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RESEARCH ARTICLE

Khintchine-type recurrence for 3-point configurations

Ethan Ackelsberg [1], Vitaly Bergelson [2] and Or Shalom [3]

[1] Department of Mathematics, Ohio State University, Columbus, OH 43210, USA; E-mail: ackelsberg.1@osu.edu
[2] Department of Mathematics, Ohio State University, Columbus, OH 43210, USA; E-mail: vitaly@math.ohio-state.edu
[3] Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, 91904, Israel; E-mail: or.shalom@mail.huji.ac.il

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Abstract
The goal of this paper is to generalise, refine and improve results on large intersections from [2, 8]. We show that if G is a countable discrete abelian group and $\varphi, \psi : G \to G$ are homomorphisms, such that at least two of the three subgroups $\varphi(G)$, $\psi(G)$ and $(\psi - \varphi)(G)$ have finite index in G, then $\{\varphi, \psi\}$ has the *large intersections property*. That is, for any ergodic measure preserving system $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$, any $A \in \mathcal{X}$ and any $\varepsilon > 0$, the set

$$\{g \in G : \mu(A \cap T^{-1}_{\varphi(g)} A \cap T^{-1}_{\psi(g)} A) > \mu(A)^3 - \varepsilon\}$$

is syndetic (Theorem 1.11). Moreover, in the special case where $\varphi(g) = ag$ and $\psi(g) = bg$ for $a, b \in \mathbb{Z}$, we show that we only need one of the groups aG, bG or $(b - a)G$ to be of finite index in G (Theorem 1.13), and we show that the property fails, in general, if all three groups are of infinite index (Theorem 1.14).

One particularly interesting case is where $G = (\mathbb{Q}_{>0}, \cdot)$ and $\varphi(g) = g$, $\psi(g) = g^2$, which leads to a multiplicative version of the Khintchine-type recurrence result in [8]. We also completely characterise the pairs of homomorphisms φ, ψ that have the large intersections property when $G = \mathbb{Z}^2$.

The proofs of our main results rely on analysis of the structure of the *universal characteristic factor* for the multiple ergodic averages

$$\frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} T_{\varphi(g)} f_1 \cdot T_{\psi(g)} f_2.$$

In the case where G is finitely generated, the characteristic factor for such averages is the *Kronecker factor*. In this paper, we study actions of groups that are not necessarily finitely generated, showing, in particular, that, by passing to an extension of $\mathbf{X}$, one can describe the characteristic factor in terms of the *Conze–Lesigne factor* and the σ-algebras of $\varphi(G)$ and $\psi(G)$ invariant functions (Theorem 4.10).

Contents

1. Introduction

Let $(G, +)$ be a countable discrete abelian group. A *probability measure-preserving G-system*, or simply *G-system* for short, is a quadruple $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$, where $(X, \mathcal{X}, \mu)$ is a standard Borel probability space (that is, up to isomorphism of measure spaces, X is a compact metric space, $\mathcal{X}$ is the Borel σ-algebra and μ is a regular Borel probability measure) and $T_g : X \to X$, $g \in G$, are measure-preserving transformations, such that $T_{g+h} = T_g \circ T_h$ for every $g, h \in G$ and $T_0 = Id$. The transformation $T_g : X \to X$ gives rise to a unitary operator on $L^2(\mu)$, which we also denote by T_g, given by the formula $T_g f(x) = f(T_g x)$. We say that a G-system is *ergodic* if the only measurable $(T_g)_{g\in G}$-invariant functions are the constant functions.

1.1. Khintchine-type recurrence and the large intersections property

The starting point for the study of recurrence in ergodic theory is the Poincaré recurrence theorem, which states that, for any measure-preserving system $(X, \mathcal{X}, \mu, T)$ and any set $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$, such that $\mu(A \cap T^{-n}A) > 0$.

Khintchine's recurrence theorem strengthens and enhances Poincaré's recurrence theorem by improving on the size of the intersections and the size of the set of return times.

Theorem 1.1 (Khintchine's recurrence theorem [24]). *For any measure-preserving system $(X, \mathcal{X}, \mu, T)$, any $A \in \mathcal{X}$ and any $\varepsilon > 0$, the set*

$$\left\{n \in \mathbb{N} : \mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon\right\}$$

has bounded gaps.

Khintchine's recurrence theorem easily extends to general semigroups, where the appropriate counterpart of 'bounded gaps' is the notion of syndeticity. In this paper, we deal with recurrence in countable discrete abelian groups. A subset A of a countable discrete abelian group G is said to be *syndetic* if there exists a finite set $F \subseteq G$, such that $A + F = \{a + f : a \in A, f \in F\} = G$.

It is natural to ask if recurrence theorems other than Poincaré's recurrence theorem also have Khintchine-type enhancements. For instance, it follows from the IP Szemerédi theorem of Furstenberg and Katznelson [20] and also from [3, Theorem B] that, for any abelian group G, any $k \in \mathbb{N}$ and any family of homomorphisms $\varphi_1, \ldots, \varphi_k : G \to G$, the following holds: if $\left(X, \mathcal{X}, \mu, (T_g)_{g \in G}\right)$ is a G-system and $A \in \mathcal{X}$ has $\mu(A) > 0$, then the set

$$\left\{g \in G : \mu\left(A \cap T^{-1}_{\varphi_1(g)} A \cap \cdots \cap T^{-1}_{\varphi_k(g)} A\right) > 0\right\}$$

is syndetic.[1] With the goal of Khintchine-type enhancements in mind, this motivates the following definition:

Definition 1.2. A family of homomorphisms $\varphi_1, \ldots, \varphi_k : G \to G$ has the *large intersections property* if the following holds: for any ergodic G-system $\left(X, \mathcal{X}, \mu, (T_g)_{g \in G}\right)$, any $A \in \mathcal{X}$ and any $\varepsilon > 0$, the set

$$\left\{g \in G : \mu\left(A \cap T^{-1}_{\varphi_1(g)} A \cap \cdots \cap T^{-1}_{\varphi_k(g)} A\right) > \mu(A)^{k+1} - \varepsilon\right\}$$

is syndetic.

The large intersections property is closely related to the phenomenon of popular differences in combinatorics (see, e.g. [1, 11, 12, 25, 26]).

Determining which families of homomorphisms have the large intersections property is a challenging problem with many surprising features. In the case $G = \mathbb{Z}$ and $\varphi_i(n) = in$, the problem was resolved in [8].

Theorem 1.3 ([8], Theorems 1.2 and 1.3). *The family $\{n, 2n, \ldots, kn\}$ has the large intersections property in $\mathbb{Z}$ if and only if $k \leq 3$.*

Later work of Frantzikinakis [18] and of Donoso et al. [15] generalised this picture for arbitrary homomorphisms $\mathbb{Z} \to \mathbb{Z}$, which take the form $n \mapsto an$ for some $a \in \mathbb{Z}$.

Theorem 1.4 ([18], special case of Theorem C; [15], Theorem 1.5).

1. *For any $a, b \in \mathbb{Z}$, the families $\{an, bn\}$ and $\{an, bn, (a + b)n\}$ have the large intersections property (in $\mathbb{Z}$).*
2. *For any $k \geq 4$ and any distinct and nonzero integers $a_1, \ldots, a_k \in \mathbb{Z}$, the family $\{a_1 n, \ldots, a_k n\}$ does not have the large intersections property (in $\mathbb{Z}$).*

Remark 1.5. Finitary combinatorial work of [26, Theorem 1.6] suggests that the family $\{a_1 n, a_2 n, a_3 n\}$ has the large intersections property if and only if $a_i + a_j = a_k$ for some permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.

In [10], Khintchine-type recurrence results are established in the infinitely generated torsion groups $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$.

[1] In fact, this set is an IP* set, which is a stronger notion of largeness that we do not address in this paper (see 20).

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Theorem 1.6 ([10], Theorems 1.12 and 1.13).

1. *Fix a prime $p > 2$. If $c_1, c_2 \in \mathbb{Z}/p\mathbb{Z}$ are distinct and nonzero, then $\{c_1 g, c_2 g\}$ has the large intersections property in $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$.*
2. *Fix a prime $p > 3$. If $c_1, c_2 \in \mathbb{Z}/p\mathbb{Z}$ are distinct and nonzero and $c_1 + c_2 \neq 0$, then $\{c_1 g, c_2 g, (c_1 + c_2)g\}$ has the large intersections property in $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$.*

Remark 1.7. It is conjectured in [10, Conjecture 1.14] that, if $c_1, c_2, c_3 \in \mathbb{Z}/p\mathbb{Z}$ are distinct and nonzero and $c_i + c_j \neq c_k$ for every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, then $\{c_1 g, c_2 g, c_3 g\}$ does not have the large intersections property in $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$.

Khintchine-type recurrence in general abelian groups was addressed in [2] and [27]. For 3-point linear configurations, the following was shown in [2]:

Theorem 1.8 ([2], Theorem 1.10). *Let G be a countable discrete abelian group. Let $\varphi, \psi : G \to G$ be homomorphisms. If all three of the subgroups $\varphi(G)$, $\psi(G)$ and $(\psi - \varphi)(G)$ have finite index in G, then $\{\varphi, \psi\}$ has the large intersections property.*

Remark 1.9. Earlier work of Chu demonstrates that at least some finite index condition is necessary for large intersections. Namely, it follows from [13, Theorem 1.2] that the pair $\{(n, 0), (0, n)\}$, does not have the large intersections property in $\mathbb{Z}^2$ (see [2, Example 10.2]). While we do not pursue optimal lower bounds for families lacking the large intersections property in this paper, Chu also showed that, for the pair $\{(n, 0), (0, n)\}$, the optimal lower bound is still polynomially large. In particular, $\mu\left(A \cap T_{(n,0)}^{-1} A \cap T_{(0,n)}^{-1} A\right) > \mu(A)^4 - \varepsilon$ for syndetically many n (see [13, Theorem 1.1]).

For more restricted 4-point configurations, the following result was shown in [2] and independently in [27]:

Theorem 1.10 ([2], Theorem 1.11; [27], Theorem 1.3). *Let G be a countable discrete abelian group. Let $a, b \in \mathbb{Z}$ be distinct, nonzero integers, such that all four of the subgroups aG, bG, $(a + b)G$ and $(b - a)G$ have finite index in G. Then $\{ag, bg, (a + b)g\}$ has the large intersections property.*

1.2. Main results

In this paper, we refine the understanding of Khintchine-type recurrence for 3-point configurations in abelian groups and make substantial progress towards characterising the pairs of homomorphisms $\varphi, \psi : G \to G$ that have the large intersections property.

Our first result shows that the large intersections property holds for any pair of homomorphisms $\{\varphi, \psi\}$ so long as at least two of the three subgroups in Theorem 1.8 have finite index in G. In particular, this shows that [2, Conjecture 10.1] is false.

Theorem 1.11. *Let G be a countable discrete abelian group. Let $\varphi, \psi : G \to G$ be homomorphisms, such that at least two of the three subgroups $\varphi(G)$, $\psi(G)$ and $(\psi - \varphi)(G)$ have finite index in G. Then for any ergodic G-system $\left(X, \mathcal{X}, \mu, (T_g)_{g \in G}\right)$, any $A \in \mathcal{X}$ and any $\varepsilon > 0$, the set*

$$\left\{g \in G : \mu\left(A \cap T_{\varphi(g)}^{-1} A \cap T_{\psi(g)}^{-1} A\right) > \mu(A)^3 - \varepsilon\right\}$$

is syndetic.

As mentioned above (see Remark 1.9), the work of Chu [13] provides a counterexample to the large intersections property when all three subgroups $\varphi(G)$, $\psi(G)$ and $(\psi - \varphi)(G)$ have infinite index in G. In this paper, we give additional counterexamples for the group $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}$ with homomorphisms $g \mapsto ag$ and $g \mapsto bg$ for some $a, b \in \mathbb{Z}$ (see Theorem 1.14 below). A natural question to ask, then, is what happens when only one of the subgroups $\varphi(G)$, $\psi(G)$ or $(\psi - \varphi)(G)$ has finite index. Namely:

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Question 1.12. Let G be a countable discrete abelian group, and let $\varphi : G \to G$, $\psi : G \to G$ be homomorphisms, such that at least one of the subgroups $\varphi(G)$, $\psi(G)$ or $(\psi - \varphi)(G)$ has finite index in G. Is it true that, for any ergodic G-system $\left(X, \mathcal{X}, \mu, (T_g)_{g\in G}\right)$, any $A \in \mathcal{X}$, and any $\varepsilon > 0$, the set

$$\left\{g \in G : \mu\left(A \cap T^{-1}_{\varphi(g)} A \cap T^{-1}_{\psi(g)} A\right) > \mu(A)^3 - \varepsilon\right\}$$

is syndetic?

Note that, by symmetry, it is enough to provide an answer to Question 1.12 under the assumption that $(\psi - \varphi)(G)$ has finite index. Indeed, suppose $\psi(G)$ has finite index in G. Then, since $(T_g)_{g\in G}$ is a measure-preserving action, we have the identity

$$\mu\left(A \cap T^{-1}_{\varphi(g)} A \cap T^{-1}_{\psi(g)} A\right) = \mu\left(A \cap T^{-1}_{-\varphi(g)} A \cap T^{-1}_{(\psi-\varphi)(g)} A\right).$$

Hence, the pair $\{\varphi, \psi\}$ has the large intersections property if and only if $\left\{\widetilde{\varphi}, \widetilde{\psi}\right\}$ has the large intersections property, where $\widetilde{\varphi} = -\varphi$ and $\widetilde{\psi} = \psi - \varphi$. Moreover, we have $(\widetilde{\psi} - \widetilde{\varphi})(G) = \psi(G)$, which is of finite index. A similar argument applies when $\varphi(G)$ has finite index.

When $G = \mathbb{Z}^2$, we can use additional tools from linear algebra to classify all pairs of homomorphisms φ and ψ, which allows us to answer Question 1.12 affirmatively in this setting. In fact, we can give a precise description of the optimal size of intersections for all 3-point configurations in $\mathbb{Z}^2$ (see Subsection 1.4 below). However, our results rely heavily on properties of 2×2 matrices, and it appears that the full generality of Question 1.12 for general abelian groups and general homomoprhisms is out of reach without developing new techniques.

On the other hand, in the special case $\varphi(g) = ag$ and $\psi(g) = bg$ for $a, b \in \mathbb{Z}$, we answer Question 1.12 affirmatively:

Theorem 1.13. *Let G be a countable discrete abelian group. Let $a, b \in \mathbb{Z}$ be integers, such that $(b-a)G$ has finite index in G. Then for any ergodic G-system $\left(X, \mathcal{X}, \mu, (T_g)_{g\in G}\right)$, any $A \in \mathcal{X}$ and any $\varepsilon > 0$, the set*

$$\left\{g \in G : \mu\left(A \cap T^{-1}_{ag} A \cap T^{-1}_{bg} A\right) > \mu(A)^3 - \varepsilon\right\}$$

is syndetic.

We also show that the assumption that $(b - a)G$ has finite index in G is necessary. To see this, we prove the following result:

Theorem 1.14. *Let $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}$. Let $l \in \mathbb{N}$. There exists a number $P = P(l)$, such that, for any $a, b \in \mathbb{N}$ with $p \mid \gcd(a, b)$ for some prime $p \geq P$, there is an ergodic G-system $\left(X, \mathcal{X}, \mu, (T_g)_{g\in G}\right)$ and a set $A \in \mathcal{X}$ with $\mu(A) > 0$, such that*

$$\mu(A \cap T^{-1}_{ag} A \cap T^{-1}_{bg} A) \leq \mu(A)^l$$

for every $g \neq 0$.

Question 1.15. Can p in the statement of Theorem 1.14 be replaced by any natural number?

1.3. Applications to geometric progressions and other multiplicative patterns

One particularly interesting corollary of Theorem 1.13 is a multiplicative version of the following large intersection theorem in [8]:

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Theorem 1.16 ([8], Corollary 1.5). *Let $E \subseteq \mathbb{Z}$ be a set of positive upper Banach density*

$$d^*(E) = \limsup_{N-M\to\infty} \frac{|E \cap \{M, M+1, \ldots, N-1\}|}{N-M} > 0.$$

Then, for any $\varepsilon > 0$, the set

$$\{n \in \mathbb{Z} : d^*(E \cap (E-n) \cap (E-2n)) > d^*(E) - \varepsilon\}$$

is syndetic.

Consider the group $G = (\mathbb{Q}_{>0}, \cdot)$. This is a multiplicative counterpart of $(\mathbb{Z}, +)$. In the group $(\mathbb{Q}_{>0}, \cdot)$, the *upper Banach density* of a set $E \subseteq \mathbb{Q}_{>0}$ is given by

$$d^*_{\mathrm{mult}}(E) = \sup_{\Phi} \limsup_{N\to\infty} \frac{|E \cap \Phi_N|}{|\Phi_N|}, \tag{1}$$

where the supremum is taken over all Følner sequences $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ in $(\mathbb{Q}_{>0}, \cdot)$. An instructive class of examples of Følner sequences in $(\mathbb{Q}_{>0}, \cdot)$ is given by sequences of the form

$$\Phi_N = \left\{ b_N \prod_{i=1}^{N} q_i^{r_i} : -N \le r_i \le N \right\},$$

where $(q_n)_{n\in\mathbb{N}}$ is a sequence of generators of $(\mathbb{Q}_{>0}, \cdot)$ and $(b_N)_{N\in\mathbb{N}}$ is any sequence in $\mathbb{Q}_{>0}$. The subscript on d^*_{mult} is to emphasise that this density is with respect to the multiplicative structure on $\mathbb{Q}_{>0}$ rather than its additive structure. Using an ergodic version of the Furstenberg correspondence principle (see [6, Theorem 2.8]), we deduce the following result as an immediate consequence of Theorem 1.13:

Theorem 1.17. *Let $E \subseteq \mathbb{Q}_{>0}$ be a set of positive multiplicative upper Banach density $d^*_{mult}(E) > 0$, and let $k \in \mathbb{Z}$. Then for any $\varepsilon > 0$, the sets*

$$\left\{ q \in \mathbb{Q}_{>0} : d^*_{mult}\left(E \cap q^{-k}E \cap q^{-(k+1)}E\right) > d^*_{mult}(E)^3 - \varepsilon \right\} \tag{2}$$

and

$$\left\{ q \in \mathbb{Q}_{>0} : d^*_{mult}\left(E \cap q^{-1}E \cap q^{-k}E\right) > d^*_{mult}(E)^3 - \varepsilon \right\} \tag{3}$$

are syndetic.

Remark 1.18. The special case where $k = 1$ in (2) or $k = 2$ in (3) is related to the existence of length 3 geometric progressions in sets of positive multiplicative density. Heuristically, if E were a random set, where each positive rational number $q \in \mathbb{Q}_{>0}$ is independently chosen to be inside E with probability α, then the expected number of geometric progressions of length 3 and quotient q would be α^3. Now, fix any set E with $d^*_{mult}(E) = \alpha$. Choosing ε sufficiently small, our result implies that E contains almost as many geometric progressions with quotient q as a random set with the same density, α, for a syndetic set of quotients.

