)$ is a bounded open set. Furthermore, [2, p. 164] shows that $\lambda \in \Omega(A)$ implies $\overline{\lambda} \in \sigma_{\text{p}}(A^*)$. Since $\sigma(A)$ is closed and $\Omega(A) \subseteq \sigma(A)$, the boundary of $\Omega(A)$ is contained in $\sigma(A) \backslash \Omega(A) = \sigma_{\text{ap}}(A)$.

Let $\lambda, \mu \in \sigma(A)$. Following [2, Proof 2], we examine four special cases.

(i) If $\lambda, \mu \in \sigma_{\text{ap}}(A)$, then (6.4) ensures that $\frac{1}{2}(\lambda + \mu) \in \sigma_{\text{ap}}(A \odot I) \subseteq \sigma(A \odot I)$.

(ii) If $\lambda, \mu \in \Omega(A)$, then $\overline{\lambda}, \overline{\mu} \in \sigma_{\text{p}}(A^*) \subseteq \sigma_{\text{ap}}(A^*)$. Then (i) ensures that

$$\tfrac{1}{2}(\overline{\lambda + \mu}) \in \sigma_{\text{ap}}(A^* \odot I) = \sigma_{\text{ap}}((A \odot I)^*) \subseteq \sigma((A \odot I)^*),$$

so $\frac{1}{2}(\lambda + \mu) \in \sigma(A \odot I)$.

(iii) Suppose that $\lambda \in \sigma_{\text{ap}}(A)$ and $\mu \in \Omega(A)$. Then $\overline{\lambda} \in \sigma(A^*)$ and $\overline{\mu} \in \sigma_{\text{ap}}(A^*)$. If $\overline{\lambda} \in \sigma_{\text{ap}}(A^*)$, then (a) ensures that

$$\tfrac{1}{2}(\overline{\lambda + \mu}) \in \sigma_{\text{ap}}(A^* \odot I) \subseteq \sigma((A \odot I)^*), \quad \text{so } \tfrac{1}{2}(\lambda + \mu) \in \sigma(A \odot I).$$

Suppose instead that $\overline{\lambda} \in \Omega(A^*)$. The openness of $\Omega(A)$ and $\Omega(A^*)$ provide $\tau > 0$ such that $\overline{\lambda} - t \in \Omega(A^*)$ and $\mu + t \in \Omega(A)$ for $0 \leqslant t < \tau$.

– If $\overline{\lambda} - \tau \in \Omega(A^*)$ and $\mu + \tau \in \sigma_{\text{ap}}(A)$, then $\lambda - \tau = \overline{\overline{\lambda} - \tau} \in \sigma_{\text{ap}}(A)$. Then (a) ensures that

$$\tfrac{1}{2}(\lambda + \mu) = \tfrac{1}{2}(\lambda - \tau + \mu + \tau) \in \sigma_{\text{ap}}(A \odot I) \subseteq \sigma(A \odot I).$$

– If $\overline{\lambda} - \tau \in \sigma_{\text{ap}}(A^*)$ and $\mu + \tau \in \Omega(A)$, this case is analogous to the previous one.

– Suppose that $\overline{\lambda} - \tau \in \sigma_{\text{ap}}(A^*)$ and $\mu + \tau \in \sigma_{\text{ap}}(A)$. If $t_n \to \tau$ and $0 < t_n < \tau$, then $\overline{\lambda} - t_n \in \Omega(A^*)$ and hence $\lambda - t_n \in \sigma_{\text{ap}}(A)$. Thus,

$$\tfrac{1}{2}(\lambda - t_n + \mu + \tau) \in \sigma(A \odot I).$$

Since $\sigma(A \odot I)$ is closed, $\frac{1}{2}(\lambda + \mu) \in \sigma(A \odot I)$.

(iv) The case $\lambda \in \Omega(A)$ and $\mu \in \sigma_{\text{ap}}(A)$ is analogous to (iii).

In all cases $\frac{1}{2}(\lambda + \mu) \in \sigma(A \odot I)$, so $\sigma(A \odot I) = \frac{1}{2}(\sigma(A) + \sigma(A))$.

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(b) The proof is similar to that of (a), so we sketch the details. Lemma 6.2 and Theorem 6.1 yield $\sigma(A \odot A) \subseteq \sigma(A \otimes A) = \sigma(A)\sigma(A)$. If $\lambda, \mu \in \sigma_{\text{ap}}(A)$, then an argument similar to that of (a) ensures that $\lambda\mu \in \sigma_{\text{ap}}(A \odot A)$. As above, we can follow [2, Proof 2] and use a case-by-case analysis to show that $\lambda\mu \in \sigma(A \odot A)$, so that $\sigma(A)\sigma(A) \subseteq \sigma(A \odot A)$. ∎

7 Diagonal operators

Since diagonal operators are among the most elementary operators one encounters in the infinite-dimensional setting [12, Chapter 2], it makes sense to consider their symmetric tensor products. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots$ be an orthonormal basis for $\mathcal{H}$ and suppose that $L, M \in \mathcal{B}(\mathcal{H})$ satisfy $L\mathbf{e}_i = \lambda_i \mathbf{e}_i$ and $M\mathbf{e}_i = \mu_i \mathbf{e}_i$ for $i \geqslant 1$. For $i, j \geqslant 1$,

$$\begin{aligned}(L \odot M)(\mathbf{e}_i \odot \mathbf{e}_j) &= \tfrac{1}{4}(L \otimes M + M \otimes L)(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i)\\ &= \tfrac{1}{4}(L\mathbf{e}_i \otimes M\mathbf{e}_j + M\mathbf{e}_i \otimes L\mathbf{e}_j + L\mathbf{e}_j \otimes M\mathbf{e}_i + M\mathbf{e}_j \otimes L\mathbf{e}_i)\\ &= \tfrac{1}{4}(\lambda_i\mathbf{e}_i \otimes \mu_j\mathbf{e}_j + \mu_i\mathbf{e}_i \otimes \lambda_j\mathbf{e}_j + \lambda_j\mathbf{e}_j \otimes \mu_i\mathbf{e}_i + \mu_j\mathbf{e}_j \otimes \lambda_i\mathbf{e}_i)\\ &= \tfrac{1}{2}(\lambda_i\mu_j + \lambda_j\mu_i)(\mathbf{e}_i \odot \mathbf{e}_j). \qquad (7.1)\end{aligned}$$

Thus, $L \odot M$ is a diagonal operator with

$$\sigma_{\text{p}}(L \odot M) = \left\{\tfrac{1}{2}(\lambda_i\mu_j + \lambda_j\mu_i) : i, j \geqslant 1\right\} \quad \text{and} \quad \sigma(L \odot M) = \sigma_{\text{p}}(L \odot M)^{-}.$$

For symmetric products of diagonal operators, we can improve upon Theorem 5.1(a).