Theorem 1.14 shows that, if n and m share a large prime factor, then $\{q^n, q^m\}$ does not have the large intersections property in $(\mathbb{Q}_{>0}, \cdot)$. What happens in the case that n and m are coprime is an interesting question that we are unable to answer with our current methods:

Question 1.19. Suppose $n, m \in \mathbb{N}$ are coprime. Does the pair $\{q^n, q^m\}$ have the large intersections property in $(\mathbb{Q}_{>0}, \cdot)$?

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Since every $\mathbb{Z}$-action can be lifted to a $(\mathbb{Q}_{>0}, \cdot)$-action (indeed, $(\mathbb{Q}_{>0}, \cdot)$ is torsion-free, so $\mathbb{Z}$ embeds as a subgroup), we see from Theorem 1.3 above that $\{q, q^2, \ldots, q^k\}$ does not have the large intersections property for $k \geq 4$. However, we can still ask about geometric progressions of length 4.

Question 1.20. Does the triple $\{q, q^2, q^3\}$ have the large intersections property in $(\mathbb{Q}_{>0}, \cdot)$?

For a discussion of where our methods come up short for answering Questions 1.19 and 1.20, see Subsection 2.7 below.

1.3.1. Patterns in $(\mathbb{N}, \cdot)$

A notion of upper Banach density can be defined in the semigroup $(\mathbb{N}, \cdot)$ by the formula in (1), where the supremum is now taken over Følner sequences in $(\mathbb{N}, \cdot)$. Examples of Følner sequences in $(\mathbb{N}, \cdot)$ include sequences of the form

$$\Phi_N = \left\{ b_N \prod_{i=1}^{N} p_i^{r_i} : 0 \leq r_i \leq N \right\},$$

where $(p_n)_{n\in\mathbb{N}}$ is an enumeration of the prime numbers and $(b_N)_{N\in\mathbb{N}}$ is any sequence in $\mathbb{N}$. In Section 8, we transfer Theorems 1.11 and 1.13 to the setting of cancellative abelian semigroups. As a consequence, we obtain the following result about geometric configurations in the multiplicative integers:

Theorem 1.21. *Let $E \subseteq \mathbb{N}$ be a set of positive multiplicative upper Banach density, and let $k \in \mathbb{Z}$. Then for any $\varepsilon > 0$, the sets*

$$\left\{ m \in \mathbb{N} : d^*_{mult}\left(E \cap E/m^k \cap E/m^{k+1}\right) > d^*_{mult}(E)^3 - \varepsilon \right\}$$

and

$$\left\{ m \in \mathbb{N} : d^*_{mult}\left(E \cap E/m \cap E/m^k\right) > d^*_{mult}(E)^3 - \varepsilon \right\}$$

are (multiplicatively) syndetic in $(\mathbb{N}, \cdot)$.

1.4. Applications to patterns in $\mathbb{Z}^2$

When $G = \mathbb{Z}^2$, we are able to give a complete picture of the phenomenon of large intersections for 3-point matrix patterns, that is, patterns of the form $\{\vec{x}, \vec{x} + M_1\vec{n}, \vec{x} + M_2\vec{n}\}$, where $\vec{x}, \vec{n} \in \mathbb{Z}^2$ and M_1, M_2 are 2×2 matrices with integer entries (note that any homomorphism $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ can be expressed as a 2×2 matrix with integer entries, so matrix patterns capture all possible configurations in $\mathbb{Z}^2$ that can be described within the framework of group homomorphisms).

Following [8], we say that the *syndetic supremum* of a bounded real-valued $\mathbb{Z}^2$-sequence $(a_{n,m})_{(n,m)\in\mathbb{Z}^2}$ is the quantity

$$\text{synd-sup}_{(n,m)\in\mathbb{Z}^2} a_{n,m} := \sup\left\{ a \in \mathbb{R} : \left\{ (n,m) \in \mathbb{Z}^2 : a_{n,m} > a \right\} \text{ is syndetic in} \mathbb{Z}^2 \right\}.$$

For 2×2 integer matrices M_1 and M_2 and $\alpha \in (0,1)$, we define the *ergodic popular difference density* by

$$\text{epdd}_{M_1,M_2}(\alpha) := \inf \text{synd-sup}_{\vec{n}\in\mathbb{Z}^2} \mu\left(A \cap T^{-1}_{M_1\vec{n}} A \cap T^{-1}_{M_2\vec{n}} A\right),$$

where the infimum is taken over all ergodic $\mathbb{Z}^2$-systems $(X, \mathcal{X}, \mu, (T_{\vec{n}})_{\vec{n}\in\mathbb{Z}^2})$ and sets $A \in \mathcal{X}$ with $\mu(A) = \alpha$. This can be seen as an ergodic-theoretic analogue to the *popular difference density* defined in [26]. It is natural to ask if $\text{epdd}_{M_1,M_2}(\alpha)$ coincides with the finitary combinatorial quantity $\text{pdd}_{M_1,M_2}(\alpha)$. Standard tools for translating between ergodic theory and combinatorics, such as Furstenberg's correspondence principle, are insufficient for resolving this question, and we do not know the

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Table 1. *Ergodic popular difference densities for 3-point matrix patterns in* $\mathbb{Z}^2$*.*

$r(M_1, M_2)$	Other conditions	$\mathrm{epdd}_{M_1,M_2}(\alpha)$	Reason
$(2,2,2)$	-	α^3	[2, Theorem 1.10]
$(2,2,1)$	-	α^3	Theorem 1.11
$(2,1,1)$	-	α^3	'Fubini' for UC - lim [9]
$(1,1,1)$	$[M_1, M_2] = 0$	$< \alpha^{c\log(1/\alpha)}$	Behrend-type construction [4, 8]
$(1,1,1)$	$[M_1, M_2] \neq 0$, 'row-like'	α^3	'Fubini' for UC - lim [9]
$(1,1,1)$	$[M_1, M_2] \neq 0$, 'column-like'	$\alpha^{4-o(1)}$	[13, Theorem 1.1], [16, Theorem 1.2]

answer in general. However, in special cases where $\mathrm{pdd}_{M_1,M_2}(\alpha)$ is known, it is in agreement with the values of $\mathrm{epdd}_{M_1,M_2}(\alpha)$ displayed in Table 1 below, and we suspect that $\mathrm{pdd}_{M_1,M_2}(\alpha) = \mathrm{epdd}_{M_1,M_2}(\alpha)$ in the remaining cases (see Subsection 7.3 below for additional remarks on (combinatorial) popular difference densities for matrix patterns in $\mathbb{Z}^2$).

Theorem 1.11 provides a sufficient condition on the matrices M_1 and M_2 to guarantee that $\mathrm{epdd}_{M_1,M_2}(\alpha) \geq \alpha^3$ for $\alpha \in (0,1)$. We now seek to describe the quantity $\mathrm{epdd}_{M_1,M_2}(\alpha)$ for any pair of 2×2 integer matrices M_1 and M_2. Table 1 summarises ergodic popular difference densities for all 3-point matrix configurations in $\mathbb{Z}^2$ (for matrices M_1, M_2, we let $r(M_1, M_2)$ be a list of the ranks of M_1, M_2 and $M_2 - M_1$ in decreasing order, and we denote by $[M_1, M_2]$ the commutator $M_1M_2 - M_2M_1$).

The cases $r(M_1, M_2) = (2,2,2)$ and $r(M_1, M_2) = (2,2,1)$ are covered directly by [2, Theorem 1.10] and Theorem 1.11, respectively. Indeed, a matrix M has full rank if and only if the subgroup $M(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$ has finite index. More precisely,

$$[\mathbb{Z}^2 : M(\mathbb{Z}^2)] = \begin{cases} |\det(M)|, & \text{if } \det(M) \neq 0; \\ \infty, & \text{if } \det(M) = 0. \end{cases}$$

The remaining cases are proved in Section 7.

1.5. Preliminary remarks on characteristic factors

In this paper, we approach multiple recurrence problems by determining and utilising the so-called *characteristic factors*, which are the factors that are responsible for the limiting behaviour of the quantity

$$\mu\left(A \cap T^{-1}_{\varphi(g)} A \cap T^{-1}_{\psi(g)} A\right)$$

in ergodic G-systems (see Subsection 2.2 for a discussion of factors in general and Definition 3.3 for a definition of characteristic factors). For $\mathbb{Z}$-actions, there are two different approaches to characteristic factors for linear averages, developed independently by Host and Kra [23] and by Ziegler [30], giving rise to factors that coincide (see [5, Appendix A]). However, in the context of G-actions, where G is an arbitrary (nonfinitely generated) countable discrete abelian group, the approaches of Host–Kra and of Ziegler may produce different factors (see Subsection 2.6 below for more details).

Our work, thus, leads to the general open question of how, in the setup of countable discrete abelian groups, the Host–Kra factors are related to the actual characteristic factors of the corresponding multiple ergodic averages (the factors obtained by Ziegler's approach). Discerning the relationship between the Host–Kra factors and the characteristic factors may lead to a better understanding of the quantities

$$\mu(A \cap T^{-1}_{\varphi_1(g)} A \cap \ldots \cap T^{-1}_{\varphi_k(g)} A),$$

where $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g \in G})$ is a G-system, $A \in \mathcal{X}$ and $\varphi_i : G \to G$ are homomorphisms or, more generally, polynomial maps.

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1.6. Structure of the paper

The paper is organised as follows. In Section 2, we introduce notation and conventions that we use throughout the paper.

Proofs of the main results appear in Sections 3–6. First, in Section 3, we establish characteristic factors for the multiple ergodic averages

$$\mathrm{UC}\text{ - }\lim_{g\in G} T_{\varphi(g)} f_1 \cdot T_{\psi(g)} f_2$$

when $(\psi - \varphi)(G)$ has finite index in G and prove Theorem 1.11. Then, in Section 4, we use an extension trick to simplify the characteristic factors, and in Section 5, prove a new limit formula for the extension system, leading to a proof of Theorem 1.13. Finally, we prove Theorem 1.14 in Section 6.

The final two sections contain applications of the main results. Using Theorem 1.11 together with additional tools from [2, 8, 9, 13, 16], we compute ergodic popular difference densities for 3-point matrix patterns in $\mathbb{Z}^2$. In Section 8, we extend the main results (Theorems 1.11 and 1.13) to the setting of cancellative abelian semigroups.

2. Preliminaries

The goal of this section is to introduce some notations and objects that will play an important role in this paper. Throughout this section, we let G denote an arbitrary countable discrete abelian group and $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ a G-system.

2.1. Uniform Cesàro limits

The large intersection property of a family $\{\varphi_1, \dots, \varphi_k\}$ is related to the limit behaviour of the multiple ergodic averages

$$\frac{1}{|\Phi_N|} \sum_{g\in\Phi_N} \prod_{i=1}^{k} T_{\varphi_i(g)} f_i, \tag{4}$$

where $(\Phi_N)_{N\in\mathbb{N}}$ is a Følner sequence[2] in G and $f_1, \dots, f_k \in L^\infty(\mu)$. By [3] and [32], the quantity in (4) converges in $L^2(\mu)$ as $N \to \infty$, and the limit is independent of the choice of Følner sequence $(\Phi_N)_{N\in\mathbb{N}}$. For more concise notation, we define the *uniform Cesàro limit* $x = \mathrm{UC}\text{ - }\lim_{g\in G} x_g$ if $\frac{1}{|\Phi_N|}\sum_{g\in\Phi_N} x_g \to x$ for every Følner sequence $(\Phi_N)_{N\in\mathbb{N}}$ in G.

One crucial tool for handling uniform Cesàro limits is the following version of the van der Corput differencing trick:

Lemma 2.1 (van der Corput lemma, cf. [2], Lemma 2.2). *Let $\mathcal{H}$ be a Hilbert space and G a countable amenable group. Let $(u_g)_{g\in G}$ be a bounded sequence in $\mathcal{H}$. If $UC\text{ - }\lim_{g\in G}\langle u_{g+h}, u_g\rangle$ exists for every $h \in G$, and*

$$UC\text{ - }\lim_{h\in G} UC\text{ - }\lim_{g\in G} \langle u_{g+h}, u_g\rangle = 0$$

then,

$$UC\text{ - }\lim_{g\in G} u_g = 0$$

strongly.

[2]A sequence $(\Phi_N)_{N\in\mathbb{N}}$ of finite subsets of G is a *Følner sequence* if, for any $x \in G$, $\frac{|(\Phi_N + x)\triangle\Phi_N|}{|\Phi_N|} \to 0$ as $N \to \infty$.

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Another useful tool for computing uniform Cesàro limits is the following 'Fubini' trick, which we use extensively in Section 7:

Lemma 2.2 ([9], special case of Lemma 1.1). *Let G and H be countable discrete amenable groups, and let $(v_{h,g})_{(h,g)\in H\times G}$ be a bounded sequence. Suppose*

$$UC\text{-}\lim_{(h,g)\in H\times G} v_{h,g}$$

exists, and for every $g \in G$,

$$UC\text{-}\lim_{h\in H} v_{h,g}$$

exists. Then

$$UC\text{-}\lim_{g\in G} UC\text{-}\lim_{h\in H} v_{h,g} = UC\text{-}\lim_{(h,g)\in H\times G} v_{h,g}.$$

2.2. Factors

A *factor* of $\mathbf{X}$ is a G-system $\mathbf{Y} = (Y, \mathcal{Y}, \nu, (S_g)_{g\in G})$ together with a measurable map $\pi : X \longrightarrow Y$, such that $\pi_*\mu = \nu$ and $\pi \circ T_g = S_g \circ \pi$ for all $g \in G$. There is a natural one-to-one correspondence between factors and $(T_g)_{g\in G}$-invariant sub-σ-algebras of $\mathcal{X}$. Throughout the paper, we freely move between the system $\mathbf{Y}$ and the σ-algebra $\pi^{-1}(\mathcal{Y})$ and refer to both of them as factors of $\mathbf{X}$. Given $f \in L^2(\mu)$, we denote by $E(f|\mathcal{Y})$ the conditional expectation of f with respect to the σ-algebra $\pi^{-1}(\mathcal{Y})$. We say that f is *measurable with respect to* $\mathcal{Y}$ if $f = E(f|\mathcal{Y})$.

2.3. Factor of invariant sets

Let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a G-system. We write $\mathcal{I}_G(X)$ for the sub-σ-algebra of G-invariant sets. We say that $\mathbf{X}$ is *ergodic* if $\mathcal{I}_G(X)$ is the σ-algebra comprised of null and conull subsets of $(X, \mathcal{X}, \mu)$. For a subgroup $H \leq G$, we denote by $\mathcal{I}_H(X)$ the sub-σ-algebra of H-invariant sets. Given a homomorphism $\varphi : G \longrightarrow G$, it is convenient to denote by $\mathcal{I}_\varphi(X)$ the σ-algebra $\mathcal{I}_{\varphi(G)}(X)$.

2.4. Host–Kra factors

The Gowers–Host–Kra seminorms are an ergodic-theoretic version of the uniformity norms introduced by Gowers in [22]. These seminorms were first introduced by Host and Kra in [23] in the case of ergodic $\mathbb{Z}$-systems, and then generalised by Chu, Frantzikinakis and Host to $\mathbb{Z}$-systems that are not necessarily ergodic in [14]. In [10, Appendix A], a general theory of Gower–Host–Kra seminorms is developed for (not necesssarily ergodic) G-systems, where G is an arbitrary countable discrete abelian group.

Definition 2.3. Let G be a countable discrete abelian group, and let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a G-system. Let $f \in L^\infty(X)$, and let $k \geq 1$ be an integer. The *Gowers–Host–Kra seminorm* $\|f\|_{U^k(G)}$ *of order k of f* is defined recursively by the formula

$$\|f\|_{U^1(G)} := \|E(\phi|\mathcal{I}_G(X))\|_{L^2}$$

for $k = 1$, and

$$\|f\|_{U^k(G)} := \text{UC}\text{-}\lim_{g\in G}\left(\|\Delta_g f\|_{U^{k-1}}^{2^{k-1}}\right)^{1/2^k}$$

for $k > 1$, where $\Delta_g f(x) = f(T_g x) \cdot \overline{f(x)}$.

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In [10, Appendix A], it is shown that the Gower–Host–Kra seminorms for general G-systems are indeed seminorms. Moreover, these seminorms correspond to factors of $\mathbf{X}$.

Proposition 2.4 (cf. [10], Proposition 1.10). *Let G be a countable discrete abelian group, let $\mathbf{X}$ be a G-system and let $k \geq 0$. There exists a unique (up to isomorphism) factor $\mathbf{Z}^k(X) = \left(Z^k(X), \mathcal{Z}^k(X), \mu_k, (T_g^{(k)})_{g\in G}\right)$ of $\mathbf{X}$ with the property that for every $f \in L^\infty(X)$, $\|f\|_{U^{k+1}(X)} = 0$ if and only if $E(f|\mathcal{Z}^k(X)) = 0$.*

The factors $\mathbf{Z}^k$ guaranteed by Proposition 2.4 are called the *Host–Kra* factors of $\mathbf{X}$.

Let $\mathbf{X} = \left(X, \mathcal{X}, \mu, (T_g)_{g\in G}\right)$ be a G-system. Then, $\mathcal{Z}^0(X)$ is the same as the σ-algebra $\mathcal{I}_G(X)$. In particular, if $\mathbf{X}$ is ergodic, then $\mathcal{Z}^0(X)$ is trivial. In the literature, $\mathbf{Z}^1(X)$ is often called the *Kronecker* factor, and $\mathbf{Z}^2(X)$ the *Conze–Lesigne* or *quasi-affine* factor of $\mathbf{X}$.

We summarise some basic results about the Host–Kra factors.

Theorem 2.5. *Let G be a countable discrete abelian group, and let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a ergodic G-system. Then,*

(i) *For every $k \geq 1$, $\mathcal{Z}^{k-1}(X) \leq \mathcal{Z}^k(X)$. In other words, $\mathbf{Z}^{k-1}(X)$ is a factor of $\mathbf{Z}^k(X)$. In particular, $\mathcal{I}(X) \leq \mathcal{Z}^k(X)$ for every $k \geq 0$.*
(ii) *The Kronecker factor of $\mathbf{X}$ is isomorphic to a rotation on a compact abelian group. Namely, there exists a homomorphism $\alpha : G \to Z$ into a compact abelian group $(Z, +)$, such that $\mathbf{Z}^1(X)$ is isomorphic to $(Z, (R_g)_{g\in G})$, where $R_g z = z + \alpha(g)$.*
(iii) *For every $k \geq 1$, if $\mathbf{X}$ is ergodic, then $\mathbf{Z}^k(X)$ is an extension of $\mathbf{Z}^{k-1}(X)$ by a compact abelian group $(H, +)$ and a cocycle $\rho : G \times Z^{k-1}(X) \to H$. Namely, $Z^k(X) = Z^{k-1}(X) \times H$ as measure spaces, and the action is given by $T_g^{(k)}(z, h) = (T_g^{(k-1)} z, h + \rho(g, z))$.*

Proof. The proof of (i) is an immediate consequence of the monotonicity of the seminorms (see [23, Corollary 4.4]). The proof of (ii) in the generality of countable discrete abelian groups can be found in [2, Lemma 2.4]. The proof of (iii) can be found for $\mathbb{Z}$-actions in [23, Proposition 6.3], and the same proof works for arbitrary countable discrete abelian groups. □

2.5. Joins and meets of factors

Let G be a countable discrete abelian group, let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a G-system and let $\varphi, \psi : G \to G$ be arbitrary homomorphisms.

1. Let $\mathcal{Z}_\varphi^1(X)$, or just $\mathcal{Z}_\varphi(X)$, denote the σ-algebra of the Kronecker factor of X with respect to the action of $\varphi(G)$, that is, the σ-algebra of the factor $\mathbf{Z}_\varphi^1(X)$ obtained by applying Proposition 2.4 for the G-system $(X, \mathcal{X}, \mu, (T_{\varphi(g)})_{g\in G})$ and $k = 1$. More generally, let H be a subgroup of G and $k \geq 1$, we let $\mathcal{Z}_H^k(X)$ denote the σ-algebra of the k-th Host–Kra factor $\mathbf{Z}_H^k(X)$ with respect to the action of H.
2. Let $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2$ be σ-algebras on X. Then,
   - We write $\mathcal{A} \leq \mathcal{X}$ if the σ-algebra $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{X}$.
   - We let $\mathcal{A}_1 \vee \mathcal{A}_2$ denote the *join* of $\mathcal{A}_1$ and $\mathcal{A}_2$, that is, the σ-algebra generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathcal{X}$.
   - We let $\mathcal{A}_1 \wedge \mathcal{A}_2$ denote the *meet* of $\mathcal{A}_1$ and $\mathcal{A}_2$, that is, the maximal σ-algebra which is also a sub-σ-algebra of $\mathcal{A}_1$ and $\mathcal{A}_2$.
   - We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *μ-independent* if their meet is trivial modulo μ-null sets.
   - More generally, we say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *relatively independent over the σ-algebra* $\mathcal{A}$ if $\mathcal{A}_1 \wedge \mathcal{A}_2 \leq \mathcal{A}$.
3. We let $\mathcal{I}_{\varphi,\psi}(X)$ denote the meet of $\mathcal{I}_\varphi(X)$ and $\mathcal{I}_\psi(X)$ and $\mathcal{Z}_{\varphi,\psi}(X)$ the meet of $\mathcal{Z}_\varphi(X)$ and $\mathcal{Z}_\psi(X)$. We let $\mathbf{Z}_{\varphi,\psi}(X)$ denote the factor of $\mathbf{X}$ which corresponds to the σ-algebra $\mathcal{Z}_{\varphi,\psi}(X)$.

The next two lemmas give convenient alternative descriptions of independent and relatively independent σ-algebras. These results are classical and can be found, for example, in [31, Proposition 1.4]; we provide short proofs for the convenience of the reader.

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Proposition 2.6 (Independent σ-algebras). *Let $\mathbf{X} = (X, \mathcal{X}, \mu)$ be a probability space. Two σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ on X are μ-independent if and only if the following equivalent conditions hold:*

(i) *Any function $f \in L^\infty(X)$ measurable with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$ simultaneously is a constant μ-almost everywhere.*
(ii) *If $f \in L^\infty(X)$ is measurable with respect to $\mathcal{A}_1$ and $g \in L^\infty(X)$ is measurable with respect to $\mathcal{A}_2$, then*

$$\int_X f \cdot g \, d\mu = \int_X f \, d\mu \cdot \int_X g \, d\mu.$$

Proof. The first definition of independence above is clearly equivalent to (i). We prove the equivalence between (i) and (ii).

(i)$\Rightarrow$ (ii).

$$\int_X f \cdot g \, d\mu = \int_X E(f|\mathcal{A}_2) \cdot g \, d\mu = \int_X E(f|\mathcal{A}_2) \, d\mu \cdot \int_X g \, d\mu = \int_X f \, d\mu \cdot \int_X g \, d\mu,$$

where the second equality holds since $E(f|\mathcal{A}_2)$ is a constant μ-a.e. by (i).

For (ii)$\Rightarrow$ (i), let $\widetilde{f} = f - \int f d\mu$. Then,

$$\|\widetilde{f}\|^2_{L^2(\mu)} = \int |\widetilde{f}|^2 \, d\mu = \left| \int_X \widetilde{f} \, d\mu \right|^2 = 0.$$

We conclude that $f = \int f d\mu$. □

Proposition 2.7 (Relatively independent σ-algebras). *Let $\mathbf{X} = (X, \mathcal{X}, \mu)$ be a probability space. Let $\mathcal{A}_1, \mathcal{A}_2$ be two σ-algebras on X, and let $\mathcal{A}$ be a third σ-algebra, such that $\mathcal{A} \preceq \mathcal{A}_1 \wedge \mathcal{A}_2$. Then, $\mathcal{A}_1$ and $\mathcal{A}_2$ are relatively independent with respect to $\mathcal{A}$ if the following equivalent conditions hold:*

(i) *Any function $f \in L^\infty(X)$ measurable with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$ simultaneously is measurable with respect to $\mathcal{A}$.*
(ii) *If f is measurable with respect to $\mathcal{A}_1$ and g is measurable with respect to $\mathcal{A}_2$, then*

$$E(fg|\mathcal{A}) = E(f|\mathcal{A}) \cdot E(g|\mathcal{A}).$$

Proof. Condition (i) is equivalent to the definition of relative independence above. Therefore, it is enough to prove the equivalence of (i) and (ii).

(i)$\Rightarrow$(ii). We have $E(fg|\mathcal{A}_1) = f \cdot E(g|\mathcal{A}_1) = f \cdot E(g|\mathcal{A})$, where the last equality follows from (i). Now, by taking the conditional expectation over $\mathcal{A}$, we have

$$E(fg|\mathcal{A}) = E(f|\mathcal{A}) \cdot E(g|\mathcal{A}).$$

(ii)$\Rightarrow$ (i). Let $\widetilde{f} = f - E(f|\mathcal{A})$. Then $E(|\widetilde{f}|^2|\mathcal{A}) = E(\widetilde{f}|\mathcal{A})^2 = 0$. In particular, $\int |\widetilde{f}|^2 d\mu = 0$, thus, $f = E(f|\mathcal{A})$. □

2.6. Characteristic factors

Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an invertible ergodic measure preserving system and $f_1, ..., f_k \in L^\infty(X)$, $k \geq 0$. The convergence of the multiple ergodic averages

$$\frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{k} T^{in} f_i \tag{5}$$

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in $L^2(\mu)$ for general k was established by Host and Kra [23] and independently, though somewhat later, by Ziegler [30].

Host and Kra proved convergence by showing that the averages in (5) are controlled by the Gowers–Host–Kra seminorms defined above. This reduces the general convergence problem to convergence under the additional assumption that each function f_i is measurable with respect to the Host–Kra factor.

Ziegler, on the other hand, studied the universal (minimal) characteristic factors for the multiple ergodic averages

$$\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{k} T^{a_i n} f_i,$$

where $a_1, \ldots, a_k \in \mathbb{Z}$ are distinct and nonzero. These are the minimal factors $\mathcal{Z}_{k-1}(X)$, such that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{k} T^{a_i n} f_i = 0$$

whenever $E(f_i|\mathcal{Z}_{k-1}(X)) = 0$ for some i.

In [5, Appendix A], Leibman proved that, for $\mathbb{Z}$-systems, the factors studied by Host and Kra coincide with the factors studied by Ziegler, thus giving these factors the name *Host–Kra–Ziegler factors*. Using Følner sequences in order to define averages, one can generalise the above to arbitrary countable discrete abelian groups (or even more generally, to amenable groups). However, in the setting of general abelian groups, Host–Kra factors may no longer coincide with the characteristic factors for averages of the form

$$\text{UC - }\lim_{g\in G}\prod_{i=1}^{k} T_{a_i g} f_i.$$

We give a very simple example. Let p be a prime number and $\mathbb{F}_p$ be the group with p elements. We denote by $\mathbb{F}_p^\infty$ the direct sum of countably many copies of $\mathbb{F}_p$. In [10], it is shown that there are many nontrivial ergodic $\mathbb{F}_p^\infty$-systems with nontrivial Host–Kra factors $\mathcal{Z}^k(X)$ for any $k \geq 0$. However, the only characteristic factor for the average

$$\text{UC - }\lim_{g\in G} T_g f_1 \cdot \ldots \cdot T_{pg} f_p$$

is $\mathcal{X}$. Indeed, since $T_{pg} = Id$, the average is nonzero for every $f_p \neq 0$, assuming that $f_1 = \ldots = f_{p-1} = 1$ (say). To overcome this technicality, one may restrict to the case where $k < p$, but the situation is not that simple for arbitrary countable discrete abelian groups, and, in general, Host–Kra factors may not coincide with the universal characteristic factors.

This phenomenon was not studied previously in the literature, but it plays an important role in this paper. More specifically, we study how the Host–Kra factor $\mathcal{Z}^1(X)$, which coincides with the classical Kronecker factor, is related to the the universal characteristic factor, $\mathcal{Z}_1(X)$, for the average

$$\text{UC - }\lim_{g\in G} T_g f_1 T_{2g} f_2,$$

where $f_1, f_2 \in L^\infty(\mu)$, in the setting of actions of countable discrete abelian groups. One of our main tools is a result which asserts, roughly speaking, that by adding eigenfunctions to the system $\mathbf{X}$, one has that the characteristic factor $\mathcal{Z}_1(X)$ is generated by the Host–Kra factor $\mathcal{Z}^1(X)$ and the σ-algebra of $2G$-invariant functions. We also give an example that illustrates the necessity of adding eigenfunctions to the system (see Example 4.1).

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2.7. Seminorms for multiplicative configurations

We now give a brief explanation of where our methods come up short of fully answering Questions 1.19 and 1.20. As discussed above, our approach to the large intersections property is to study families of seminorms and their corresponding characteristic factors. However, in the case of Questions 1.19 and 1.20, these seminorms have somewhat exotic behaviour.

For example, Question 1.19 is related to the averages

$$\text{UC - }\lim_{q\in\mathbb{Q}_{>0}} f_1(T_{q^n}x)f_2(T_{q^m}x) \tag{6}$$

for some ergodic $(\mathbb{Q}_{>0},\cdot)$-system. An application of the van der Corput lemma (Lemma 2.1) shows that (6) is equal to zero if

$$\text{UC - }\lim_{q\in\mathbb{Q}_{>0}} \left|\int \Delta_{q^m} f_1 \cdot E(\Delta_{q^m} f_2 | \mathcal{I}_{q^{n-m}}(X))\, d\mu\right| = 0.$$

If the action of $T_{q^{n-m}}$, $q\in(\mathbb{Q}_{>0},\cdot)$, were ergodic (e.g. if $n=m+1$), then the above expression is manageable as we will see in this paper. Presumably, if n and m are coprime, then this expression may also be manageable, but we do not see how.

Question 1.20 is related to the average

$$\text{UC - }\lim_{q\in\mathbb{Q}_{>0}} T_q f_1 T_{q^2} f_2 T_{q^3} f_3. \tag{7}$$

Using the van der Corput lemma, the Cauchy–Schwarz inequality and then the van der Corput lemma again, we see that the average in (7) is zero if

$$\text{UC - }\lim_{q_1\in\mathbb{Q}_{>0}} \left|\text{UC - }\lim_{q_2\in\mathbb{Q}_{>0}} \int \Delta_{q_1^2}\Delta_{q_2^3} f_3\, d\mu\right| = 0.$$

If in the expression above we had q_1^2, q_2^2, or q_1^3, q_2^3, then this expression would be related to the Gowers–Host–Kra seminorm of f_3 with respect to the action of all squares or cubes of $(\mathbb{Q}_{>0},\cdot)$. The above quantity is therefore some combination of the two. Again, presumably, the fact that 2 and 3 are coprime may be useful to analyse these seminorms. Studying the structure of these new peculiar seminorms is an interesting problem that we do not pursue in this paper.

3. Theorem 1.11

We first give a brief overview of the proof of Theorem 1.11. Let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic G-system, and let $\varphi, \psi : G \to G$ be arbitrary homomorphisms, such that $(\psi - \varphi)(G)$ has finite index in G. The key component in the proof of Theorem 1.11 is the analysis of the limit behaviour of the multiple ergodic averages

$$\text{UC - }\lim_{g\in G} f_1(T_{\varphi(g)}x)\cdot f_2(T_{\psi(g)}x) \tag{8}$$

for $f_1, f_2 \in L^\infty(X)$. Standard arguments using the van der Corput lemma (Proposition 3.5) show that

$$\begin{aligned}&\text{UC - }\lim_{g\in G} f_1(T_{\varphi(g)}x)\cdot f_2(T_{\psi(g)}x) = \\ &\qquad \text{UC - }\lim_{g\in G} E(f_1|\mathcal{Z}_\varphi(X))(T_{\varphi(g)}(x))E(f_2|\mathcal{Z}_\psi(X))(T_{\psi(g)}(x)),\end{aligned} \tag{9}$$

where $\mathcal{Z}_\varphi(X)$ and $\mathcal{Z}_\psi(X)$ are the σ-algebras of the Kronecker factors of X with respect to the actions of $\varphi(G)$ and $\psi(G)$, respectively (see Subsection 2.5).

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In Theorem 1.11, we assume, furthermore, that $\varphi(G)$ has finite index in G. In this case, the factor $\mathcal{Z}_{\varphi}(X)$ coincides with $\mathcal{Z}^1_G(X)$, the Kronecker factor of X with respect to the action of G (see Lemma 3.6). Our main observation is that one can replace $\mathcal{Z}_{\psi}(X)$ in (9) with a smaller factor. As an illustration, we give the following example:

Example 3.1. Consider the additive group $G = \bigoplus_{j=1}^{\infty} \mathbb{Z}/4\mathbb{Z}$. We use $i \in \mathbb{C}$ to denote the square root of -1, and for every natural number $n \in \mathbb{N}$, we let C_n denote the group of roots of unity of degree n. We define an action of G on $X = \left(\prod_{j\in\mathbb{N}} C_4\right) \times C_2$ by

$$T_g(\mathbf{x}, y) = \left((i^{g_j} x_j)_{j\in\mathbb{N}}, y \cdot \prod_{j\in\mathbb{N}} (x_j^{2g_j} \cdot i^{g_j^2 - g_j})\right),$$

where $\mathbf{x} = (x_1, x_2, \ldots) \in \prod_{j\in\mathbb{N}} C_4$ and $g = (g_1, g_2, \ldots)$ is any representation of g in $\bigoplus_{j=1}^{\infty} \mathbb{Z}/4\mathbb{Z}$. The system $(X, (T_g)_{g\in G})$ is a group extension of its Kronecker factor $Z_G(X) = \prod_{j\in\mathbb{N}} C_4$ by the cocycle

$$\sigma : G \times \prod_{j\in\mathbb{N}} C_4 \to C_2,$$
$$\sigma(g, \mathbf{x}) = \prod_{j\in\mathbb{N}} (x_j^{2g_j} \cdot i^{g_j^2 - g_j}).$$

Let $\psi(g) = 2g$. We observe that the function $f(\mathbf{x}, y) = y$ is orthogonal to $L^2(Z^1(X))$. On the other hand, we have

$$T_{2g} f(\mathbf{x}, y) = \sigma(2g, \mathbf{x}) \cdot y = \prod_{j\in\mathbb{N}} (x_j^{4g_j} \cdot i^{4g_j^2 - 2g_j}) \cdot y = \prod_{j\in\mathbb{N}} (-1)^{g_j} y = \prod_{j\in\mathbb{N}} (-1)^{g_j} f(\mathbf{x}, y).$$

In other words, f is an eigenfunction with respect to the action of $\psi(G)$ on X with eigenvalue $\lambda(2g) = \prod_{j\in\mathbb{N}} (-1)^{g_j}$. Therefore, f is measurable with respect to $\mathcal{Z}_{\psi}(X)$, and we see that $\mathcal{Z}^1(X) \neq \mathcal{Z}_{\psi}(X)$. Now, let $\varphi(g) = g$. We claim that f does not contribute to (8). Namely, we have that

$$\text{UC - }\lim_{g\in G} T_g f_1 T_{2g} f = 0$$

for every bounded function f_1. Indeed, by (9), it is enough to check this equality in the case where f_1 is an eigenfunction with respect to the action of G. Let $\chi(g)$ be the eigenvalue of f_1, we see that

$$\text{UC - }\lim_{g\in G} T_g f_1 T_{2g} f = f_1 \cdot f \cdot \text{UC - }\lim_{g\in G} \chi(g) \cdot \lambda(2g).$$

The eigenfunctions of X take the form $h(\mathbf{x}, y) = \prod_{i=1}^{n} x_i^{l_i}$ for some $n \in \mathbb{N}$ and $l_1, \ldots, l_n \in \{0, 1, 2, 3\}$. Therefore, $g \mapsto \chi(g)\lambda(2g)$ is a nontrivial character of G and so

$$\text{UC - }\lim_{g\in G} \chi(g)\lambda(2g) = 0.$$

Remark 3.2. In the example above, the factor $\mathbf{Z}^1(X)$ is isomorphic to $\prod_{j\in\mathbb{N}} C_4$ equipped with the action $T_g^{(1)} x = (i^{g_j} \cdot x_j)_{j\in\mathbb{N}}$, while $\mathbf{Z}^2(X) = \mathbf{X}$. On the other hand, for the $2G$-system $\left(X, (T_g)_{g\in 2G}\right)$, we have $\mathbf{Z}^1_{2G}(X) = \mathbf{X}$.

Example 3.1 suggests that a $\psi(G)$-eigenfunction contributes to (8) if and only if its eigenvalue coincides with an eigenvalue of the G-action. In practice, we use a result of Frantzikinakis and Host [19] to decompose f_2 into a linear combination of eigenfunctions (see Proposition 3.12). However, since

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the action of $\psi(G)$ may not be ergodic, we have to include in our analysis the case where the $\psi(G)$-eigenvalue, $\lambda(\psi(g))$, is not a constant in X, but rather a $\psi(G)$-invariant function. We let $\widetilde{\mathcal{Z}}_\psi(X)$ be the sub-σ-algebra of $\mathcal{Z}_\psi(X)$ generated by all the $\psi(G)$-eigenfunctions with eigenvalues $\lambda(\psi(\cdot),x) : X \to \widehat{G}$ that coincide with an eigenvalue with respect to the G-action for μ-a.e. $x \in X$. We show that one can replace $\mathcal{Z}_\psi(X)$ with $\widetilde{\mathcal{Z}}_\psi(X)$ in (9). After replacing $\mathcal{Z}_\psi(X)$ by $\widetilde{\mathcal{Z}}_\psi(X)$, the remainder of the proof of Theorem 1.11 follows by modifying previous arguments used for deducing Khintchine-type recurrence from knowledge of relevant characteristic factors (see, e.g. [2, Section 8]).

3.1. Characteristic factors

We start with a definition of characteristic factors (cf. [21, Section 3]).