Proposition 7.2 *Let L, M be diagonal operators as above. Then $\|L\|\|M\|(\sqrt{2}-1) \leqslant \|L \odot M\| \leqslant \|L\|\|M\|$ and these inequalities are sharp.*

Proof The computation (7.1) shows that it suffices to prove the result for 2×2 diagonal matrices. Let $L = \operatorname{diag}(\lambda_1, \lambda_2)$ and $M = \operatorname{diag}(\mu_1, \mu_2)$. By linearity we may assume $\|L\| = \max\{|\lambda_1|, |\lambda_2|\} = 1$ and $\|M\| = \max\{|\mu_1|, |\mu_2|\} = 1$, Consider $L \odot M$, which by (4.4) we identify with $\operatorname{diag}(\lambda_1\mu_1, \frac{\lambda_1\mu_2 + \lambda_2\mu_1}{2}, \lambda_2\mu_2)$. Then

$$\|L \odot M\| = \max\left\{|\lambda_1\mu_1|, \tfrac{1}{2}|\lambda_1\mu_2 + \lambda_2\mu_1|, |\lambda_2\mu_2|\right\} \qquad (7.3)$$

and one of the following holds:

(a) $|\lambda_1| = |\mu_1| = 1$ or $|\lambda_2| = |\mu_2| = 1$.
(b) $|\lambda_1| = |\mu_2| = 1$ or $|\lambda_2| = |\mu_1| = 1$.

If (a) holds, then $\|L \odot M\| \geqslant \max\{|\lambda_1\mu_1|, |\lambda_2\mu_2|\} = 1$. If (b) holds, then without loss of generality assume that $|\lambda_1| = |\mu_2| = 1$. From (7.3),

$$\begin{aligned}\|L \odot M\| &\geqslant \inf_{|\mu_1|, |\lambda_2| \leqslant 1} \max\left\{|\mu_1|, \tfrac{1}{2}|1 + \lambda_2\mu_1|, |\lambda_2|\right\}\\ &\geqslant \inf_{0 \leqslant s \leqslant 1} \max\{s, \tfrac{1}{2}(1 - s^2)\} = \sqrt{2} - 1.\end{aligned}$$

The lower bound is attained by $L = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2}-1 \end{bmatrix}$ and $M = \begin{bmatrix} -\sqrt{2}+1 & 0 \\ 0 & 1 \end{bmatrix}$. The upper bound is attained by $L = M = I$. ∎

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Proposition 7.4 (a) *There exists a self-adjoint diagonal operator D such that $\sigma(D)$ has measure zero in $\mathbb{R}$ and $\sigma(D \odot D)$ has positive measure in $\mathbb{R}$.*

(b) *There exists a diagonal operator D such that $\sigma(D)$ has planar Lebesgue measure zero and $\sigma(D \odot D)$ has positive planar Lebesgue measure.*

Proof (a) Let $\mathscr{C}$ denote the Cantor set, which has measure zero. The exponential function is differentiable, so by [28, Lemma 7.25] $\{e^{\mu} : \mu \in \mathscr{C} \cap \mathbb{Q}\}^{-} = \{e^{c} : c \in \mathscr{C}\}$ has measure zero in $\mathbb{R}$. Let D be a diagonal operator with point spectrum $\{e^{\lambda} : \lambda \in \mathscr{C} \cap \mathbb{Q}\}$, so that $\sigma(D) = \{e^{c} : c \in \mathscr{C}\}$ has measure zero. Since $\mathscr{C} + \mathscr{C} = [0, 2]$, Theorem 6.3(b) ensures that $\sigma(D \odot D) = \sigma(D)\sigma(D) = \{e^{c+d} : c, d \in \mathscr{C}\} = [1, e^{2}]$ has positive measure in $\mathbb{R}$.

(b) Let D be a diagonal operator with point spectrum $\{e^{\lambda + i\mu} : \lambda \in \mathscr{C} \cap \mathbb{Q}, \mu \in \mathbb{Q} \cap [0, 2\pi)\}$. Then $\sigma(D)$ has planar measure zero, but an argument similar to that in (a) ensures that $\sigma(D \odot D)$ is the annulus centered at 0 with radii 1 and e^{2}, which has positive measure. ■

Below the brackets $\{\!\{$ and $\}\!\}$ indicate a *multiset*; that is, a set that permits multiplicity.

Proposition 7.5 *Let $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ be commuting diagonal operators with $\sigma_{\mathrm{p}}(A_i) = \{\!\{\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots\}\!\}$ allowing for repetition. Then*

$$\sigma_{\mathrm{p}}(A_1 \odot A_2 \odot \cdots \odot A_n) = \left\{\!\!\left\{ \frac{1}{n!} \sum_{\pi \in \Sigma_n} \lambda_{i_1}^{(\pi(1))} \lambda_{i_2}^{(\pi(2))} \cdots \lambda_{i_n}^{(\pi(n))} : i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \right\}\!\!\right\}$$

and $\sigma(A_1 \odot A_2 \odot \cdots \odot A_n) = \sigma_{\mathrm{p}}(A_1 \odot A_2 \odot \cdots \odot A_n)^{-}$.

8 The shift operator and its adjoint

In this section, we find the spectrum of the symmetric tensor product of the unilateral shift and its adjoint. Let $(Sf)(z) = zf(z)$ denote the unilateral shift on $H^2(\mathbb{D})$ [12, Chapter 5]. Its adjoint is the backward shift $(S^* f)(z) = (f(z) - f(0))/z$.

Theorem 8.1 *The self-adjoint operators $S \odot S^*$ and $S \wedge S^*$ satisfy*

$$\sigma_{\mathrm{p}}(S \odot S^*) = \left\{\!\!\left\{ \cos\left(\tfrac{(2j-1)\pi}{k+2}\right) : k \geqslant 0 \text{ and } 1 \leqslant j \leqslant \left\lfloor \tfrac{k+2}{2} \right\rfloor \right\}\!\!\right\}$$

and

$$\sigma_{\mathrm{p}}(S \wedge S^*) = \left\{\!\!\left\{ \cos\left(\tfrac{2j\pi}{k+2}\right) : k \geqslant 1 \text{ and } 1 \leqslant j \leqslant \left\lfloor \tfrac{k+1}{2} \right\rfloor \right\}\!\!\right\},$$

with the eigenvalues in these multisets repeated by multiplicity. Moreover, $\sigma(S \odot S^*) = \sigma_{\mathrm{ap}}(S \odot S^*) = \sigma(S \wedge S^*) = \sigma_{\mathrm{ap}}(S \wedge S^*) = [-1, 1]$ and $\|S \odot S^*\| = \|S \wedge S^*\| = 1$.

Proof Identify $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$ with $H^2(\mathbb{D}^2)$ as in Example 2.5 and consider

$$T = \frac{1}{2}(S \otimes S^* + S^* \otimes S)(z^i w^j) = \begin{cases} \frac{1}{2}(z^{i+1}w^{j-1} + z^{i-1}w^{j+1}), & \text{if } i, j \geqslant 1, \\ \frac{1}{2}z^{i+1}w^{j-1}, & \text{if } i = 0 \text{ and } j \geqslant 1, \\ \frac{1}{2}z^{i-1}w^{j+1}, & \text{if } i \geqslant 1 \text{ and } j = 0, \\ 0, & \text{if } i = j = 0. \end{cases}$$

Define $\mathcal{V}_0 = \mathcal{V}_0^+ = \operatorname{span}\{1\}$ and $\mathcal{V}_0^- = \{0\}$. For $k \geqslant 1$, let

$$\begin{aligned} \mathcal{V}_k &= \operatorname{span}\{z^i w^{k-i} : 0 \leqslant i \leqslant k\}, && \text{so } \dim \mathcal{V}_k = k+1, \\ \mathcal{V}_k^+ &= \operatorname{span}\{z^i w^{k-i} + z^{k-i}w^i : 0 \leqslant i \leqslant \lfloor \tfrac{k}{2} \rfloor\}, && \text{so } \dim \mathcal{V}_k^+ = \lfloor \tfrac{k}{2} \rfloor + 1, \\ \mathcal{V}_k^- &= \operatorname{span}\{z^i w^{k-i} - z^{k-i}w^i : 0 \leqslant i \leqslant \lfloor \tfrac{k-1}{2} \rfloor\}, && \text{so } \dim \mathcal{V}_k^- = \lfloor \tfrac{k-1}{2} \rfloor + 1, \end{aligned}$$

and note that $\dim \mathcal{V}_k = \dim \mathcal{V}_k^+ + \dim \mathcal{V}_k^-$ for $k \geqslant 1$ by a parity argument (or Proposition 2.11). Recall from (2.13) that $H^2(\mathbb{D}^2) = H^2_{\text{sym}}(\mathbb{D}^2) \oplus H^2_{\text{asym}}(\mathbb{D}^2)$ is an orthogonal direct sum. We have $\mathcal{V}_k = \mathcal{V}_k^+ \oplus \mathcal{V}_k^-$ for $k \geqslant 1$, in which each $\mathcal{V}_k, \mathcal{V}_k^+, \mathcal{V}_k^-$ is T-invariant, and

$$(8.2) \qquad H^2(\mathbb{D}^2) = \bigoplus_{k=0}^{\infty} \mathcal{V}_k, \qquad H^2_{\text{sym}}(\mathbb{D}^2) = \bigoplus_{k=0}^{\infty} \mathcal{V}_k^+, \qquad H^2_{\text{asym}}(\mathbb{D}^2) = \bigoplus_{k=1}^{\infty} \mathcal{V}_k^-.$$

With respect to the orthonormal basis $\{z^{k-i}w^i\}_{i=0}^k$ of $\mathcal{V}_k$, we identify the restriction $T|_{\mathcal{V}_k}$ with the $(k+1) \times (k+1)$ matrix (by convention $A_0 = [0]$)

$$A_k = \begin{bmatrix} 0 & \frac{1}{2} & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & \frac{1}{2} & 0 & \ddots & & \\ & & \ddots & 0 & \frac{1}{2} & \\ & & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & & \frac{1}{2} & 0 \end{bmatrix}.$$

From [21, Proposition 2.1], we have

$$(8.3) \qquad \sigma(A_k) = \left\{ \cos\left(\tfrac{j\pi}{k+2}\right) : j = 1, 2, \ldots, k+1 \right\}.$$

For k odd, with respect to the orthonormal basis $\{\frac{1}{\sqrt{2}}(z^{k-i}w^i + z^i w^{k-i})\}_{i=0}^{\frac{k-1}{2}}$ of $\mathcal{V}_k^+$, we identify the restriction $T|_{\mathcal{V}_k^+}$ with the $\frac{k+1}{2} \times \frac{k+1}{2}$ matrix

$$B_k = \begin{bmatrix} 0 & \frac{1}{2} & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & \frac{1}{2} & 0 & \ddots & & \\ & & \ddots & 0 & \frac{1}{2} & \\ & & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

For k even, with respect to the orthonormal basis $\{\frac{1}{\sqrt{2}}(z^{k-i}w^i + z^i w^{k-i})\}_{i=0}^{\frac{k}{2}-1} \cup \{z^{k/2}w^{k/2}\}$ of $\mathcal{V}_k^+$, we identify the restriction $T|_{\mathcal{V}_k^+}$ with the $\left(\frac{k}{2}+1\right) \times \left(\frac{k}{2}+1\right)$ matrix

$$B_k = \begin{bmatrix} 0 & \frac{1}{2} & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & \frac{1}{2} & 0 & \ddots & & \\ & & \ddots & 0 & \frac{1}{2} & \\ & & & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ & & & & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

with the convention $B_0 = [0]$. We identify the spectrum of the B_k later.