Definition 3.3. Let G be a countable discrete abelian group, let $\varphi, \psi : G \to G$ be arbitrary homomorphisms and let $X = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a G-system. A factor $\mathbf{Y} = (Y, \mathcal{Y}, \nu, (S_g)_{g\in G})$ of $\mathbf{X}$ is called a *partial characteristic factor for the pair* (φ, ψ) *with respect to* φ if

$$\text{UC - }\lim_{g\in G} T_{\varphi(g)} f_1 T_{\psi(g)} f_2 = \text{UC - }\lim_{g\in G} T_{\varphi(g)} E(f_1|\mathcal{Y}) T_{\psi(g)} f_2$$

for every $f_1, f_2 \in L^\infty(X)$. We define a partial characteristic factor with respect to ψ similarly and say that $\mathbf{Y}$ is a *characteristic factor* if it is a partial characteristic factor with respect to both φ and ψ, that is

$$\text{UC - }\lim_{g\in G} T_{\varphi(g)} f_1 T_{\psi(g)} f_2 = \text{UC - }\lim_{g\in G} T_{\varphi(g)} E(f_1|\mathcal{Y}) T_{\psi(g)} E(f_2|\mathcal{Y})$$

for every $f_1, f_2 \in L^\infty(X)$.

In other words, a factor of a measure preserving system $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ is a characteristic factor for a certain multiple ergodic average, if the study of the limit behaviour of the average can be reduced to this factor. The following easy lemma is related to the well-known result of Furstenberg, which asserts that a system $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ is weakly mixing if and only if the Kronecker factor, $\mathcal{Z}^1(X)$, is trivial.

Lemma 3.4. *Let* $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ *be a* G*-system, let* $\varphi : G \to G$ *be a homomorphism and let* $f \in L^2(X)$*. If* $E(f|\mathcal{Z}_\varphi(X)) = 0$*, then for every* $h \in L^2(X)$*, we have*

$$UC\text{ - }\lim_{g\in G}\left|\int_X T_{\varphi(g)} f \cdot h \, d\mu\right| = 0.$$

Proof. Assume $E(f \mid \mathcal{Z}_\varphi(X)) = 0$. Then by Proposition 2.4, $\|f\|_{U^2(\varphi(G))} = 0$, that is,

$$\text{UC - }\lim_{g\in G}\left|\int_X \Delta_{\varphi(g)} f \, d\mu\right| = 0.$$

Since $\text{UC - }\lim_{g\in G} |a_g| = 0 \iff \text{UC - }\lim_{g\in G} a_g^2 = 0$ for every bounded complex-valued sequence $g \mapsto a_g$, we have

$$\text{UC - }\lim_{g\in G} \int_{X\times X} (T_{\varphi(g)} \times T_{\varphi(g)}) f \otimes f \cdot \overline{f \otimes f} \, d(\mu\times\mu) = 0.$$

The mean ergodic theorem implies that

$$\int_{X^2} E(f\otimes f|\mathcal{I}_{\varphi\times\varphi}(X\times X)) \cdot f\otimes f \, d(\mu\times\mu) = 0$$

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and $E(f\otimes f|\mathcal{I}_{\varphi\times\varphi}(X\times X))=0$. Therefore, for every $h\in L^2(X)$, we have,

$$\text{UC - }\lim_{g\in G}\left(\int_X T_{\varphi(g)}f\cdot h\,d\mu\right)^2=\int_{X^2}E(f\otimes f|\mathcal{I}_{\varphi\times\varphi}(X\times X))\cdot h\otimes h\,d\mu\times\mu=0,$$

which implies that

$$\text{UC - }\lim_{g\in G}\left|\int_X T_{\varphi(g)}f\cdot h\,d\mu\right|=0$$

as required. □

Using the van der Corput lemma (Lemma 2.1), we show that $\mathcal{Z}_\varphi(X)$ and $\mathcal{Z}_\psi(X)$ are partial characteristic factors for the pair (φ,ψ) with respect to φ and ψ, respectively.

Proposition 3.5. *Let $X=(X,\mathcal{X},\mu,(T_g)_{g\in G})$ be an ergodic G-system. Let $\varphi,\psi:G\to G$ be homomorphisms, such that $(\psi-\varphi)(G)$ has finite index in G. Then, for any $f_1,f_2\in L^\infty(\mu)$, one has*

$$UC\text{ - }\lim_{g\in G}T_{\varphi(g)}f_1\cdot T_{\psi(g)}f_2=UC\text{ - }\lim_{g\in G}T_{\varphi(g)}E(f_1|\mathcal{Z}_\varphi(X))\cdot T_{\psi(g)}E(f_2|\mathcal{Z}_\psi(X))$$

in $L^2(\mu)$.

Proof. We follow the argument of Furstenberg and Weiss [21]. By linearity and symmetry, it is enough to show that

$$\text{UC - }\lim_{g\in G}T_{\varphi(g)}f_1\cdot T_{\psi(g)}f_2=0$$

whenever $E(f_1|\mathcal{Z}_\varphi(X))=0$. Dividing through by a constant, we may assume that $\|f_i\|_\infty\le 1$ for $i=1,2$.

We use the van der Corput lemma with $u_g=T_{\varphi(g)}f_1\cdot T_{\psi(g)}f_2$. For every $g,h\in G$, we have

$$\langle u_{g+h},u_g\rangle=\int_X T_{\varphi(g+h)}f_1\cdot T_{\psi(g+h)}f_2\cdot T_{\varphi(g)}\overline{f_1}\cdot T_{\psi(g)}\overline{f_2}\,d\mu. \tag{10}$$

Since the measure μ is $T_{\varphi(g)}$-invariant, (10) is equal to

$$\int_X T_{\varphi(h)}f_1\cdot\overline{f_1}\cdot T_{(\psi-\varphi)(g)}\left(T_{\psi(h)}f_1\cdot\overline{f_2}\right)d\mu.$$

Hence, by the mean ergodic theorem, we have

$$\text{UC - }\lim_{g\in G}\langle u_{g+h},u_g\rangle=\int_X T_{\varphi(h)}f_1\cdot\overline{f_1}\cdot E(T_{\psi(h)}f_2\cdot\overline{f_2}|\mathcal{I}_{\psi-\varphi}(X)).$$

Since $H:=(\psi-\varphi)(G)$ has finite index in G and the action of G on X is ergodic, we can find a partition $X=\bigcup_{i=1}^l A_i$ to H-invariant sets, where l is at most the index of H in G. Since f_2 is bounded by 1,

$$\left|\text{UC - }\lim_{g\in G}\langle u_{g+h},u_g\rangle\right|\le\sum_{i=1}^k\left|\int_X T_{\varphi(h)}f_1\cdot\overline{f_1}\cdot 1_{A_i}\,d\mu\right|.$$

Now, since $E(f_1|Z_\varphi(X))=0$, Lemma 3.4 implies that $\text{UC - }\lim_{h\in G}\left|\int_X T_{\varphi(h)}f_1\cdot\overline{f_1}\cdot 1_{A_i}d\mu\right|=0$, for every $1\le i\le k$. The van der Corput lemma (Lemma 2.1) then implies that

$$\text{UC - }\lim_{g\in G}T_{\varphi(g)}f_1\cdot T_{\psi(g)}f_2=0,$$

and this completes the proof. □

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In [5, Appendix A], Leibman proved the following result in the special case where $G = \mathbb{Z}$. For the sake of completeness, we give a proof for arbitrary countable discrete abelian G in Appendix A.

Lemma 3.6. *Let $(X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a G-system, and let $H \le G$ be a subgroup of finite index. Then, for every $k \ge 1$, one has $\mathcal{Z}^k_H(X) = \mathcal{Z}^k_G(X)$.*

In particular, if $\varphi(G)$ has finite index in G, then the factor $\mathcal{Z}_\varphi(X)$ coincides with $\mathcal{Z}(X)$.

Corollary 3.7. *Let G be a countable discrete abelian group, let $X = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a G-system and let $\varphi, \psi : G \to G$ be arbitrary homomorphisms, such that $\varphi(G)$ and $(\psi - \varphi)(G)$ have finite index in G. Then, for any bounded functions $f_1, f_2 \in L^\infty(X)$,*

$$UC\text{-}\lim_{g\in G} T_{\varphi(g)} f_1 \cdot T_{\psi(g)} f_2 = UC\text{-}\lim_{g\in G} T_{\varphi(g)} E(f_1|\mathcal{Z}(X)) \cdot T_{\psi(g)} E(f_2|\mathcal{Z}_\psi(X)).$$

Let G be a countable discrete abelian group and $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic G-system. By Theorem 2.5(ii), the Kronecker factor of $\mathbf{X}$, $\mathbf{Z}^1(X)$ is isomorphic to an ergodic rotation. Therefore, it is convenient to identify the Kronecker factor with the system $\mathbf{Z} = (Z, \alpha)$, where Z is a compact abelian group and $\alpha : G \to Z$ is a homomorphism, such that $T^1_g z = z + \alpha_g$, where T^1 is the G-action on Z. The following corollary of Proposition 3.5 will be useful later on in this paper.

Proposition 3.8. *Let $\mathbf{X} = \big(X, \mathcal{X}, \mu, (T_g)_{g\in G}\big)$ be an ergodic G-system with Kronecker factor $\mathbf{Z} = (Z, \alpha)$. Let $\varphi, \psi : G \to G$ be homomorphisms, such that $(\psi - \varphi)(G)$ has finite index in G. Then, for any $f_0, f_1, f_2 \in L^\infty(\mu)$ and any continuous function $\eta : Z^2 \to \mathbb{C}$, we have*

$$\begin{aligned} &UC\text{-}\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big) \int_X f_0 \cdot T_{\varphi(g)} f_1 \cdot T_{\psi(g)} f_2 \, d\mu \\ &\quad = UC\text{-}\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big) \int_X f_0 \cdot T_{\varphi(g)} E(f_1|\mathcal{Z}_\varphi(X)) \cdot T_{\psi(g)} E(f_2|\mathcal{Z}_\psi(X)) \, d\mu. \end{aligned}$$

Proof. By the Stone–Weierstrass theorem and linearity, we may assume $\eta(u, v) = \lambda_1(u)\lambda_2(v)$ for some characters $\lambda_1, \lambda_2 \in \widehat{Z}$. Let $\pi : X \to Z$ be the factor map, and let $\chi_i := \lambda_i \circ \pi$. Note that $T_g \chi_i = \lambda_i(\alpha_g)\chi_i$, so χ_i is a G-eigenfunction with eigenvalue $\lambda_i \circ \alpha$.

Now, set

$$\begin{aligned} h_0 &:= \overline{\chi_1\chi_2} f_0, \\ h_1 &:= \chi_1 f_1, \\ h_2 &:= \chi_2 f_2. \end{aligned}$$

Since χ_1 and χ_2 are measurable with respect to the Kronecker factor $\mathcal{Z}(X)$, which is a sub-σ-algebra of $\mathcal{Z}_\varphi(X)$ and $\mathcal{Z}_\psi(X)$, we have the identities

$$\begin{aligned} E(h_1|\mathcal{Z}_\varphi(X)) &= \chi_1 \cdot E(f_1|\mathcal{Z}_\varphi(X)), \\ E(h_2|\mathcal{Z}_\varphi(X)) &= \chi_2 \cdot E(f_2|\mathcal{Z}_\varphi(X)). \end{aligned}$$

Thus, applying Proposition 3.5 for the functions h_1, h_2 and integrating against h_0, we have

$$\begin{aligned} &\text{UC}\text{-}\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big) \int_X f_0 \cdot T_{\varphi(g)} f_1 \cdot T_{\psi(g)} f_2 \, d\mu \\ &\quad = \text{UC}\text{-}\lim_{g\in G} \int_X h_0 \cdot T_{\varphi(g)} h_1 \cdot T_{\psi(g)} h_2 \, d\mu \end{aligned}$$

$$= \text{UC - }\lim_{g\in G}\int_X h_0\cdot T_{\varphi(g)}E(h_1|\mathcal{Z}_{\varphi}(X))\cdot T_{\psi(g)}E(h_2|\mathcal{Z}_{\psi}(X))\,d\mu$$
$$= \text{UC - }\lim_{g\in G}\eta\big(\alpha_{\varphi(g)},\alpha_{\psi(g)}\big)\int_X f_0\cdot T_{\varphi(g)}E(f_1|\mathcal{Z}_{\varphi}(X))\cdot T_{\psi(g)}E(f_2|\mathcal{Z}_{\psi}(X))\,d\mu. \qquad \square$$

In the next section, we will study the factor $\mathcal{Z}_{\psi}(X)$ further.

3.2. *Relative orthonormal basis*

Let G be a countable discrete abelian group, and let $\mathbf{X} = (X,\mathcal{X},\mu,(T_g)_{g\in G})$ be a G-system. Under the assumption that the system is ergodic, it is well known that the factor $\mathcal{Z}^1(X)$ admits an orthonormal basis of eigenfunctions. The following example demonstrates that this may fail for nonergodic systems.

Example 3.9. Let $S^1 = \{z\in\mathbb{C} : |z| = 1\}$. Consider $X = S^1\times S^1$ equipped with the Borel σ-algebra, the Haar probability measure μ and the measure-preserving transformation $T(x,y) = (x, y\cdot x)$. Any function $f\in L^2(X)$ takes the form

$$f(x,y) = \sum_{n,m\in\mathbb{N}} a_{n,m}x^n y^m$$

for some $a_{n,m}\in\mathbb{C}$ with

$$\sum_{n,m\in\mathbb{N}} |a_{n,m}|^2 < \infty. \tag{11}$$

Now suppose that there exists some constant $c\in S^1$, such that $Tf(x,y) = c\cdot f(x,y)$ for μ-a.e. $(x,y)\in S^1\times S^1$. By the uniqueness of the Fourier series, we deduce that

$$a_{n+m,m} = c\cdot a_{n,m}$$

for every $n,m\in\mathbb{N}$. If $m\neq 0$, this is a contradiction to (11) unless $a_{n,m} = 0$. We conclude that f is an eigenfunction if and only if it is independent of the y coordinate. In particular, $L^2(X)$ is not generated by the eigenfunctions of X.

On the other hand, the functions $\{x^n\}_{n\in\mathbb{N}}$ are invariant and therefore measurable with respect to $\mathcal{Z}^1(X)$. Moreover, the functions $\{y^m\}_{n\in\mathbb{N}}$ satisfy $\Delta_n(y^m) = T^n(y^m)\cdot y^{-m} = x^{n\cdot m}$, which is an invariant function. Hence, y^m is also measurable with respect to $\mathcal{Z}^1(X)$. We thus conclude that X coincides with $\mathcal{Z}^1(X)$.

In order to handle nonergodic systems, Frantzikinakis and Host [19] came up with the following definition.

Definition 3.10. Let H be a countable discrete abelian group acting on a probability space $(X,\mathcal{X},\mu,(T_h)_{h\in H})$. A *relative orthonormal system* is a countable family $(\phi_j)_{j\in\mathbb{N}}$ belonging to $L^2(\mu)$, such that

(i) $\mathbb{E}(\left|\phi_j\right|^2|\mathcal{I}_H(X))$ has value 0 or 1 μ-a.e. for every $j\in\mathbb{N}$;
(ii) $\mathbb{E}(\phi_j\overline{\phi_k}|\mathcal{I}_H(X)) = 0$ μ-a.e. for all $j,k\in\mathbb{N}$ with $j\neq k$.

The family $(\phi_j)_{j\in\mathbb{N}}$ is also a *relative orthonormal basis* if it also satisfies

(iii) The linear space spanned by the set of functions

$$\big\{\phi_j\psi : j\in\mathbb{N}, \psi\in L^\infty(\mu) \text{ is } H\text{-invariant}\big\}$$

is dense in $L^2(\mu)$.

We also give a definition of eigenfunctions that applies to nonergodic systems.

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Definition 3.11 (H-eigenfunctions). Let H be a countable discrete abelian group and $X = (X, \mathcal{X}, \mu, (T_h)_{h\in H})$ be an H-system. We say that $f : X \to \mathbb{C}$ is an *H-eigenfunction* if there exists an H-invariant function $\lambda : X \to \widehat{H}$, such that $T_h f(x) = \lambda(x,h) \cdot f(x)$ for all $h \in H$ and μ-a.e. $x \in X$. In this case, we also say that λ is the *eigenvalue* of f.

Note that under the assumption that the H-action is ergodic, this definition coincides with the standard definition of an eigenfunction. Observe, moreover, that the functions $\{y^m\}_{m\in\mathbb{N}}$ from Example 3.9 are eigenfunctions according to this definition.

Frantzikinakis and Host proved the following result:

Theorem 3.12 ([19], Theorem 5.2). *Let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_h)_{h\in H})$ be an H-system. Then $\mathcal{Z}_H(X)$ admits a relative orthonormal basis of eigenfunctions.*

The proof of Theorem 3.12 is given for $\mathbb{Z}$-actions in [19], but the same argument can be easily generalised for arbitrary group actions.

3.3. *Proof of Theorem 1.11*

In this subsection, we prove Theorem 1.11. Example 3.1 is a good example to have in mind while reading this section.

Let $\mathbf{X} = \big(X, \mathcal{X}, \mu, (T_g)_{g\in G}\big)$ be an ergodic G-system, and let $\mathbf{Z} = (Z, \alpha)$ be the Kronecker factor of $\mathbf{X}$. Let $A \in \mathcal{X}$ and $f = \mathbf{1}_A$. We can write

$$f_c := E(f|\mathcal{Z}(X)) = \sum_{i\in\mathbb{N}} a_i \zeta_i,$$

where $\{\zeta_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of eigenfunctions and $a_i \in \mathbb{C}$. Moreover, using Theorem 3.12,

$$f_\psi := E(f_\psi|\mathcal{Z}_\psi(X)) = \sum_{i\in\mathbb{N}} b_i \xi_i,$$

where $\{\xi_i\}_{i\in\mathbb{N}}$ is a relative orthonormal basis of $\psi(G)$-eigenfunctions and $b_i = E(f \cdot \overline{\xi}_i|\mathcal{I}_\psi(X))$ are $\psi(G)$-invariant functions.

Choose $N_1 \in \mathbb{N}$ sufficiently large so that

$$\left\| f_c - \sum_{i=1}^{N_1} a_i \zeta_i \right\|_2 < \frac{\varepsilon}{8}$$

and

$$\left\| f_\psi - \left(\sum_{i=1}^{N_1} b_i \xi_i \right) \right\|_2 < \frac{\varepsilon}{8}.$$

For each $j \in \mathbb{N}$, the function ξ_j is a $\psi(G)$-eigenfunction, so we can write $\xi_j\big(T_{\psi(g)}x\big) = \mu_j(x, \psi(g))\xi_j(x)$ for some $\psi(G)$-invariant function $\mu_j : X \to \widehat{\psi(G)}$. The group Z is compact, so $\widehat{Z}$ is countable and we can write $\widehat{Z} = \bigcup_{n\in\mathbb{N}} F_n$, where $F_1 \subseteq F_2 \subseteq \cdots$ are finite sets. Let

$$C_n := \big\{ g \mapsto \chi_1(\alpha_{\varphi(g)})\chi_2(\alpha_{\psi(g)}) : \chi_1, \chi_2 \in F_n \big\},$$

and let $C = \bigcup_{n\in\mathbb{N}} C_n$. Finally, let

$$E_{j,n} := \Big\{ x \in X : \mu_j(x, \cdot) \in C_n \cup \Big(\widehat{G} \setminus C\Big) \Big\}.$$

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Note that the complement of $E_{j,n}$ consists of all $x \in X$, such that $\mu_j(x,\cdot)$ belongs to a finite set. Since μ_j is measurable, we conclude that so is the complement of $E_{j,n}$. Hence, $E_{j,n}$ are measurable. Since $\bigcup_{n=1}^{\infty} E_{j,n} = X$ for every $j \in \mathbb{N}$, there exists sufficiently large $N_2 \in \mathbb{N}$, such that

$$\left(\int_{X\setminus E_{j,N_2}} |b_j\xi_j|^2 \, d\mu\right)^{1/2} < \frac{\varepsilon}{16N_1}$$

for $j = 1,\ldots,N_1$. Then, let $N \geq \max\{N_1, N_2\}$, such that: if $T_g\zeta_i = \chi(\alpha_g)\zeta_i$ for some $i = 1,\ldots,N_1$, then $\chi \in F_N$.