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With respect to the orthonormal basis $\{\frac{1}{\sqrt{2}}(z^i w^{k-i} - z^{k-i} w^i)\}_{i=0}^{\lfloor \frac{k-1}{2} \rfloor}$ of $\mathcal{V}_k^-$, we identify the restriction $T|_{\mathcal{V}_k^-}$ with the $\lfloor \frac{k+1}{2} \rfloor \times \lfloor \frac{k+1}{2} \rfloor$ matrix (note that $C_1 = [-\frac{1}{2}]$ and $C_2 = [0]$)

$$C_k = \begin{bmatrix} 0 & \frac{1}{2} & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & \frac{1}{2} & 0 & \ddots & & \\ & & \ddots & 0 & \frac{1}{2} & \\ & & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & & \frac{1}{2} & \frac{(-1)^k - 1}{4} \end{bmatrix}.$$

For $k \geqslant 2$ even, $\sigma(C_k) = \{\cos(\frac{2j\pi}{k+2}) : j = 1, 2, \ldots, \frac{k}{2}\}$ [21, Proposition 2.1]. Suppose $k \geqslant 1$ is odd. By [21, equation (11)], $\lambda \in \sigma(2C_k)$ if and only if $\lambda = -2x$, where $x \in [-1, 1]$ solves

$$(-1 + 2x)\frac{\sin(\frac{1}{2}(k+1)\cos^{-1}(x))}{\sin(\cos^{-1}(x))} - \frac{\sin(\frac{1}{2}(k-1)\cos^{-1}(x))}{\sin(\cos^{-1}(x))} = 0.$$

Since $\cos(\frac{(2\ell-1)\pi}{k+2})$ for $\ell = 1, 2, \ldots, \frac{k+1}{2}$ are the distinct solutions to this equation, $\sigma(C_k) = \{-\cos(\frac{(2\ell-1)\pi}{k+2}) : \ell = 1, 2, \ldots, \frac{k+1}{2}\}$. Since $-\cos(x) = \cos(\pi - x)$, we can reindex and rewrite this as $\sigma(C_k) = \{\cos(\frac{2j\pi}{k+2}) : j = 1, 2, \ldots, \frac{k+1}{2}\}$. Regardless of the parity of k,

$$\sigma(C_k) = \left\{\cos\left(\frac{2j\pi}{k+2}\right) : j = 1, 2, \ldots, \left\lfloor \frac{k+1}{2} \right\rfloor\right\}.$$

Since $\mathcal{V}_k = \mathcal{V}_k^+ \oplus \mathcal{V}_k^-$, up to unitary equivalence $A_k = B_k \oplus C_k$. Thus,

$$\sigma(A_k) = \sigma(B_k) \cup \sigma(C_k). \tag{8.4}$$

From (8.3)–(8.4), we obtain

$$\sigma(B_k) = \left\{\cos\left(\frac{(2j-1)\pi}{k+2}\right) : j = 1, 2, \ldots, \left\lfloor \frac{k+2}{2} \right\rfloor\right\}.$$

Since $S \odot S^*$ and $S \wedge S^*$ are self-adjoint and have norm at most 1, their spectra are contained in $[-1, 1]$. Up to unitary equivalence, (2.13) and (8.2) imply that

$$S \odot S^* = \bigoplus_{k=0}^{\infty} B_k \quad \text{and} \quad S \wedge S^* = \bigoplus_{k=1}^{\infty} C_k.$$

This yields the claimed point spectra of $S \odot S^*$ and $S \wedge S^*$. A density argument reveals that $[-1, 1] = \sigma_{\mathrm{p}}(S \odot S^*)^- \subseteq \sigma_{\mathrm{ap}}(S \odot S^*) \subseteq \sigma(S \odot S^*) \subseteq [-1, 1]$, so equality holds throughout. A similar argument treats $S \wedge S^*$. ■

9 Shifts and diagonal operators

We consider here the symmetric tensor product of shift operators and diagonal operators. This setting suggests working on the sequence space ℓ^2 instead of $H^2(\mathbb{D})$ [12, Section 1.2]. Let $\mathbf{e}_0, \mathbf{e}_1, \ldots$ be the standard basis of ℓ^2 and consider the unilateral shift $S\mathbf{e}_i = \mathbf{e}_{i+1}$ [12, Chapter 3]. Its adjoint is given by $S^*\mathbf{e}_i = \mathbf{e}_{i-1}$ for $i \geqslant 1$ and $S^*\mathbf{e}_0 = 0$.

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Theorem 9.1 *Let* $M = \operatorname{diag}(\mu_0, \mu_1, \ldots)$ *be a bounded diagonal operator on* ℓ^2.

(a) $\frac{1}{\sqrt{2}}\|M\| \leqslant \|S \odot M\| \leqslant \|M\|$. *Both inequalities are sharp.*
(b) *If some* $\mu_i = 0$ *or the set of nonzero* μ_i *is bounded away from* 0, *then* $0 \in \sigma_{\mathrm{p}}(S \odot M)$.
(c) $\sigma_{\mathrm{p}}(S \odot M) \subseteq \{0\}$.

Proof (a) Since $M^* = \operatorname{diag}(\overline{\mu_0}, \overline{\mu_1}, \ldots)$ is a diagonal operator and $(S \odot M^*)^* = S^* \odot M$ by Proposition 4.6, this follows from Theorem 9.2(a) below.

(b) Suppose that the set of nonzero μ_i is bounded away from zero. Note that for all $i, j \geqslant 0$,

$$(S \odot M)(\mathbf{e}_i \odot \mathbf{e}_j) = \frac{\mu_j}{2}\mathbf{e}_{i+1} \odot \mathbf{e}_j + \frac{\mu_i}{2}\mathbf{e}_i \odot \mathbf{e}_{j+1}. \tag{9.2}$$

If some $\mu_i = 0$, then (9.2) ensures that $0 \in \sigma_{\mathrm{p}}(S \odot M)$ since $(S \odot M)(\mathbf{e}_i \odot \mathbf{e}_i) = 0$. Thus, we may assume that $|\mu_i| \geqslant \delta > 0$ for all $i \geqslant 0$. Define $C = \sqrt{\|M\|/\delta}$.

Let $\sum_{i \leqslant j} |a_{ij}|^2 < \infty$ and let $\mathbf{v} = 2\sum_{0 \leqslant i \leqslant j < \infty} a_{ij}\mathbf{e}_i \odot \mathbf{e}_j$, which is well defined by Lemma 2.14. Then (9.2) ensures that

$$(S \odot M)\mathbf{v} = \sum_{0 \leqslant i \leqslant j < \infty} a_{ij}\left(\mu_j \mathbf{e}_{i+1} \odot \mathbf{e}_j + \mu_i \mathbf{e}_i \odot \mathbf{e}_{j+1}\right). \tag{9.3}$$

When (9.3) is expanded, the coefficient of $\mathbf{e}_k \odot \mathbf{e}_\ell$ for $k \leqslant \ell$ is

$$0, \qquad \text{if } k = \ell = 0,$$

$$2\mu_0 a_{0,0}, \qquad \text{if } k = 0 \text{ and } \ell = 1,$$

$$\mu_0 a_{0,\ell-1}, \qquad \text{if } k = 0 \text{ and } \ell \geqslant 2, \tag{9.4}$$

$$\mu_k a_{k-1,k}, \qquad \text{if } 1 \leqslant k = \ell, \tag{9.5}$$

$$2\mu_k a_{k,k} + \mu_{k+1} a_{k-1,k+1}, \qquad \text{if } k \geqslant 1 \text{ and } \ell = k+1, \tag{9.6}$$

$$\mu_k a_{k,\ell-1} + \mu_\ell a_{k-1,\ell}, \qquad \text{if } k \geqslant 1 \text{ and } \ell \geqslant k+2. \tag{9.7}$$

Then $(S \odot M)\mathbf{v} = \mathbf{0}$ if and only if the $a_{k,\ell}$ are square summable and (9.4)–(9.7) vanish for all $\ell \geqslant k \geqslant 0$. We define such $a_{k,\ell}$, not all zero, in four steps (see Figure 1).

(1) Let $a_{0,0} = 0$. For $\ell \geqslant 1$, let $a_{0,\ell-1} = 0$ so that (9.4) vanishes.
(2) For each $k \geqslant 2$, let $a_{k-1,k} = a_{k-1,k+2} = a_{k-1,k+4} = \cdots = 0$. Then (9.5) and (9.7) vanish for $k \geqslant 2$ and even $\ell \geqslant k+2$.
(3) Let $a_{1,1} = 0$ and, for each $k \geqslant 2$, let $a_{k,k}$ and $a_{k-1,k+1}$ be such that

$$\begin{bmatrix} 2a_{k,k} \\ a_{k-1,k+1} \end{bmatrix} \perp \begin{bmatrix} \overline{\mu_k} \\ \overline{\mu_{k+1}} \end{bmatrix} \quad \text{and} \quad 0 < \left\| \begin{bmatrix} 2a_{k,k} \\ a_{k-1,k+1} \end{bmatrix} \right\| < \frac{1}{(k+1)^{3/2}}. \tag{9.8}$$

Then (9.6) vanishes for $k \geqslant 2$ and $\ell = k+1$.
(4) For $k \geqslant 2$, let

$$a_{k-1,k+3} = -a_{k,k+2}\frac{\mu_k}{\mu_{k+3}}, \qquad a_{k-1,k+5} = -a_{k,k+4}\frac{\mu_k}{\mu_{k+5}}, \ldots. \tag{9.9}$$

Then (9.7) vanishes for all $k \geqslant 1$ with odd $\ell \geqslant k+3$.

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$$\begin{bmatrix}
a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & a_{0,4} & a_{0,5} & a_{0,6} & a_{0,7} & a_{0,8} \\
a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} \\
a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} \\
a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} \\
a_{4,0} & a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} & a_{4,8} \\
a_{5,0} & a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} & a_{5,8} \\
a_{6,0} & a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} & a_{6,8} \\
a_{7,0} & a_{7,1} & a_{7,2} & a_{7,3} & a_{7,4} & a_{7,5} & a_{7,6} & a_{7,7} & a_{7,8} \\
a_{8,0} & a_{8,1} & a_{8,2} & a_{8,3} & a_{8,4} & a_{8,5} & a_{8,6} & a_{8,7} & a_{8,8}
\end{bmatrix}$$

Figure 1: Colors denote the step where the $a_{k,\ell}$ are fixed: Step (1) is in violet; (2) is in red; (3) is in green; and (4) is in blue. The symmetry of symmetric tensors permits us to focus on $\ell \geqslant k \geqslant 0$. The violet and red values are zero.

This completes the definition of the $a_{k,\ell}$. We must prove that they are square summable.