Now, let $B_0 \in Z$ be a small neighborhood of 0 in Z, such that if $z \in B_0$ and $\chi \in F_N$, then

$$|\chi(z) - 1| < \frac{\varepsilon}{16N}.$$

Let $\eta_0 : Z \longrightarrow [0,\infty)$ be a continuous function supported on B_0 normalised so that

$$\text{UC - }\lim_{g\in G} \eta_0(\alpha_{\varphi(g)})\eta_0(\alpha_{\psi(g)}) = 1.$$

Put $\eta(u,v) := \eta_0(u)\eta_0(v)$. Then, by Proposition 3.8, we have

$$\begin{aligned}
&\text{UC - }\lim_{g\in G} \eta\big(\alpha_{\varphi(g)},\alpha_{\psi(g)}\big)\ \mu\Big(A\cap T_{\varphi(g)}^{-1}A\cap T_{\psi(g)}^{-1}A\Big)\\
&\quad= \text{UC - }\lim_{g\in G} \eta\big(\alpha_{\varphi(g)},\alpha_{\psi(g)}\big) \int_X f\cdot T_{\varphi(g)}f_c\cdot T_{\psi(g)}f_\psi \, d\mu\\
&\quad= \int_X f\cdot \text{UC - }\lim_{g\in G} \eta_0(\alpha_{\varphi(g)})T_{\varphi(g)}f_c\cdot \eta_0(\alpha_{\psi(g)})T_{\psi(g)}f_\psi \, d\mu.
\end{aligned}$$

From the definition of B_0, if $\alpha_{\varphi(g)} \in B_0$, then $\left\|T_{\varphi(g)}\zeta_i - \zeta_i\right\|_\infty < \frac{\varepsilon}{16N}$ for $i = 1,\ldots,N_1$. Hence, for every $g \in G$, since η_0 is supported on B_0, we have

$$\begin{aligned}
\left\|\eta_0(\alpha_{\varphi(g)})T_{\varphi(g)}f_c - \eta_0(\alpha_{\varphi(g)})f_c\right\|_2 &\leq \left\|\eta_0(\alpha_{\varphi(g)})\left(T_{\varphi(g)}f_c - \sum_{i=1}^{N_1} a_iT_{\varphi(g)}\zeta_i\right)\right\|_2\\
&\quad+\left\|\eta_0(\alpha_{\varphi(g)})\left(\sum_{i=1}^{N_1} a_iT_{\varphi(g)}\zeta_i - \sum_{i=1}^{N_1} a_i\zeta_i\right)\right\|_2\\
&\quad+\left\|\eta_0(\alpha_{\varphi(g)})\left(\sum_{i=1}^{N_1} a_i\zeta_i - f_c\right)\right\|_2\\
&\leq \eta_0(\alpha_{\varphi(g)})\left(\left\|f_c - \sum_{i=1}^{N_1} a_i\zeta_i\right\|_2 + N_1\frac{\varepsilon}{16N} + \left\|f_c - \sum_{i=1}^{N_1} a_i\zeta_i\right\|_2\right)\\
&< \eta_0(\alpha_{\varphi(g)})\left(\frac{\varepsilon}{8} + \frac{\varepsilon}{16} + \frac{\varepsilon}{8}\right) = \frac{5\varepsilon}{16}\eta_0(\alpha_{\varphi(g)}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\int_X f\cdot \eta_0(\alpha_{\varphi(g)})T_{\varphi(g)}f_c\cdot \eta_0(\alpha_{\psi(g)})T_{\psi(g)}f_\psi \, d\mu - \int_X f_c\cdot f\cdot \eta\big(\alpha_{\varphi(g)},\alpha_{\psi(g)}\big)T_{\psi(g)}f_\psi \, d\mu\right|\\
&\qquad= \left|\int_X f\cdot \eta_0(\alpha_{\psi(g)})T_{\psi(g)}f_\psi\cdot \big(\eta_0(\alpha_{\varphi(g)})T_{\varphi(g)}f_c - \eta_0(\alpha_{\varphi(g)})f_c\big) \, d\mu\right|
\end{aligned}$$

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$$\leq \eta_0(\alpha_{\psi(g)})\left\|\eta_0(\alpha_{\varphi(g)})T_{\varphi(g)}f_c - \eta_0(\alpha_{\varphi(g)})f_c\right\|_1$$
$$< \frac{5\varepsilon}{16}\eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big).$$

Taking a Cesàro average, we have the inequality

$$\begin{aligned}\text{UC-}\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big)\ \mu\Big(A \cap T_{\varphi(g)}^{-1}A \cap T_{\psi(g)}^{-1}A\Big) \\ > \int_X f_c \cdot f \cdot \text{UC-}\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big)T_{\psi(g)}f_\psi\, d\mu - \frac{5\varepsilon}{16}.\end{aligned} \tag{12}$$

Now, we estimate the average

$$\text{UC-}\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big)T_{\psi(g)}f_\psi.$$

First, for each $i = 1, \ldots, N_1$, we have

$$\left\|\eta_0(\alpha_{\psi(g)})\big(T_{\psi(g)}(b_i\xi_i) - b_i\xi_i\big)\right\|_\infty = \left\|b_i \cdot \eta_0(\alpha_{\psi(g)})\big(T_{\psi(g)}\xi_i - \xi_i\big)\right\|_\infty < \frac{\varepsilon}{16N}\eta_0(\alpha_{\psi(g)}).$$

Next, let $1 \leq j \leq N_1$. Write $T_{\psi(g)}(b_j\xi_j) = b_j\mu_j(x, \psi(g))\psi_j$. If $\mu_j(x, \cdot) \notin C$, then for any $\chi_1, \chi_2 \in \widehat{Z}$, the character $g \mapsto \chi_1(\alpha_{\varphi(g)})\chi_2(\alpha_{\psi(g)})\mu_j(x, \psi(g))$ is nontrivial, so

$$\text{UC-}\lim_{g\in G} \chi_1(\alpha_{\varphi(g)})\chi_2(\alpha_{\psi(g)})\mu_j(x, \psi(g)) = 0.$$

Hence, by the Stone–Weierstrass theorem,

$$\text{UC-}\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big)\mu_j(x, \psi(g)) = 0.$$

Therefore,

$$\text{UC-}\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big)T_{\psi(g)}f_\psi = \text{UC-}\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big)T_{\psi(g)}\widetilde{f_\psi},$$

where $\widetilde{f_\psi} = E(f|\widetilde{\mathcal{Z}}_\psi(X))$ and $\widetilde{\mathcal{Z}}_\psi(X)$ is the factor generated by $\psi(G)$-eigenfunctions whose eigenvalues come from C. Note that

$$\widetilde{f_\psi} = \sum_{i\in\mathbb{N}} b_i\widetilde{\xi}_i,$$

where

$$\widetilde{\xi}_j(x) = \begin{cases}\xi_j(x), & \mu_j(x, \cdot) \in C; \\ 0, & \mu_j(x, \cdot) \notin C.\end{cases}$$

We note that since C is at most countable, $\widetilde{\chi}_j$ is measurable. Moreover,

$$\widetilde{f_\psi} - \sum_{i=1}^{N_1} b_i\widetilde{\xi}_i = E(f - \sum_{i=1}^{N_1} b_i\xi_i|\widetilde{\mathcal{Z}}_\psi(X)),$$

so

$$\left\|\widetilde{f_\psi} - \sum_{i=1}^{N_1} b_i\widetilde{\xi}_i\right\|_2 < \frac{\varepsilon}{8}.$$

If $x \in E_{j,N}$, then we must have $\mu_j(x,\cdot) \in C_N$. That is, $\mu_j(x,\psi(g)) = \chi_1(\alpha_{\varphi(g)})\chi_2(\alpha_{\psi(g)})$ for some $\chi_1, \chi_2 \in F_N$. Thus,

$$\begin{aligned}\left|\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\left(\mu_j(x,\psi(g)) - 1\right)\right| &= \left|\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\left(\chi_1(\alpha_{\varphi(g)})\chi_2(\alpha_{\psi(g)}) - 1\right)\right| \\ &\leq \left|\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\left(\chi_1(\alpha_{\varphi(g)})\chi_2(\alpha_{\psi(g)}) - \chi_2(\alpha_{\psi(g)})\right)\right| \\ &\quad + \left|\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\left(\chi_2(\alpha_{\psi(g)}) - 1\right)\right| \\ &\leq \eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\left(\frac{\varepsilon}{16N} + \frac{\varepsilon}{16N}\right) = \frac{\varepsilon}{8N}\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\|\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\left(T_{\psi(g)}(b_j\widetilde{\xi}_j) - b_j\widetilde{\xi}_j\right)\right\|_2^2 \\ &\quad = \int_X \left|\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\left(T_{\psi(g)}(b_j\widetilde{\xi}_j) - b_j\widetilde{\xi}_j\right)\right|^2 d\mu \\ &\quad = \int_X \left|b_j(x)\widetilde{\xi}_j(x)\right|^2 \left|\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\left(\mu_j(x,\psi(g)) - 1\right)\right|^2 d\mu(x) \\ &\quad \leq \eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)^2 \left(\int_{E_{j,N}} \left(\frac{\varepsilon}{8N}\right)^2 \left|b_j\widetilde{\xi}_j\right|^2 d\mu + 4\int_{X\setminus E_{j,N}} \left|b_j\widetilde{\xi}_j\right|^2 d\mu\right) \\ &\quad \leq \eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)^2 \left(\left(\frac{\varepsilon}{8N}\right)^2 + 4\left(\frac{\varepsilon}{16N_1}\right)^2\right) \leq 2\left(\frac{\varepsilon}{8N_1}\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\right)^2.\end{aligned}$$

Putting together our estimates, we have

$$\begin{aligned}&\left\|\text{UC - }\lim_{g\in G}\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)T_{\psi(g)}f_\psi - \widetilde{f}_\psi\right\|_2 \\ &\quad = \left\|\text{UC - }\lim_{g\in G}\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)T_{\psi(g)}\widetilde{f}_\psi - \widetilde{f}_\psi\right\|_2 \\ &\quad \leq \left\|\text{UC - }\lim_{g\in G}\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)T_{\psi(g)}\widetilde{f}_\psi - T_{\psi(g)}\sum_{i=1}^{N_1} b_i\widetilde{\xi}_i\right\|_2 \\ &\qquad + \left\|\text{UC - }\lim_{g\in G}\sum_{i=1}^{N_1}\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\left(T_{\psi(g)}(b_i\widetilde{\xi}_i) - b_j\widetilde{\xi}_j\right)\right\|_2 \\ &\qquad + \left\|\sum_{i=1}^{N_1} b_i\widetilde{\xi}_i - \widetilde{f}_\psi\right\|_2 \\ &\quad < \frac{\varepsilon}{8} + N_1\frac{\sqrt{2}\varepsilon}{8N_1} + \frac{\varepsilon}{8} \leq \frac{(2\sqrt{2}+5)\varepsilon}{16} < \frac{\varepsilon}{2}.\end{aligned}$$

Substituting back into (12), we have

$$\begin{aligned}\text{UC - }\lim_{g\in G}\eta\left(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\right)\mu\left(A \cap T_{\varphi(g)}^{-1}A \cap T_{\psi(g)}^{-1}A\right) &> \int_X f_c \cdot f \cdot \widetilde{f}_\psi \, d\mu - \frac{13\varepsilon}{16} \\ &\geq \mu(A)^3 - \frac{13\varepsilon}{16}. \qquad (13)\end{aligned}$$

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Since UC - $\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big) = 1$, it follows that the set

$$\left\{g \in G : \mu\Big(A \cap T^{-1}_{\varphi(g)} A \cap T^{-1}_{\psi(g)} A\Big) > \mu(A)^3 - \varepsilon\right\}$$

is syndetic in G. If not, there exists a Følner sequence $(\Phi_N)_{N\in\mathbb{N}}$, such that $\mu(A \cap T^{-1}_{\varphi(g)} A \cap T^{-1}_{\psi(g)} A) \le \mu(A)^3 - \varepsilon$ for every $g \in \bigcup_{N\in\mathbb{N}} \Phi_N$. But then,

$$\text{UC - }\lim_{g\in G} \eta\big(\alpha_{\varphi(g)}, \alpha_{\psi(g)}\big)\ \mu\Big(A \cap T^{-1}_{\varphi(g)} A \cap T^{-1}_{\psi(g)} A\Big) \le \mu(A)^3 - \varepsilon,$$

which contradicts the inequality in (13).

4. Extensions

As we have observed in Subsection 3.3, the partial characteristic factors obtained in Proposition 3.5 are not the minimal characteristic factors. For example, in Subsection 3.3, we proved that one can replace $\mathcal{Z}_\psi(X)$ with the smaller factor $\widetilde{\mathcal{Z}}_\psi(X)$. In this section, we develop an extension trick that will be used to further simplify the characteristic factors. These results will be useful in the proof of Theorem 1.13, where $\varphi(G)$ is no longer assumed to have finite index in G. In the example below, we illustrate our main result in the simpler case where $\varphi(g) = g$, $\psi(g) = 2g$. The following example is based on Example 3.1.

Example 4.1. Let $G = \bigoplus_{j=1}^\infty \mathbb{Z}/4\mathbb{Z}$, and let $X = \Big(\prod_{j\in\mathbb{N}} C_4\Big) \times C_2 \times C_2$, where the action of $g \in G$ on X is given by

$$T_g(\mathbf{x}, x_\infty, y) = \left((i^{g_j} x_j)_{j\in\mathbb{N}}, x_\infty \cdot \prod_{k=1}^\infty (-1)^{g_k}, y \cdot \prod_{j\in\mathbb{N}} (x_j^{2g_j} \cdot i^{g_j^2 - g_j})\right) \tag{14}$$

for $\mathbf{x} = (x_1, x_2, \dots) \in \prod_{j\in\mathbb{N}} C_4$, $x_\infty \in C_2$ and $y \in C_2$. Note that for $g = (g_1, g_2, \dots) \in G$, only finitely many of the coordinates $g_j \in \mathbb{Z}/4\mathbb{Z}$ are nonzero, so (14) is well defined.

As in Example 3.1, the function $f(\mathbf{x}, x_\infty, y) = y$ is a $2G$-eigenfunction with eigenvalue $2g \mapsto \prod_{j=1}^\infty (-1)^{g_j}$. However, this time, f may have a nontrivial contribution for the average. Indeed, if we let $f_1(\mathbf{x}, x_\infty, y) = x_\infty$, then f_1 is a G-eigenfunction with eigenvalue $g \mapsto \prod_{k=1}^\infty (-1)^{g_k}$ and

$$\text{UC - }\lim_{g\in G} T_g f_1(\mathbf{x}, x_\infty, y) T_{2g} f(\mathbf{x}, x_\infty, y) = x_\infty \cdot y$$

is nonzero. Let $\varphi(g) = g$ and $\psi(g) = 2g$. The above computation shows that f is measurable with respect to $\widetilde{\mathcal{Z}}_\psi$ where $\widetilde{\mathcal{Z}}_\psi$ is defined in Subsection 3.3. As a result, we deduce that $\mathcal{Z}(X) \vee \mathcal{I}_\psi(X) \prec \widetilde{\mathcal{Z}}_\psi(X)$ is a strict inclusion.

Consider the homomorphism $\lambda : G \to S^1$, $\lambda(g) = \prod_{j=1}^\infty i^{g_j}$ and observe that $\lambda(2g) = \prod_{j=1}^\infty (-1)^{g_i}$ is the eigenvalue of f_2. We extend X to a new system $\widetilde{X}$, where λ is an eigenvalue. Let $\widetilde{X} = \Big(\prod_{j\in\mathbb{N}} C_4\Big) \times C_4 \times C_2$, and let the action of $g \in G$ on $\widetilde{X}$ be given by

$$S_g(\mathbf{x}, x_\infty, y) = \left((i^{g_j} x_j)_{j\in\mathbb{N}}, \lambda(g) x_\infty, y \cdot \prod_{j\in\mathbb{N}} (x_j^{2g_j} \cdot i^{g_j^2 - g_j})\right)$$

for $\mathbf{x} = (x_1, x_2, \dots) \in \prod_{j\in\mathbb{N}} C_4$, $x_\infty \in C_4$ and $y \in C_2$. It is easy to see that $\widetilde{\mathbf{X}} = (\widetilde{X}, (S_g)_{g\in G})$ is an extension of $\mathbf{X}$ with respect to the factor map $\pi(\mathbf{x}, x_\infty, y) = (\mathbf{x}, x_\infty^2, y)$. Observe that now the function $h(\mathbf{x}, x_\infty, y) = x_\infty$ on $\widetilde{X}$ is an eigenfunction with eigenvalue λ and we deduce that $h \cdot \overline{f} \circ \pi$ is a $2G$-invariant function on $\widetilde{X}$. This means that $\overline{f} \circ \pi$ is measurable with respect to the σ-algebra $\widetilde{Z}(\widetilde{X}) \vee \mathcal{I}_\psi(\widetilde{X})$. In fact, one can show that now we have an equality $\mathcal{Z}(X) \vee \mathcal{I}_\psi(\widetilde{X}) = \widetilde{\mathcal{Z}}_\psi(\widetilde{X})$.

Definition 4.2. Let G be a countable discrete abelian group, and let $\varphi : G \to G$ be a homomorphism. We say that a character $\chi \in \widehat{G}$ *factors through* φ is $\chi = \lambda \circ \varphi$ for some $\lambda \in \widehat{G}$.

The main result in this section is the following theorem.

Theorem 4.3. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic G-system. Let $\varphi, \psi : G \to G$ be homomorphisms. For any countable set $C \subseteq \widehat{G}$ of characters that factor through φ and ψ, there exists an ergodic extension $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ with the following property: for any $\chi \in C$, there exist G-eigenvalues λ, μ of $\widetilde{\mathbf{X}}$, such that $\lambda(\varphi(g)) = \mu(\psi(g)) = \chi(g)$.*

The fact that $\widetilde{\mathbf{X}}$ in Theorem 4.3 is ergodic will be important in our proof. In preparation for proving that $\widetilde{\mathbf{X}}$ is ergodic, we need the following definition.

Definition 4.4. Let (X, G) be an ergodic system and U a compact abelian group. A *cocycle* is a measurable map $\rho : G \times X \to U$ satisfying $\rho(g + g', x) = \rho(g, x) \cdot \rho(g', T_g x)$ for every $g, g' \in G$ and μ-a.e. $x \in X$. Two cocycles $\rho, \rho' : G \times X \to U$ are said to be *cohomologous* if there exists a measurable map $F : X \to U$, such that $\rho(g, x) \cdot \rho'(g, x)^{-1} = \Delta_g F(x)$ for all $g \in G$ and μ-a.e. $x \in X$. The *image* of ρ, U_ρ, is defined to be the minimal closed subgroup generated by $\{\rho(g, x) : g \in G, x \in X\}$. The cocycle ρ is said to be *minimal* if it is not cohomologous to any cocycle ρ' with $U_{\rho'} \subsetneq U_\rho$.

In [31], Zimmer proved that every cocycle is cohomologous to a minimal cocycle and established the following criterion for ergodicity.

Lemma 4.5 ([31], Corollary 3.8). *Let $\mathbf{X} = \left(X, \mathcal{X}, \mu, (T_g)_{g\in G}\right)$ be an ergodic G-system, U a compact abelian group and $\rho : G \times X \to U$ a cocycle. Then, $\mathbf{X} \times_\rho U$ is ergodic if and only if ρ is minimal and $U = U_\rho$.*

We are now set to prove Theorem 4.3.

Proof of Theorem 4.3. Let $\{\chi_i : i \in \mathbb{N}\}$ be an enumeration of the elements in C. By assumption, for every $i \in \mathbb{N}$, there exist homomorphisms $\chi_i^\varphi, \chi_i^\psi : G \to S^1$, such that $\chi_i^\varphi(\varphi(g)) = \chi_i^\psi(\psi(g)) = \chi_i(g)$. Let $I = \mathbb{N} \times \{\varphi, \psi\}$, and let $\widetilde{\chi} : G \to (S^1)^I$ be the homomorphism whose (i, φ)-coordinate is χ_i^φ and (j, ψ)-coordinate is χ_j^ψ for every $i, j \in \mathbb{N}$. By Zimmer's theory, there exists a minimal cocycle $\rho : G \times X \to (S^1)^I$ which is cohomologous to $\widetilde{\chi}$, where the latter is viewed as a $G \times X \to (S^1)^I$ function that is independent of $x \in X$. This means that there exists a measurable map $F : X \to (S^1)^I$, such that $\rho_g = \widetilde{\chi}(g) \cdot \Delta_g F$. Let V be the image of ρ, then, by Lemma 4.5, $\widetilde{X} = X \times_\rho V$ is ergodic. Now, for every coordinate $t \in I$, consider the projection map $\pi_t : (S^1)^I \to S^1$. By restricting π_t to V, we get a homomorphism $\tau_t : V \to S^1$. Then, the function $\phi_{i,\varphi}(x, v) := \tau_{i,\varphi}(v) \cdot \pi_{i,\varphi} F(x)$ is an eigenfunction with eigenvalue $\Delta_g \phi_{i,\varphi}(x, v) = \chi_i^\varphi(g)$ and $\phi_{j,\psi}(x, v) = \tau_{j,\psi}(v) \cdot \pi_{j,\psi} F(x)$ is an eigenfunction with eigenvalue $\Delta_g \phi_{j,\psi}(x, v) = \chi_j^\psi(g)$. This completes the proof. □

4.1. Characteristic factors related to Theorem 1.13

The goal of this subsection is to prove a stronger version of Propositions 3.5 and 3.8 with smaller characteristic factors. We will use the above extension theorem in order to express these characteristic factors in terms of $\mathcal{Z}_{\varphi,\psi}(X)$ and the invariant σ-algebras, $\mathcal{I}_\varphi(X)$ and $\mathcal{I}_\psi(X)$. Then, using a result of Tao and Ziegler [29] (see Theorem 4.8 below), we will reduce matters further to studying the Conze–Lesigne factor $\mathcal{Z}^2(X)$ with respect to the action of G, which is already well understood for arbitrary countable discrete abelian groups (see [2], [27]).

We start with a lemma.

Lemma 4.6. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic G-system. Let $\mathcal{I}_{\varphi\times\psi}(X \times X)$ denote the σ-algebra of $(T_{\varphi(g)} \times T_{\psi(g)})_{g\in G}$-invariant sets in $X \times X$. Then,*

$$\mathcal{I}_{\varphi\times\psi}(X \times X) \preceq \mathcal{Z}_{\varphi}(X) \times \mathcal{Z}_{\psi}(X).$$

Proof. Let $f_1, f_2 \in L^\infty(X)$ be arbitrary functions and $f(x, y) = f_1(x)f_2(y)$. Then, by the mean ergodic theorem, we have that

$$E(f|\mathcal{I}_{\varphi\times\psi}(X \times X))(x, y) = \text{UC - }\lim_{g\in G} T_{\varphi(g)} f_1(x) \cdot T_{\psi(g)} f_2(y)$$

in $L^2(\mu \times \mu)$. By the van der Corput lemma, $E(f|\mathcal{I}_{\varphi\times\psi}(X \times X)) = 0$ if

$$\text{UC - }\lim_{h\in G}\left|\text{UC - }\lim_{g\in G}\int_{X\times X} T_{\varphi(g+h)} f_1(x) \cdot T_{\psi(g+h)} f_2(y) \cdot \overline{T_{\varphi(g)} f_1(x)} \cdot \overline{T_{\psi(g)} f_2(y)} d(\mu \times \mu)(x, y)\right| = 0.$$

Since $\varphi(G) \times \psi(G)$ is measure-preserving, the above is equal to

$$\text{UC - }\lim_{h\in G}\left(\left|\int_X \Delta_{\varphi(h)} f_1(x) d\mu(x)\right|\right)\left(\left|\int_X \Delta_{\psi(h)} f_2(y) d\mu(y)\right|\right)$$

which by the Cauchy–Schwarz inequality is bounded above by

$$\left(\|f_1\|_{U^2(\varphi(G))} \cdot \|f_2\|_{U^2(\psi(G))}\right)^{1/2}.$$

We deduce that if $E(f|\mathcal{Z}_{\varphi}(X) \times \mathcal{Z}_{\psi}(X)) = 0$, then $E\big(f|\mathcal{I}_{\varphi\times\psi}(X \times X)\big) = 0$. Since linear combinations of functions of the form $f_1 \otimes f_2$ with $f_1, f_2 \in L^\infty(X)$ are dense in $L^\infty(X \times X)$, we deduce that the same holds for every bounded function on $X \times X$, and this completes the proof. $\square$

Using Theorem 4.3, we can now prove the following useful result.

Lemma 4.7. *Let G be a countable discrete abelian group, and let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic G-system. Suppose that $\varphi, \psi : G \to G$ are arbitrary homomorphisms, such that $(\psi - \varphi)(G)$ has finite index in G. Then, there exists an ergodic extension $\pi : \widetilde{X} \to X$, such that*

$$\pi^{-1}(\mathcal{I}_{\varphi\times\psi}(X)) \preceq \left(\mathcal{Z}(\widetilde{X}) \vee \mathcal{I}_{\varphi}(\widetilde{X})\right) \otimes \left(\mathcal{Z}(\widetilde{X}) \vee \mathcal{I}_{\psi}(\widetilde{X})\right).$$

Proof. Let $\{\zeta_i\}_{i\in\mathbb{N}}$ be a relative orthonormal basis of eigenfunctions for $\mathcal{Z}_{\varphi}(X)$ and $\{\xi_i\}_{i\in\mathbb{N}}$ be the same for $\mathcal{Z}_{\psi}(X)$. For every $i, j \in \mathbb{N}$, let $\lambda_i : \varphi(G) \times X \to \mathbb{C}$ and $\mu_j : \psi(G) \times X \to \mathbb{C}$ denote the eigenvalues of ζ_i and ξ_j, respectively. Our goal is to study the functions $f \in L^\infty(X^2)$ which are $(T_{\varphi(g)} \times T_{\psi(g)})_{g\in G}$-invariant. By Lemma 4.6, we can write any such function as

$$f(x, y) = \sum_{i,j\in\mathbb{N}} c_{i,j}(x, y)\zeta_i(x)\overline{\xi_j(y)},$$

where $c_{i,j}$ is a $\varphi(G) \times \psi(G)$-invariant function. Since f is $T_{\varphi(g)} \times T_{\psi(g)}$-invariant, we deduce that

$$c_{i,j}(x, y)\lambda_i(\varphi(g), x)\overline{\mu_j(\psi(g), y)} = c_{i,j}(x, y).$$

Hypothetically, if $c_{i,j}$ was a constant, then unless it is zero (and then can be removed from the summation), the equation above implies that $\lambda_i(\varphi(g), \cdot) = \mu_j(\psi(g), \cdot) = \chi(g)$ for some character $\chi \in \widehat{G}$. In this special case, we can apply Theorem 4.3 in order to find an extension where λ_i and μ_j are eigenvalues. This means that we can express the lift of $\zeta_i \otimes \xi_j$ to $\widetilde{X}$ as a product of a tensor product of G-eigenfunctions (whose eigenvalues are λ_i and μ_j) and a $\varphi(G)\times\psi(G)$-invariant function, which completes the proof in this special case. Below, we generalise the above to arbitrary $c_{i,j}$.

Let $C_{i,j}=\{(x,y)\in X\times X : c_{i,j}(x,y)\neq 0\}$. Then, $\lambda_i(\varphi(g),x)\overline{\mu_j(\psi(g),y)}=1$ for every $(x,y)\in C_{i,j}$ and all $g\in G$. Hence, $g\mapsto\lambda_i(\varphi(g),x)$ and $g\mapsto\mu_j(\psi(g),y)$ are equal to the same character $\chi\in\widehat{G}$ which factors through φ and ψ simultaneously for all $(x,y)\in C_{i,j}$. Now, for every $\chi\in\widehat{G}$, we let

$$J_\chi=\{(i,j)\in\mathbb{N}^2 : (\mu\times\mu)(\{(x,y)\in X\times X : \forall g\ \lambda_i(\varphi(g),x)=\mu_j(\psi(g),y)=\chi(g)\}>0\}$$

and set

$$C:=\{\chi\in\widehat{G} : J_\chi\neq\emptyset\}\text{ and }J:=\bigcup_{\chi\in C}J_\chi.$$

Our first observation is that

$$f(x,y)=\sum_{(i,j)\in J}c_{i,j}(x,y)\zeta_i(x)\xi_j(y). \tag{15}$$

Indeed, if $(i,j)\notin J$, then for every χ, $(i,j)\notin J_\chi$, but then from the computation above $\mu(C_{i,j})=0$ and $c_{i,j}=0$ for $(\mu\times\mu)$-a.e. $(x,y)\in X\times X$.

Claim. *The set C is at most countable.*

Proof of the claim. We use the fact that in a probability space there can be at most countably many disjoint sets of positive measure. Assume by contradiction that C is uncountable. Since there are only countably many $(i,j)\in\mathbb{N}^2$, we deduce that there exists some (i_0,j_0) which belongs to J_χ for all χ in an uncountable subset of $\widehat{G}$. But since the sets

$$\{(x,y)\in X\times X : \forall g\in G,\ \lambda_i(\varphi(g),x)=\mu_j(\psi(g),y)=\chi(g)\}$$

are disjoint for different χ's and of positive measure, we obtain a contradiction. This proves the claim. □

Now, we return to the proof of the lemma. Since C is at most countable, we can apply Theorem 4.3. We see that there exists an ergodic extension $\pi:\widetilde{X}\to X$, such that for every $\chi\in C$, there exist G-eigenvalues $\chi^\varphi,\chi^\psi : G\to S^1$ with $\chi^\varphi(\varphi(g))=\chi(g)$ and $\chi^\psi(\psi(g))=\chi(g)$. Let $m_\chi^\varphi,m_\chi^\psi:\widetilde{X}\to S^1$ be the corresponding eigenfunctions. Now, fix some $(i,j)\in J$, and let $\chi\in C$ be, such that $\lambda_i(\varphi(g),x)=\mu_j(\psi(g),y)=\chi(g)$ whenever $c_{i,j}(x,y)\neq 0$. We deduce that $\left(c_{i,j}\cdot\zeta_i\otimes\xi_j\right)\circ\pi\cdot\overline{m_\chi^\varphi\otimes m_\chi^\psi}$ is a $\varphi(G)\times\psi(G)$-invariant function. Since $c_{i,j}$ is also $\varphi(G)\times\psi(G)$-invariant, we deduce by equation (15) that $f\circ\pi$ is a linear combination of products of eigenfunctions $m_\chi^\varphi\otimes m_\chi^\psi$ and some $\varphi(G)\times\psi(G)$-invariant functions. Equivalently, the lift of f to $\widetilde{X}\times\widetilde{X}$ is measurable with respect to the σ-algebra

$$\left(\mathcal{Z}^1(\widetilde{X})\vee\mathcal{I}_\varphi(\widetilde{X})\right)\otimes\left(\mathcal{Z}^1(\widetilde{X})\vee\mathcal{I}_\psi(\widetilde{X})\right)$$

as required. □

The following result of Tao and Ziegler [29] plays in important role in our work.

Theorem 4.8 ([29], Theorem 1.19). *Let G be a countable discrete abelian group, and let $\mathbf{X}=(X,\mathcal{X},\mu,(T_g)_{g\in G})$ be a G-system. Let H_1,H_2 be two subgroups of G, and denote by H_1+H_2 the subgroup of G generated by H_1 and H_2. Then, for every $d_1,d_2\in\mathbb{N}$, one has*

$$\mathcal{Z}_{H_1}^{d_1}(X)\wedge\mathcal{Z}_{H_2}^{d_2}(X)\leq\mathcal{Z}_{H_1+H_2}^{d_1+d_2}(X).$$

In particular, by setting $d_1=d_2=1$ and using Lemma 3.6, we deduce:

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Lemma 4.9. *Let G be a countable discrete abelian group and $(X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a G-system, and let $\varphi, \psi : G \to G$ be homomorphisms, such that $(\psi-\varphi)(G)$ has finite index in G. Then, $\mathcal{Z}_{\varphi,\psi}(X) \preceq \mathcal{Z}^2_G(X)$.*

We combine this with the results in Section 3 to deduce the following version of Theorem 3.5.

Theorem 4.10. *Let G be a countable discrete abelian group and $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic G-system. Suppose that $\varphi, \psi : G \to G$ are arbitrary homomorphisms, such that $(\psi - \varphi)(G)$ has finite index in G. There exists an ergodic extension $\pi : (\widetilde{X}, \widetilde{\mu}) \to (X, \mu)$, such that for any $f_0, f_1, f_2 \in L^\infty(\mu)$*

$$UC\text{-}\lim_{g\in G} \int_X f_0 \cdot T_{\varphi(g)} f_1 \cdot T_{\psi(g)} f_2 \, d\mu =$$

$$UC\text{-}\lim_{g\in G} \int_{\widetilde{X}} \widetilde{f_0} \cdot T_{\varphi(g)} E(\widetilde{f_1} | \mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_\varphi(\widetilde{X})) \cdot T_{\psi(g)} E(\widetilde{f_2} | \mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_\psi(\widetilde{X})) \, d\widetilde{\mu}$$

in $L^2(\widetilde{X})$, where $\widetilde{f_i} := f_i \circ \pi$ denotes the lift of f_i to the extension $\widetilde{X}$.

Recall that the factors $\mathcal{Z}_\varphi(X)$ and $\mathcal{Z}_\psi(X)$ are relatively independent over $\mathcal{Z}_{\varphi,\psi}(X)$. To put this fact to use, we need to introduce a construction known as a fibre product:

Definition 4.11 (The fibre product over a factor.). For $i = 1, 2$, let $\mathbf{Y}_i = (Y_i, \mathcal{Y}_i, \mu_i, (S^{(i)}_g)_{g\in G})$ be G-systems. Suppose that $\mathbf{Y} = (Y, \mathcal{Y}, \nu, (S_g)_{g\in G})$ is a common factor, and let $\pi_i : Y_i \to Y$, $i = 1, 2$ denote the factor maps. The *fibre product of $\mathbf{Y}_1$ and $\mathbf{Y}_2$ over $\mathbf{Y}$* is the system $\mathbf{Y}_1 \times_{\mathbf{Y}} \mathbf{Y}_2 = \left(Y_1 \times_Y Y_2, \mathcal{Y}_1 \otimes \mathcal{Y}_2, \mu_1 \times_Y \mu_2, (S^{(1)}_g \times S^{(2)}_g)_{g\in G}\right)$, where

$$Y_1 \times_Y Y_2 = \{(y_1, y_2) \in Y_1 \times Y_2 : \pi_1(y_1) = \pi_2(y_2)\}$$

and

$$\mu_1 \times_Y \mu_2 = \int_Y \mu_{1,y} \times \mu_{2,y} d\nu(y),$$

where

$$\mu_i = \int_Y \mu_{i,y} d\nu(y)$$

is the disintegration of the measure μ_i over Y for $i = 1, 2$.

We will use the following result from [31]:

Theorem 4.12. *Let G be a countable discrete abelian group, and let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a G-system. Let $\mathbf{Y}_1 = (Y_1, \mathcal{A}_1, \mu_1, (T^{(1)}_g)_{g\in G})$ and $\mathbf{Y}_2 = (Y_2, \mathcal{A}_2, \mu_2, (T^{(2)}_g)_{g\in G})$ be two factors of X with factor maps $\pi_i : X \to Y_i$ for $i = 1, 2$, and let $\mathbf{Y} = (Y, \nu)$ be their meet. Then, the σ-algebra $\mathcal{A}_1 \vee \mathcal{A}_2$ corresponds to the fibre product $\mathbf{Y}_1 \times_{\mathbf{Y}} \mathbf{Y}_2$.*

Remark 4.13. In particular, Theorem 4.12 implies that $\mathbf{Y}_1 \times_{\mathbf{Y}} \mathbf{Y}_2$ is a factor of $\mathbf{X}$. We note that Zimmer also proved the other direction, namely, that two factors $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are relatively independent over a third factor $\mathcal{Y}$ if and only if the fibre product $\mathbf{Y}_1 \times_{\mathbf{Y}} \mathbf{Y}_2$ is a factor of $\mathbf{X}$ (see [31, Proposition 1.5]).

We also need the following result:

Theorem 4.14 (cf. [23], Proposition 4.6). *Let $\pi : (Y, \mathcal{Y}, \nu, (S_g)_{g\in G}) \to (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a factor map between G-systems, and let $k \geq 1$. Then, $\pi^{-1}(\mathcal{Z}^k(X)) = \mathcal{Z}^k(Y) \wedge \pi^{-1}(\mathcal{X})$.*

Host and Kra [23] proved Theorem 4.14 for $\mathbb{Z}$-actions, but the argument extends easily to arbitrary countable discrete abelian groups.

We now have all the requisite tools to prove Theorem 4.10.