For $k \geqslant 1$, (9.8) yields

$$|a_{k,k}|^2 < \frac{1}{(k+1)^3} \quad \text{and} \quad |a_{k-1,k+1}|^2 < \frac{1}{(k+1)^3}. \tag{9.10}$$

Then (9.9) and then (9.10) with $k+1$ in place of k ensure that

$$|a_{k-1,k+3}|^2 = |a_{k,k+2}|^2 \left|\frac{\mu_k}{\mu_{k+3}}\right|^2 \leqslant C|a_{k,k+2}|^2 \leqslant \frac{C}{(k+2)^3} \tag{9.11}$$

for $k \geqslant 1$. Next (9.9) and then (9.11) with $k+1$ in place of k imply that

$$|a_{k-1,k+5}|^2 = |a_{k,k+4}|^2 \left|\frac{\mu_k}{\mu_{k+5}}\right|^2 \leqslant C|a_{k,k+4}|^2 \leqslant \frac{C}{(k+3)^3},$$

so induction yields

$$|a_{k,k+2r}|^2 \leqslant \frac{C}{(k+r)^3}. \tag{9.12}$$

Then Step 1, Step 2, and (9.12) ensure that

$$\sum_{0 \leqslant k \leqslant \ell} |a_{k,\ell}|^2 = \sum_{k=1}^{\infty} \sum_{\ell \geqslant k} |a_{k,\ell}|^2 = \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} |a_{k,k+2r}|^2 \leqslant C \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \frac{1}{(k+r)^3},$$

which is finite by a standard argument in the study of elliptic functions [31, Proposition 10.4.2].[1] Thus, $\mathbf{v}$ is a well-defined vector in the kernel of $S \odot M$.

(c) Suppose that $\lambda \neq 0$ and $(S \odot M)\mathbf{v} = \lambda \mathbf{v}$, in which $\mathbf{v} = 2\sum_{0 \leqslant i \leqslant j < \infty} a_{ij} \mathbf{e}_i \odot \mathbf{e}_j$ and $\sum_{0 \leqslant i \leqslant j < \infty} |a_{ij}|^2 < \infty$. Then (9.2) ensures that

[1] The double sum can be explicitly evaluated. Write the summands in an array with r indexing the columns and k the rows. Sum each column and simplify to reduce the double sum to the well-known $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

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$$\mathbf{0} = ((S \odot M) - \lambda I)\mathbf{v} = 2 \sum_{0 \leqslant i \leqslant j < \infty} a_{i,j}((S \odot M) - \lambda I)(\mathbf{e}_i \odot \mathbf{e}_j)$$

$$= \sum_{0 \leqslant i \leqslant j < \infty} a_{i,j}\mu_j \mathbf{e}_{i+1} \odot \mathbf{e}_j + \sum_{0 \leqslant i \leqslant j < \infty} a_{i,j}\mu_i \mathbf{e}_i \odot \mathbf{e}_{j+1} - \sum_{0 \leqslant i \leqslant j < \infty} \lambda a_{i,j} \mathbf{e}_i \odot \mathbf{e}_j. \tag{9.13}$$

When (9.13) is expanded, the coefficient of $\mathbf{e}_k \odot \mathbf{e}_\ell$ for $k \leqslant \ell$ is

$$0 = -\lambda a_{0,0}, \quad \text{if } k = \ell = 0, \tag{9.14}$$

$$0 = 2\mu_0 a_{0,0} - \lambda a_{0,1}, \quad \text{if } k = 0 \text{ and } \ell = 1,$$

$$0 = \mu_0 a_{0,\ell-1} - \lambda a_{0,\ell}, \quad \text{if } k = 0 \text{ and } \ell \geqslant 2, \tag{9.15}$$

$$0 = \mu_k a_{k-1,k} - \lambda a_{k,k}, \quad \text{if } 1 \leqslant k = \ell, \tag{9.16}$$

$$0 = 2\mu_k a_{k,k} + \mu_{k+1} a_{k-1,k+1} - \lambda a_{k,k+1}, \quad \text{if } k \geqslant 1 \text{ and } \ell = k+1, \tag{9.17}$$

$$0 = \mu_k a_{k,\ell-1} + \mu_\ell a_{k-1,\ell} - \lambda a_{k,\ell}, \quad \text{if } k \geqslant 1 \text{ and } \ell \geqslant k+2. \tag{9.18}$$

We use induction to prove that $a_{k,\ell} = 0$ for $0 \leqslant k \leqslant \ell < \infty$. For $k \geqslant 0$, let $\mathrm{P}(k)$ be the statement "$a_{k,k+i} = 0$ for all $i \geqslant 0$." The truth of $\mathrm{P}(0)$ follows from (9.14), which ensures that $a_{0,0} = 0$, and induction on ℓ using (9.15), which yields $a_{0,\ell} = 0$ for $\ell \geqslant 0$.

Suppose $\mathrm{P}(k-1)$ is true: $a_{k-1,\ell} = 0$ for $\ell \geqslant k-1$. Then (9.16) yields $\lambda a_{k,k} = \mu_k a_{k-1,k} = 0$, so $a_{k,k} = 0$. Next, (9.17) ensures that $\lambda a_{k,k+1} = 2\mu_k a_{k,k} + \mu_{k+1} a_{k-1,k+1} = 0$, so $a_{k,k+1} = 0$. Finally, (9.18) and induction on ℓ tell us that $\lambda a_{k,\ell} = \mu_k a_{k,\ell-1} + \mu_\ell a_{k-1,\ell} = 0$ for $\ell \geqslant k+2$. Thus, $\mathrm{P}(k)$ is true, so $\mathbf{v} = \mathbf{0}$ and $\lambda \notin \sigma_{\mathrm{p}}(S \odot M)$. ∎

Theorem 9.2 *Let $M = \operatorname{diag}(\mu_0, \mu_1, \ldots)$ be a bounded diagonal operator on ℓ^2.*

(a) $\frac{1}{\sqrt{2}}\|M\| \leqslant \|S^* \odot M\| \leqslant \|M\|$. *Both inequalities are sharp.*

(b) $\{|z| < \frac{1}{2}|\mu_0|\} \cup \{0\} \subset \sigma_{\mathrm{p}}(S^* \odot M)$.

Proof (a) Since $\|S^*\| = 1$, Theorem 3.4 yields $\|S^* \odot M\| \leqslant \|M\|$. Equality holds for $M = I$ because $\sigma(S^*) = \mathbb{D}^-$ [12, Proposition 5.2.4.a] and $\sigma(S^* \odot I) = \frac{1}{2}(\sigma(S^*) + \sigma(S^*)) \subseteq \mathbb{D}^-$ by Theorem 6.3. Thus, $\|S^* \odot I\| \geqslant 1 = \|I\|$.

Suppose that $M\mathbf{e}_i = \mu_i \mathbf{e}_i$ for $i \geqslant 0$. Then

$$(S^* \odot M)(\mathbf{e}_i \odot \mathbf{e}_j) = \begin{cases} \frac{1}{2}(\mu_j \mathbf{e}_{i-1} \odot \mathbf{e}_j + \mu_i \mathbf{e}_i \odot \mathbf{e}_{j-1}), & \text{if } i, j \neq 0, \\ \frac{1}{2}(\mu_i \mathbf{e}_i \odot \mathbf{e}_{j-1}), & \text{if } 0 = i < j, \\ \frac{1}{2}(\mu_j \mathbf{e}_{i-1} \odot \mathbf{e}_j), & \text{if } 0 = j < i, \\ \mathbf{0}, & \text{if } i = j = 0. \end{cases} \tag{9.20}$$

For each $\varepsilon > 0$, there is a μ_i such that $\|M\| - \varepsilon \leqslant |\mu_i|$. Then (9.20) ensures that

$$\frac{\|M\| - \varepsilon}{\sqrt{2}} \leqslant \frac{|\mu_i|}{\sqrt{2}} = \|\mu_i \mathbf{e}_{i-1} \odot \mathbf{e}_i\| = \|(S^* \odot M)(\mathbf{e}_i \odot \mathbf{e}_i)\| \leqslant \|S^* \odot M\|.$$

Let $\varepsilon \to 0$ to obtain the desired lower bound.

If $\mu_i = \delta_{i0}$ for all $i \geqslant 0$, then $\|(S^* \odot M)(\sqrt{2}\mathbf{e}_0 \odot \mathbf{e}_1)\| = \|\frac{1}{\sqrt{2}}(\mathbf{e}_0 \odot \mathbf{e}_0)\| = \frac{1}{\sqrt{2}}$ and $\|M\| = 1$, so the lower bound is sharp.

(b) If $\mu_0 = 0$, the last line of (9.20) ensures that $0 \in \sigma_{\mathrm{p}}(S^* \odot M)$. Let $\mu_0 \neq 0$ and $|\lambda| < \frac{1}{2}|\mu_0|$. Lemma 2.14 permits us to define $\mathbf{v} = \sum_{j=0}^{\infty} \frac{(2\lambda)^j}{\mu_0^j}\mathbf{e}_0 \odot \mathbf{e}_j$. Then (9.20) ensures that $\lambda \in \sigma_{\mathrm{p}}(S^* \odot M)$ since

$$(S^* \odot M)\mathbf{v} = (S^* \odot M)\Big(\sum_{j=0}^{\infty} \frac{(2\lambda)^j}{\mu_0^j}\mathbf{e}_0 \odot \mathbf{e}_j\Big) = \sum_{j=0}^{\infty} \frac{(2\lambda)^j}{\mu_0^j}(S^* \odot M)(\mathbf{e}_0 \odot \mathbf{e}_j)$$
$$= \frac{1}{2}\sum_{j=0}^{\infty} \frac{(2\lambda)^j}{\mu_0^j}\mu_0\mathbf{e}_0 \odot \mathbf{e}_{j-1} = \lambda \sum_{j=1}^{\infty} \frac{(2\lambda)^{j-1}}{\mu_0^{j-1}}\mathbf{e}_0 \odot \mathbf{e}_{j-1} = \lambda\mathbf{v}. \qquad \blacksquare$$

10 Questions for further research

We conclude with questions to spur future research. Some are general, others specific. Perhaps the answers to a few are buried in the literature, although we did not find them.

Lemma 2.15 prompts us to consider symmetric tensor products of more than two vectors. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathcal{H}$, then $\|\mathbf{x}_1 \odot \mathbf{x}_2 \odot \cdots \odot \mathbf{x}_n\| = \|\mathsf{S}_n(\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \cdots \otimes \mathbf{x}_n)\| \leqslant \|\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \cdots \otimes \mathbf{x}_n\| = \|\mathbf{x}_1\|\|\mathbf{x}_2\|\cdots\|\mathbf{x}_n\|$. Equality occurs if $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_n$. Thus, only lower bounds on $\|\mathbf{x}_1 \odot \mathbf{x}_2 \odot \cdots \odot \mathbf{x}_n\|$ are of interest. Here is a partial answer.

Lemma 10.1 $\frac{1}{\sqrt{6}}\|\mathbf{x}_1\|\|\mathbf{x}_2\|\|\mathbf{x}_3\| \leqslant \|\mathbf{x}_1 \odot \mathbf{x}_2 \odot \mathbf{x}_3\| \leqslant \|\mathbf{x}_1\|\|\mathbf{x}_2\|\|\mathbf{x}_3\|$ *for* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathcal{H}$. *These inequalities are sharp.*

Proof The upper bound is discussed above. Without loss of generality, suppose $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ have unit norm. Then

$$\begin{aligned} 36\|\mathbf{x}_1 \odot \mathbf{x}_2 \odot \mathbf{x}_3\|^2 &= \sum_{\tau,\pi\in\Sigma_3} \langle \mathbf{x}_{\tau(1)} \otimes \mathbf{x}_{\tau(2)} \otimes \mathbf{x}_{\tau(3)}, \mathbf{x}_{\pi(1)} \otimes \mathbf{x}_{\pi(2)} \otimes \mathbf{x}_{\pi(3)} \rangle \\ &= 6 + \underbrace{\sum_{\tau\neq\pi} \langle \mathbf{x}_{\tau(1)} \otimes \mathbf{x}_{\tau(2)} \otimes \mathbf{x}_{\tau(3)}, \mathbf{x}_{\pi(1)} \otimes \mathbf{x}_{\pi(2)} \otimes \mathbf{x}_{\pi(3)} \rangle}_{c}, \end{aligned} \tag{10.2}$$

in which $c \in \mathbb{R}$ is of the form

$$\begin{aligned} c = 6\big(&|\langle \mathbf{x}_2, \mathbf{x}_3 \rangle|^2 + |\langle \mathbf{x}_1, \mathbf{x}_2 \rangle|^2 + |\langle \mathbf{x}_1, \mathbf{x}_3 \rangle|^2\big) \\ &+ \underbrace{6\langle \mathbf{x}_1, \mathbf{x}_2 \rangle\langle \mathbf{x}_2, \mathbf{x}_3 \rangle\langle \mathbf{x}_3, \mathbf{x}_1 \rangle + 6\langle \mathbf{x}_1, \mathbf{x}_3 \rangle\langle \mathbf{x}_3, \mathbf{x}_2 \rangle\langle \mathbf{x}_2, \mathbf{x}_1 \rangle}_{d}. \end{aligned}$$