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Proof of Theorem 4.10. By the previous result, we see that if f_0, f_1 or f_2 are orthogonal to functions measurable with respect to the σ-algebra $\mathcal{Z}_\varphi(X) \vee \mathcal{Z}_\psi(X)$, then the averages above are zero. Therefore, by Theorem 4.12, the factor $\mathbf{Z}_\varphi(X) \times_{\mathbf{Z}_{\varphi,\psi}(X)} \mathbf{Z}_\psi(X)$ is a characteristic factor. We may therefore assume without loss of generality that $\mathbf{X} = \mathbf{Z}_\varphi(X) \times_{\mathbf{Z}_{\varphi,\psi}(X)} \mathbf{Z}_\psi(X)$. For the sake of simplicity of notations, we write $\mu_{\varphi,\psi}$ for the measure $\mu_{Z_\varphi(X)} \times_{Z_{\varphi,\psi}(X)} \mu_{Z_\psi(X)}$ on $Z_\varphi(X) \times_{Z_{\varphi,\psi}(X)} Z_\psi(X)$. By linearity, it suffices to prove the theorem in the case where $f_1 = f_1^\varphi \otimes f_1^\psi$ and $f_2 = f_2^\varphi \otimes f_2^\psi$ for some $f_1^\varphi, f_2^\varphi : Z_\varphi(X) \to \mathbb{C}$ and $f_1^\psi, f_2^\psi : Z_\psi(X) \to \mathbb{C}$. Then,

$$\begin{aligned}
&\text{UC - }\lim_{g \in G} \int_X f_0 T_{\varphi(g)} f_1 \cdot T_{\psi(g)} f_2 \, d\mu \\
&\quad = \text{UC - }\lim_{g \in G} \int\limits_{Z_\varphi(X) \times Z_\psi(X)} f_0 \cdot T_{\varphi(g)} \left(f_1^\varphi \otimes f_1^\psi\right) \cdot T_{\psi(g)} \left(f_2^\varphi \otimes f_2^\psi\right) d\mu_{\varphi,\psi}.
\end{aligned} \tag{16}$$

By Proposition 3.5, (16) is equal to

$$\begin{aligned}
\text{UC - }\lim_{g \in G} \int\limits_{Z_\varphi(X) \times Z_\psi(X)} & f_0(x,y) \cdot T_{\varphi(g)} \left(f_1^\varphi \cdot E(f_1^\psi | \mathcal{Z}_{\varphi,\psi}(X))\right)(x) \\
& T_{\psi(g)} \left(E(f_2^\varphi | \mathcal{Z}_{\varphi,\psi}(X)) \cdot f_2^\psi\right)(y) \, d\mu_{\varphi,\psi}(x,y).
\end{aligned} \tag{17}$$

Note that we used the fact that $E(h|\mathcal{Z}_{\varphi,\psi}(X))(x) = E(h|\mathcal{Z}_{\varphi,\psi}(X))(y)$ for $\mu_{\varphi,\psi}$ a.e. x, y. By the mean ergodic theorem, applied to the transformation $T_\varphi \times T_\psi$, the limit in (17) converges to

$$\int\limits_{Z_\varphi(X) \times Z_\psi(X)} f_0 \cdot E\left(\left(f_1^\varphi \cdot E(f_1^\psi | \mathcal{Z}_{\varphi,\psi}(X)) \otimes E(f_2^\varphi | \mathcal{Z}_{\varphi,\psi}(X)) \cdot f_2^\psi\right) \middle| \mathcal{I}_{\varphi \times \psi}(X)\right) d\mu_{\varphi,\psi}.$$

By Lemma 4.7, we can find an ergodic extension $\pi : \widetilde{X} \to X$ (independent of f_0, f_1, f_2), such that $\pi^{-1}(\mathcal{I}_{\varphi \times \psi}(X))$ is a sub-σ-algebra of $\left(\mathcal{Z}(\widetilde{X}) \vee \mathcal{I}_\varphi(\widetilde{X})\right) \otimes \left(\mathcal{Z}(\widetilde{X}) \vee \mathcal{I}_\psi(\widetilde{X})\right)$. Now, by applying the same argument as above with $\widetilde{f_0}, \widetilde{f_1}$ and $\widetilde{f_2}$ instead of f_0, f_1 and f_2, and using Theorem 4.14 in order to replace $\pi^{-1}(\mathcal{Z}_{\varphi,\psi}(X))$ with $\mathcal{Z}_{\varphi,\psi}(\widetilde{X})$, we deduce that:

$$\begin{aligned}
&\text{UC - }\lim_{g \in G} \int_{\widetilde{X}} \widetilde{f_0} \cdot T_{\varphi(g)} \widetilde{f_1} \cdot T_{\psi(g)} \widetilde{f_2} \, d\widetilde{\mu} = \\
&\int_{\widetilde{X}} \widetilde{f_0} \cdot E\left(\left(\widetilde{f_1^\varphi} \cdot E(\widetilde{f_1^\psi} | \mathcal{Z}_{\varphi,\psi}(\widetilde{X})) \otimes E(\widetilde{f_2^\varphi} | \mathcal{Z}_{\varphi,\psi}(\widetilde{X})) \cdot \widetilde{f_2^\psi}\right) \middle| \pi^{-1}(\mathcal{I}_{\varphi \times \psi}(X))\right) d\widetilde{\mu}_{\varphi,\psi},
\end{aligned} \tag{18}$$

where $\widetilde{\mu}_{\varphi,\psi}$ is the lift of $\mu_{\varphi,\psi}$ to $\widetilde{X}$.

We return to the proof of the theorem. By linearity, it is enough to show that if $E(\widetilde{f_1} | \mathcal{Z}_G^2(\widetilde{X}) \vee \mathcal{I}_\varphi(\widetilde{X})) = 0$ or $E(\widetilde{f_2} | \mathcal{Z}_G^2(\widetilde{X}) \vee \mathcal{I}_\psi(\widetilde{X})) = 0$, then (18) is zero. By symmetry and Lemma 4.9, we may assume without loss of generality that $E(\widetilde{f_1} | \mathcal{Z}_{\varphi,\psi}(\widetilde{X}) \vee \mathcal{I}_\varphi(\widetilde{X})) = 0$. Since $\mathcal{Z}_\varphi(\widetilde{X}), \mathcal{Z}_\psi(\widetilde{X})$ are relatively independent over $\mathcal{Z}_{\varphi,\psi}(\widetilde{X})$, they are also relatively independent over the larger σ-algebra $\mathcal{Z}_{\varphi,\psi}(\widetilde{X}) \vee \mathcal{I}_\varphi(\widetilde{X})$. We deduce, by Proposition 2.7, that

$$E(\widetilde{f_1^\varphi} | \mathcal{Z}_{\varphi,\psi}(\widetilde{X}) \vee \mathcal{I}_\varphi(\widetilde{X})) \cdot E(\widetilde{f_1^\psi} | \mathcal{Z}_{\varphi,\psi}(\widetilde{X}) \vee \mathcal{I}_\varphi(\widetilde{X})) = 0. \tag{19}$$

Claim. $E(\widetilde{f_1^\psi} | \mathcal{Z}_{\varphi,\psi}(\widetilde{X}) \vee \mathcal{I}_\varphi(\widetilde{X})) = E(\widetilde{f_1^\psi} | \mathcal{Z}_{\varphi,\psi}(\widetilde{X}))$.

Proof of the claim. $\mathcal{Z}_{\varphi,\psi}(\widetilde{X}) \vee \mathcal{I}_\varphi(\widetilde{X})$ is a factor of $\mathcal{Z}_\varphi(\widetilde{X})$. By Theorem 4.14, $\widetilde{f_1^\psi}$ is measurable with respect to $\mathcal{Z}_\psi(\widetilde{X})$, and this and $\mathcal{Z}_\varphi(\widetilde{X})$ are relatively independent over $\mathcal{Z}_{\varphi,\psi}(\widetilde{X})$, so the claim follows. □

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Equation (19) and the claim imply that

$$\widetilde{f_1}^{\varphi} \cdot E(\widetilde{f_1}^{\psi}|\mathcal{Z}_{\varphi,\psi}(\widetilde{X})) = \left(\widetilde{f_1}^{\varphi} - E(\widetilde{f_1}^{\varphi}|\mathcal{Z}_{\varphi,\psi}(\widetilde{X}) \vee \mathcal{I}_{\varphi}(\widetilde{X}))\right)E(\widetilde{f_1}^{\psi}|Z_{\varphi,\psi}(\widetilde{X}))$$

is orthogonal to all functions measurable with respect to $\mathcal{Z}_{\varphi,\psi}(\widetilde{X}) \vee \mathcal{I}_{\varphi}(\widetilde{X})$, and so it is also orthogonal to those measurable with respect to $\mathcal{Z}(\widetilde{X}) \vee \mathcal{I}_{\varphi}(\widetilde{X})$. Since $\pi^{-1}(\mathcal{I}_{\varphi\times\psi}(X))$ is a sub-σ-algebra of $\left(\mathcal{Z}(\widetilde{X}) \vee \mathcal{I}_{\varphi}(\widetilde{X})\right) \otimes \left(\mathcal{Z}(\widetilde{X}) \vee \mathcal{I}_{\psi}(\widetilde{X})\right)$, this implies that (18) is equal to zero as required. □

As a corollary, we also have the following stronger counterpart of Proposition 3.8.

Corollary 4.15. *In the settings of Theorem 4.10. Let $\eta : Z(\widetilde{X}) \to \mathbb{C}$ be a continuous function and $f_0, f_1, f_2 \in L^\infty(X)$. Let α_g denote the rotation of $g \in G$ on $Z(\widetilde{X})$. If $a, b \in \mathbb{Z}$ are coprime, then*

$$UC\text{ - }\lim_{g\in G} \eta(\alpha_g) \int_{\widetilde{X}} \widetilde{f_0} \cdot T_{ag}\widetilde{f_1} \cdot T_{bg}\widetilde{f_2}\, d\widetilde{\mu} =$$

$$UC\text{ - }\lim_{g\in G} \eta(\alpha_g) \int_{\widetilde{X}} \widetilde{f_0} \cdot T_{ag}E(\widetilde{f_1}|\mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_a(\widetilde{X})) \cdot T_{bg}E(\widetilde{f_2}|\mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_b(\widetilde{X}))\, d\widetilde{\mu},$$

where $\widetilde{f_i} = f_i \circ \pi$ is the lift of f_i to $\widetilde{X}$ for $i = 0, 1, 2$.

Proof. Since η is measurable with respect to $\mathcal{Z}(\widetilde{X})$, it is a linear combination of characters. Therefore, it is enough to prove the equality in the special case where η itself is a character. Then, since a and b are coprime, we can find $t, s \in \mathbb{Z}$, such that $ta + sb = 1$. Set $h_0 = \widetilde{f_0} \cdot \eta^{-(t+s)}$, $h_1 = \widetilde{f_1} \cdot \eta^s$ and $h_2 = \widetilde{f_2} \cdot \eta^t$. Arguing as in Theorem 4.10, we have

$$\begin{aligned}&\text{UC - }\lim_{g\in G} \int_{\widetilde{X}} h_0 \cdot T_{ag}h_1 \cdot T_{bg}h_2\, d\widetilde{\mu} =\\ &\text{UC - }\lim_{g\in G} \int_{\widetilde{X}} h_0 \cdot T_{ag}E(h_1|\mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_a(\widetilde{X})) \cdot T_{bg}E(h_2|\mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_b(\widetilde{X}))\, d\widetilde{\mu}.\end{aligned} \tag{20}$$

Now, since η is measurable with respect to $\mathcal{Z}(\widetilde{X})$, it is also measurable with respect to $\mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_a(\widetilde{X})$ and $\mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_b(\widetilde{X})$, so the claim follows by rewriting h_i in terms of η and $\widetilde{f_i}$ on both sides of equation (20). □

5. A limit formula for $\{ag, bg\}$

Let G be a countable discrete abelian group and $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic G-system. In this section, we restrict ourselves to the homomorphisms $\varphi(g) = ag, \psi(g) = bg$, where $a, b \in \mathbb{Z}$. By Theorem 4.10, we see that it is enough to analyse the ergodic average

$$\text{UC - }\lim_{g\in G} T_{ag}f_1 \cdot T_{bg}f_2 \tag{21}$$

in the case where X is a Conze–Lesigne system (i.e. $X = Z^2(X)$).

Under certain assumptions on a and b, two different (but related) formulas were obtained previously in [2] and in [27] (see Theorems 5.1 and 5.2 below). Neither of the previously obtained formulas is sufficient for our purposes, so we prove a new one in this section.

5.1. *Previous limit formulas*

Assuming all of the subgroups aG, bG, $(a+b)G$ and $(b-a)G$ have finite index in G, a limit formula was obtained in [2] for the multiple ergodic averages in (21) by analysing a Mackey group associated to

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the abelian extension corresponding to the Conze–Lesigne factor (the relevant terminology is defined in the next subsection). For compact groups Z and H, let $\mathcal{M}(Z,H)$ denote the space of measurable functions $f : Z \to H$ equipped with the topology of convergence in measure (with respect to the Haar probability measure).

Theorem 5.1 ([2], Theorem 7.1). *Let G be a countable discrete abelian group. Let $a, b \in \mathbb{Z}$, such that aG, bG, $(a+b)G$ and $(b-a)G$ have finite index in G. Let $k_1' = -ab(a+b)$, $k_2' = ab(a+b)$ and $k_3' = -ab(b-a)$. Set $D = gcd(k_1', k_2', k_3')$ and $k_i = \frac{k_i'}{D}$ for $i = 1, 2, 3$. Let $c_1, c_2, c_3 \in \mathbb{Z}$ so that $\sum_{i=1}^3 k_i c_i = 1$. Let $\boldsymbol{X} = \mathbf{Z} \times_\sigma H$ be as in Theorem 2.5(iii). There is a function $\psi : Z \times Z \to H$, such that $\psi(0,z) = 0$ for every $z \in Z$ and $t \mapsto \psi(t,\cdot)$ is a continuous map from Z to $\mathcal{M}(Z,H)$, and for every $f_1, f_2, f_3 \in L^\infty(\mu)$,*

$$UC\text{-}\lim_{g\in G} f_1(T_{ag}x) f_2(T_{bg}x) f_3(T_{(a+b)g}x) = \int_{Z\times H^2} \prod_{i=1}^{3} f_i(z + a_i t, h + d_i u + a_i^2 v + c_i \psi(t,z)\; du\, dv\, dt,$$

in $L^2(\mu)$, where $x = (z,h) \in Z \times H$, and $a_1 = a, a_2 = b, a_3 = a+b$.

Assuming that $(b-a)$ is even, the last author proved the following result.

Theorem 5.2 ([27], Corollary 6.2). *Let G be a countable discrete abelian group. Let $a, b \in \mathbb{Z}$ be, such that $(b-a)$ is even and $(b-a)G$ has finite index in G. Let $\boldsymbol{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic G-system, such that $\boldsymbol{X} = \mathbf{Z}^2(X)$. Then, there exists an ergodic extension $\pi : Y \to X$ which is isomorphic to a 2-step nilpotent coset system[3] and for every $f_1, f_2, f_3 \in L^\infty(X)$,*

$$UC\text{-}\lim_{g\in G} \widetilde{f_1}(T_{ag}y\Gamma) f_2(T_{bg}y\Gamma) f_3(T_{(a+b)g}y\Gamma) =$$
$$\int_{\mathcal{G}/\Gamma}\int_{\mathcal{G}_2} \widetilde{f_1}(yy_1^a y_2^{\binom{a}{2}}) \widetilde{f_2}(yy_1^b y_2^{\binom{b}{2}}\Gamma) \widetilde{f_3}(yy_1^{a+b} y_2^{\binom{a+b}{2}}\Gamma)\; d\mu_{\mathcal{G}_2}(y_2)\; d\mu_{\mathcal{G}/\Gamma}(y\Gamma).$$

The above formula fails if $b-a$ is odd (see [27, Example 6.3]).

Observe that in the formulas in Theorems 5.1 and 5.2, we can take $f_3 \equiv 1$ and get a limit formula for the averages we are interested in. However, for the sake of our argument, we need a limit formula for every $a, b \in \mathbb{Z}$ regardless of the indices of the subgroups aG, bG and $(a\pm b)G$ and the parity of $b-a$. Below, we remove the finite index assumptions in Theorem 5.1.

5.2. *Mackey group*

Let G be a countable discrete abelian group, and let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic G-system. Suppose that $X = Z^2(X)$, then, by Theorem 2.5, we can write $\mathbf{X} = \mathbf{Z} \times_\sigma H$, where $\mathbf{Z} = (Z, \alpha)$ is the Kronecker factor, H is a compact abelian group and $\sigma : G \times Z \to H$ is a cocycle.

We now define a Mackey group associated to the cocycle σ. Let

$$W = W(a,b) := \{(z + at, z + bt) : z, t \in Z\},$$

and define $S_g w = (w_1 + \alpha_{ag}, w_2 + \alpha_{bg})$ for $g \in G$, $w = (w_1, w_2) \in W$. Let $\widetilde{\sigma}_g(w) := \big(\sigma_{ag}(w_1), \sigma_{bg}(w_2)\big)$. Then the *Mackey group* $M = M(a,b)$ is the closed subgroup of H with annihilator given by

$$M^\perp := \left\{\widetilde{\chi} \in \widehat{H^2} : \widetilde{\chi} \circ \widetilde{\sigma} \text{ is a coboundary over } (W,S)\right\}.$$

[3]The exact definition is given in [27]. We do not use this notion elsewhere in the paper.

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We will show that the Mackey group is a product of subgroups of H. For $c \in \mathbb{Z}$, let $M_c \leq H$ be the closed subgroup with annihilator

$$M_c^{\perp} := \left\{ \chi \in \widehat{H} : (g, z) \mapsto \chi\big(\sigma_{cg}(z)\big) \text{ is a coboundary over } (Z, \alpha) \right\}.$$

Proposition 5.3. *Let $a, b \in \mathbb{Z}$ be coprime, and let $M = M(a, b)$ be the Mackey group. Then $M = M_a \times M_b$.*

The proof of Proposition 5.3 relies heavily on results from [2, Section 7], which we restate here for ease of reference.

5.3. Cocycle identities

The following result gives a convenient characterisation of coboundaries (recall that a cocycle $\rho : G \times Z \to S^1$ is a coboundary if $\rho_g = \Delta_g F$ for some measurable function $F : Z \to S^1$).

Proposition 5.4 ([2], Proposition 7.12). *Let $\mathbf{Z}$ be a Kronecker system and $\rho : G \times Z \to S^1$ a cocycle. The following are equivalent:*

(i) *ρ is a coboundary;*
(ii) *for any sequence $(g_n)_{n \in \mathbb{N}}$ in G with $\alpha_{g_n} \to 0$ in Z, we have $\rho_{g_n}(z) \to 1$ in $L^2(Z)$.*

The next proposition gives three equivalent characterisations of Conze–Lesigne (or quasi-affine) cocycles.

Proposition 5.5 ([2], Proposition 7.15). *Let $\mathbf{Z}$ be an ergodic Kronecker system and $\rho : G \times Z \to S^1$ a cocycle. The following are equivalent:*

(i) *for any sequence $(g_n)_{n \in \mathbb{N}}$ in G with $\alpha_{g_n} \to 0$ in Z, there is a sequence $(\omega_n)_{n \in \mathbb{N}}$ of affine functions, such that $\omega_n \rho_{g_n}(z) \to 1$ in $L^2(Z)$;*
(ii) *for every $t \in Z$,*

$$\frac{\rho_g(z + t)}{\rho_g(z)}$$

is cohomologous to a character;
(iii) *there is a Borel set $A \subseteq Z$ with $m_Z(A) > 0$, such that*

$$\frac{\rho_g(z + t)}{\rho_g(z)}$$

is cohomologous to a character for every $t \in A$.