Muirhead's inequality [17, Chapter 2, Section 18] shows that for $x, y, z \in [0,1]$,

$$x^2 + y^2 + z^2 \geqslant 2xyz. \tag{10.3}$$

Let $x = |\langle \mathbf{x}_1, \mathbf{x}_2 \rangle|$, $y = |\langle \mathbf{x}_2, \mathbf{x}_3 \rangle|$, and $z = |\langle \mathbf{x}_3, \mathbf{x}_1 \rangle|$ in (10.3) and get (since $d \in \mathbb{R}$)

$$6\big(|\langle \mathbf{x}_2, \mathbf{x}_3 \rangle|^2 + |\langle \mathbf{x}_1, \mathbf{x}_2 \rangle|^2 + |\langle \mathbf{x}_1, \mathbf{x}_3 \rangle|^2\big) \geqslant 12|\langle \mathbf{x}_1, \mathbf{x}_2 \rangle||\langle \mathbf{x}_2, \mathbf{x}_3 \rangle||\langle \mathbf{x}_3, \mathbf{x}_1 \rangle| \geqslant -d.$$

Thus, $c \geqslant 0$ and we obtain the desired lower bound. If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are pairwise orthogonal, then $c = 0$ in (10.2), so the lower bound is sharp. $\blacksquare$

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Problem 1 *Is* $\frac{1}{\sqrt{n!}}\|\mathbf{x}_1\|\|\mathbf{x}_2\|\cdots\|\mathbf{x}_n\| \leqslant \|\mathbf{x}_1 \odot \mathbf{x}_2 \odot \cdots \odot \mathbf{x}_n\|$ *for* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathcal{H}$*?*

Lemma 10.1 leads to an analogue of Theorem 5.1(a) for three operators.

Theorem 10.4 $\displaystyle \frac{1}{\sqrt{6}} \sup_{\substack{\mathbf{x}\in\mathcal{H}\\ \|\mathbf{x}\|=1}} \big\{\|A\mathbf{x}\|\|B\mathbf{x}\|\|C\mathbf{x}\|\big\} \leqslant \|A \odot B \odot C\|$ *for* $A, B, C \in \mathcal{B}(\mathcal{H})$.

Problem 2 *For* $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ *is*

$$\frac{1}{\sqrt{n!}} \sup_{\substack{\mathbf{x}\in\mathcal{H}\\ \|\mathbf{x}\|=1}} \big\{\|A_1\mathbf{x}\|\|A_2\mathbf{x}\|\cdots\|A_n\mathbf{x}\|\big\} \leqslant \|A_1 \odot A_2 \odot \cdots \odot A_n\|?$$

Proposition 7.2 provides the sharp inequalities $\|L\|\|M\|(\sqrt{2}-1) \leqslant \|L \odot M\| \leqslant \|L\|\|M\|$ for diagonal operators L, M (with respect to the same orthonormal basis). Since the upper bound easily generalizes, the lower bound is of greater interest.

Problem 3 *Let* $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ *be diagonal operators (with respect to the same orthonormal basis). Find a sharp lower bound, in the spirit of Proposition 7.2, on* $\|A_1 \odot A_2 \odot \cdots \odot A_n\|$ *in terms of* $\|A_1\|, \|A_2\|, \ldots, \|A_n\|$.

The Weyl–von Neumann–Berg theorem asserts that every normal operator on a separable Hilbert space is the sum of a diagonal operator and a compact operator of arbitrarily small norm [8, Corollary II.4.2]. This suggests possible extensions to normal operators.

Problem 4 *Let* $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ *be commuting normal operators. Find a sharp lower bound, in the spirit of Proposition 7.2, on* $\|A_1 \odot A_2 \odot \cdots \odot A_n\|$ *in terms of* $\|A_1\|, \|A_2\|, \ldots, \|A_n\|$.

Proposition 7.5 suggests the following.

Problem 5 *Let* $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ *be commuting normal operators. Describe* $\sigma(A_1 \odot A_2 \odot \cdots \odot A_n)$ *(and its parts).*

Let us now consider the unilateral shift S and its adjoint. Theorem 8.1 identified the norm and spectrum of $S \odot S^*$ and $S \wedge S^*$. What can be said about other combinations?

Problem 6 *Identify the norm and spectrum of arbitrary symmetric or antisymmetric tensor products of* S *and* S^* *(for example, consider* $S^2 \odot S \odot S^{*3}$ *and* $S^2 \wedge S \wedge S^{*3}$*).*

Problem 7 *Describe the norm and spectrum of* $S_\alpha \odot S_\alpha^*$ *and* $S_\alpha \wedge S_\alpha^*$*, in which* S_α *is a weighted shift operator. What can be said if more factors are included?*

Theorems 9.1 and 9.2 answer some questions about $S \odot M$ and $S^* \odot M$, in which $M = \operatorname{diag}(\mu_0, \mu_1, \ldots)$ is a diagonal operator. However, a complete picture eludes us.

Problem 8 *Identify the norm and spectrum (and its parts) for* $S \odot M$ *and* $S^* \odot M$.

The general problem suggested by the previous questions is the following.

Problem 9 *For* $A_1, A_2, \ldots, A_n \in B(\mathcal{H})$*, describe the norm and spectrum (and its parts) of* $A_1 \odot A_2 \odot \cdots \odot A_n$ *and* $A_1 \wedge A_2 \wedge \cdots \wedge A_n$.

There are countless other questions that can be raised. For example, what can be said about symmetric tensor products of composition operators?

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