Lemma 5.6 ([2], Lemma 7.19). *Let $\mathbf{Z}$ be an ergodic Kronecker system and $\rho : G \times Z \to S^1$ a cocycle. Suppose (α_{g_n}) converges (to 0) in Z and $\omega_n(z) = c_n \lambda_n(z)$ are affine functions, such that $(\omega_n \rho_{g_n})$ converges (to 1) in $L^2(Z)$. Then, for every $a \in \mathbb{N}$,*

$$c_n^a \lambda_n\left(\binom{a}{2} \alpha_{g_n}\right) \lambda_n^a(z) \rho_{a g_n}(z)$$

converges (to 1) in $L^2(Z)$.

Lemma 5.7 ([2], Lemma 7.23). *Let $\mathbf{Z} \times_\sigma H$ be an ergodic Conze–Lesigne G-system. Suppose $a \in \mathbb{Z}$ and aG has finite index in G. Then, $aH = H$.*

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Lemma 5.8 ([2], Lemma 7.25). *Let Z be a compact abelian group. Let $c_1, c_2 \in S^1$ and $\lambda_1, \lambda_2 \in \widehat{Z}$. If $\lambda_1 \neq \lambda_2$, then*

$$\|c_1\lambda_1 - c_2\lambda_2\|_{L^2(Z)} = \sqrt{2}.$$

5.4. Proof of Proposition 5.3

We will prove Proposition 5.3 via the next three lemmas. Rather than proving directly that $M = M_a \times M_b$, we will instead show the dual identity $M^\perp = M_a^\perp \times M_b^\perp$. First, we show $M_a^\perp \times M_b^\perp \subseteq M^\perp$:

Lemma 5.9. *In the setup of Proposition 5.3, $M_a^\perp \times M_b^\perp \subseteq M^\perp$.*

Proof. Let $\chi_1 \in M_a^\perp$ and $\chi_2 \in M_b^\perp$. We want to show $\widetilde{\chi} = \chi_1 \otimes \chi_2 \in M^\perp$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in G, such that $(\alpha_{ag_n}, \alpha_{bg_n}) \to 0$ in W. By Proposition 5.4, it suffices to show

$$\widetilde{\chi} \circ \widetilde{\sigma}_{g_n}(w) \to 1 \tag{22}$$

in $L^2(W)$. Now, since a and b are coprime, we have $\alpha_{g_n} \to 0$ in Z. Since $\chi_1 \in M_a^\perp$, it follows that

$$\chi_1\big(\sigma_{ag_n}(z)\big) \to 1 \tag{23}$$

in $L^2(Z)$ by Proposition 5.4. Similarly,

$$\chi_2\big(\sigma_{bg_n}(z)\big) \to 1 \tag{24}$$

in $L^2(Z)$. Combining (23) and (24), we have

$$\chi_1\big(\sigma_{ag_n}(z+at)\big)\chi_2\big(\sigma_{bg_n}(z+bt)\big) \to 1$$

in $L^2(Z \times Z)$. That is, (22) holds. □

Before establishing the reverse inclusion, $M^\perp \subseteq M_a^\perp \times M_b^\perp$, we need the following result:

Lemma 5.10. *In the setup of Proposition 5.3,*

$$M^\perp \subseteq \left\{\chi_1 \otimes \chi_2 \in \widehat{H^2} : \chi_1^a = \chi_2^b = 1\right\}.$$

Proof. Let $\widetilde{\chi} = \chi_1 \otimes \chi_2 \in M^\perp$. By the argument in the proof of [2, Theorem 7.26], we have $\chi_1^a\chi_2^b = \chi_1^{a^2}\chi_2^{b^2} = 1$. Therefore,

$$\chi_1^{a(b-a)} = \chi_1^{ab}\chi_1^{-a^2} = \left(\chi_1^a\chi_2^b\right)^b\left(\chi_1^{a^2}\chi_2^{b^2}\right)^{-1} = 1.$$

By assumption, $(b-a)G$ has finite index in G. It follows that $\widehat{H}$ does not contain any $(b-a)$-torsion elements (see Lemma 5.7), so $\chi_1^a = 1$. We immediately deduce $\chi_2^b = \chi_1^{-a} = 1$ as well. □

Lemma 5.11. *In the setup of Proposition 5.3, $M^\perp \subseteq M_a^\perp \times M_b^\perp$.*

Proof. Let $\widetilde{\chi} = \chi_1 \otimes \chi_2 \in M^\perp$. We want to show $\chi_1 \in M_a^\perp$ and $\chi_2 \in M_b^\perp$. For notational convenience, let $a_1 = a$ and $a_2 = b$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in G, such that $\alpha_{g_n} \to 0$ in Z. By Proposition 5.4, it suffices to show

$$\chi_i\big(\sigma_{a_ig_n}(z)\big) \to 1 \tag{25}$$

in $L^2(Z)$ for $i = 1, 2$.

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Now, $(\alpha_{ag_n}, \alpha_{bg_n}) \to 0$ in W, so

$$\widetilde{\chi} \circ \widetilde{\sigma}_{g_n}(w) \to 1 \tag{26}$$

in $L^2(W)$ by Proposition 5.4. Moreover, since $\chi_i \circ \sigma$ is a Conze–Lesigne cocycle, we have

$$c_{i,n}\lambda_{i,n}(z)\chi_i\big(\sigma_{g_n}(z)\big) \to 1 \tag{27}$$

in $L^2(Z)$ for some sequences $(c_{i,n})_{n\in\mathbb{N}}$ in S^1 and $\big(\lambda_{i,n}\big)_{n\in\mathbb{N}}$ in $\widehat{Z}$ (see Proposition 5.5).

It follows by Lemma 5.6 that

$$c_{i,n}^{a_i}\lambda_{i,n}^{\binom{a_i}{2}}(\alpha_{g_n})\lambda_{i,n}^{a}(z)\chi_i\big(\sigma_{a_ig_n}(z)\big) \to 1 \tag{28}$$

in $L^2(Z)$. On the other hand, by Lemma 5.10, we have $\chi_i^{a_i} = 1$, so raising (27) to the a_i-th power gives

$$c_{i,n}^{a_i}\lambda_{i,n}^{a_i}(z) \to 1$$

in $L^2(Z)$. Hence, by Lemma 5.8, $\lambda_{i,n}^{a_i} = 1$ for all sufficiently large n, and $c_{i,n}^{a_i} \to 1$. Therefore, (28) simplifies to

$$d_{i,n}\chi_i\big(\sigma_{a_ig_n}(z)\big) \to 1 \tag{29}$$

in $L^2(Z)$, where $d_{i,n} = \lambda_{i,n}^{\binom{a_i}{2}}(\alpha_{g_n})$.

The numbers a and b are coprime, so at least one of them is odd. Without loss of generality, assume a is odd. Then a divides $\binom{a}{2}$, so $\lambda_{1,n}^{\binom{a}{2}} = 1$. Hence, $d_{1,n} = 1$ for all large n, so (25) follows from (29) for $i = 1$. It remains to show if (25) holds for $i = 2$.

Combining the identities in (29) for $i = 1, 2$ and using $d_{1,n} = 1$, we have

$$d_{2,n}\chi_1\big(\sigma_{ag_n}(z + at)\big)\chi_2\big(\sigma_{bg_n}(z + bt)\big) \to 1$$

in $L^2(Z \times Z)$. That is,

$$d_{2,n}\widetilde{\chi} \circ \widetilde{\sigma}_{g_n}(w) \to 1$$

in $L^2(W)$. Comparing with (26), this implies $d_{2,n} \to 1$. Therefore, (25) follows from (29) for $i = 2$. □

Proposition 5.3 follows immediately from Lemmas 5.9 and 5.11.

5.5. *Limit formula*

With the help of Proposition 5.3, we will now prove a limit formula for the averages $\text{UC-}\lim_{g\in G} T_{ag}f_1T_{bg}f_2$. We need to define one more object related to the cocycle σ before stating the limit formula. For a compact space K, let $\mathcal{M}(Z, K)$ denote the space of measurable functions $Z \to K$ equipped with the topology of convergence in measure.

Proposition 5.12. *Let $X = \mathbf{Z} \times_\sigma H$ be an ergodic Conze–Lesigne system. Let $c \in \mathbb{Z}$. There exists a function $\psi_c : Z \times Z \to H/M_c$, such that*

(1) *for every $g \in G$,*

$$\psi_c(\alpha_g, z) \equiv \sigma_{cg}(z) (\text{mod } M_c),$$

and

(2) *the map $Z \ni t \mapsto \psi_c(t, \cdot) \in \mathcal{M}(Z, H/M_c)$ is continuous.*

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In order to prove Proposition 5.12, we use the following characterisation of convergence in measure:

Lemma 5.13 ([2], 7.28). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{M}(Z,H)$. Then $f_n \to f$ in $\mathcal{M}(Z,H)$ if and only if $\chi\circ f_n \to \chi\circ f$ in $L^2(Z)$ for every character $\chi\in\widehat{H}$.*

Proof of Proposition 5.12. Given a sequence $(g_n)_{n\in\mathbb{N}}$ in G, such that $(\alpha_{g_n})_{n\in\mathbb{N}}$ is convergent in Z, we want to show that the sequence

$$\big(\sigma_{cg_n}(z)\big)_{n\in\mathbb{N}}$$

converges in $\mathcal{M}(Z,H/M_c)$. Equivalently, by Lemma 5.13, we must show that

$$\big(\chi\big(\sigma_{cg_n}(z)\big)\big)_{n\in\mathbb{N}}$$

converges in $L^2(Z)$ for every $\chi\in\widehat{H/M_c}=M_c^{\perp}$.

Let $\chi\in M_a^{\perp}$. By the definition of M_c, the cocycle $\chi\big(\sigma_{cg}(z)\big)$ is a coboundary over (Z,α). Hence, by Proposition 5.4, there is a continuous map $t\mapsto\varphi(t,\cdot)\in L^2(Z)$, such that $\varphi(\alpha_g,z)=\chi\big(\sigma_{cg}(z)\big)$. Therefore,

$$\chi\big(\sigma_{cg_n}(z)\big)\to\varphi(t,z)$$

in $L^2(Z)$, where $t=\lim_{n\to\infty}\alpha_{g_n}\in Z$. □

By the Kuratowski and Ryll-Nardzewski measurable selection theorem (see [28, Section 5.2]), there exists a measurable map $\iota_a: H/M_a\to H$, such that $\pi_a(\iota_a(x))=x$, where π_a is the canonical projection $\pi_a: H\to H/M_a$. Let $\psi_1=\iota_a\circ\psi_a$ and $\psi_2=\iota_b\circ\psi_b$. We can now state and prove a general limit formula for Conze–Lesigne systems:

Theorem 5.14. *Let $X=\mathbf{Z}\times_{\sigma}H$ be an ergodic Conze–Lesigne system. Let $a,b\in Z$. Let $M=M(a,b)=M_a\times M_b$. Then for any $f_1,f_2\in L^{\infty}(\mu)$, we have*

$$\begin{aligned}
&UC\text{-}\lim_{g\in G} f_1(T_{ag}(z,x))f_2(T_{bg}(z,x))\\
&\qquad=\int_{Z\times M_a\times M_b} f_1(z+at,x+u+\psi_1(t,z))f_2(z+bt,x+v+\psi_2(t,z))\,dt\,du\,dv
\end{aligned}\tag{30}$$

in $L^2(Z\times H)$.

Remark 5.15. We have defined the functions ψ_i by lifting ψ_a and ψ_b to the group H from H/M_a and H/M_b respectively. If ψ_1' is another functions with $\pi_a(\psi_1')=\psi_a$, then for any $t,z\in Z$, we have $\psi_1'(t,z)-\psi_1(t,z)\in M_a$. Since the Haar measure on M_a is invariant under shifts coming from M_a, the expression on the right-hand side of (30) is unchanged when ψ_1 is replaced by ψ_1'. The same is true for replacing ψ_2 by ψ_2', so it does not matter which lifts of ψ_a and ψ_b we choose.

Proof. For notational convenience, let $\psi=(\psi_1,\psi_2): Z\times Z\to H^2$, and let m_M denote the Haar measure on the Mackey group $M=M_a\times M_b$.

It suffices to prove the formula in (30) for functions of the form $f_i(z,x)=\omega_i(z)\chi_i(x)$ with $\omega_i\in L^{\infty}(Z)$ and $\chi_i\in\widehat{H}$. In this case, the right-hand side of (30) is equal to

$$\int_Z \omega_1(z+at)\omega_2(z+bt)\chi_1(x)\chi_2(x)\widetilde{\chi}(\psi(t,z))\,dt\int_M\widetilde{\chi}\,dm_M,$$

where $\widetilde{\chi}=\chi_1\otimes\chi_2\in\widehat{H^2}$.

We now consider two cases. First, if $\widetilde{\chi} \notin M^{\perp}$, then $\int_M \widetilde{\chi}\, dm_M = 0$, so the right-hand side of (30) is equal to zero. Moreover, for every $\lambda \in M^{\perp}$ and almost every $z, t \in Z$, we have

$$\int_{H^2} f_1(z+at, x) f_2(z+bt, y)\lambda(x,y)\, dx\, dy = \omega_1(z+at)\omega_2(z+bt) \int_{H^2} \widetilde{\chi}(x,y)\lambda(x,y) = 0.$$

Therefore, the left-hand side of (30) is also zero (see [2, Proposition 7.10]).

Now suppose $\widetilde{\chi} \in M^{\perp}$ so that $\int_M \widetilde{\chi}\, dm_M = 1$. For $g \in G$ and $(z,x) \in Z \times H$, we can write

$$f_1(T_{ag}(z,x)) f_2(T_{bg}(z,x)) = \omega_1(z+\alpha_{ag})\omega_2(z+\alpha_{bg})\chi_1(x)\chi_2(x)\widetilde{\chi}(\sigma_{ag}(z), \sigma_{bg}(z)).$$

Thus, letting

$$\varphi_t(z,x) := \omega_1(z+at)\omega_2(z+bt)\chi_1(x)\chi_2(x)\widetilde{\chi}(\psi(t,z)),$$

we have

$$f_1(T_{ag}(z,x)) f_2(T_{bg}(z,x)) = \varphi_{\alpha_g}(z,x).$$

Since $\widetilde{\chi}$ annihilates the Mackey group M, we see by Proposition 5.12(ii) that $Z \ni t \mapsto \widetilde{\chi}(\psi(t,\cdot)) \in L^2(Z)$ is continuous, and so $Z \ni t \mapsto \varphi_t \in L^2(Z \times H)$ is also continuous. Therefore, for any $\xi \in L^2(Z \times H)$, since the system (Z, α) is uniquely ergodic, we have

$$\text{UC - }\lim_{g \in G} \langle \varphi_{\alpha_g}, \xi \rangle = \int_Z \langle \varphi_t, \xi \rangle\, dt.$$

That is

$$\text{UC - }\lim_{g \in G} \varphi_{\alpha_g}(z,x) = \int_Z \varphi_t(z,x)\, dt \tag{31}$$

weakly in $L^2(Z \times H)$. By more general results on norm convergence on multiple ergodic averages (see [3, 32]), it follows that (31) holds strongly. The right-hand side of (30) is also equal to $\int_Z \varphi_t(z,x)\, dt$, so the formula in (30) holds when $\widetilde{\chi} \in M^{\perp}$. □

5.6. *Proof of Theorem 1.13*

We first prove the theorem in the special case where a and b are coprime.

Let $f = \mathbf{1}_A$. By Theorem 4.15, there is an extension $\widetilde{\mathbf{X}}$ of $\mathbf{X}$, such that

$$\begin{aligned}
&\text{UC - }\lim_{g \in G} \eta(\alpha_g) \int_{\widetilde{X}} \widetilde{f} \cdot T_{ag}\widetilde{f} \cdot T_{bg}\widetilde{f}\, d\widetilde{\mu} \\
&\quad = \text{UC - }\lim_{g \in G} \eta(\alpha_g) \int_{\widetilde{X}} \widetilde{f} \cdot T_{ag} E(\widetilde{f} | \mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_a(\widetilde{X})) \cdot T_{ag} E(\widetilde{f} | \mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_b(\widetilde{X}))\, d\widetilde{\mu},
\end{aligned}$$

where $\widetilde{f}$ is the lift of f to $\widetilde{X}$. For notational convenience, let $\widetilde{f_a} := E(\widetilde{f} | \mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_a(\widetilde{X}))$ and $\widetilde{f_b} := E(\widetilde{f} | \mathcal{Z}^2_G(\widetilde{X}) \vee \mathcal{I}_b(\widetilde{X}))$. We can therefore write

$$\begin{aligned}
\widetilde{f_a} &= \sum_{i \in \mathbb{N}} c_i h_i, \\
\widetilde{f_b} &= \sum_{j \in \mathbb{N}} d_j k_j,
\end{aligned}$$

where each c_i is aG-invariant, d_j is bG-invariant and h_i, k_j are $\mathcal{Z}^2_G(\widetilde{X})$-measurable. By Theorem 2.5(iii), we can write $\mathbf{Z}^2_G(\widetilde{X}) = \widetilde{\mathbf{Z}} \times_\sigma H$. Then, by Theorem 5.14,

$$\begin{aligned}
&\mathrm{UC}\text{-}\lim_{g\in G} \eta(\alpha_g)\, \mu\Big(A\cap T_{ag}^{-1}A\cap T_{bg}^{-1}A\Big)\\
&= \mathrm{UC}\text{-}\lim_{g\in G} \eta(\alpha_g)\, \int_{\widetilde{X}} \widetilde{f}\cdot T_{ag}\widetilde{f_a}\cdot T_{ag}\widetilde{f_b}\, d\widetilde{\mu}\\
&= \sum_{i,j\in\mathbb{N}} \int_{\widetilde{X}} c_i d_j \widetilde{f}\cdot \mathrm{UC}\text{-}\lim_{g\in G} \eta(\alpha_g)\, T_{ag}h_i\cdot T_{bg}k_j\, d\widetilde{\mu}\\
&= \sum_{i,j\in\mathbb{N}} \int_{\widetilde{X}\times Z\times M_a\times M_b} c_i(x)d_j(x)\widetilde{f}(x)\eta(t)h_i(\pi_Z(x)+at, \pi_H(x)+u+\psi_1(t,z))\\
&\qquad\qquad k_j(\pi_Z(x)+bt, \pi_H(x)+v+\psi_2(t,z))\, d\widetilde{\mu}(x)\, dt\, du\, dv,
\end{aligned}$$

where $(\pi_Z(x), \pi_H(x))\in Z\times H$ is the projection of $x\in\widetilde{X}$ onto the Conze–Lesigne factor $Z\times H$. By choosing $\eta: Z\to[0,\infty)$ concentrated on a small neighborhood of 0 (as in the proof of Theorem 1.11; see Subsection 3.3), it remains to show the inequality:

$$\sum_{i,j} \int_{\widetilde{X}\times M_a\times M_b} c_i(x)d_j(x)\widetilde{f}(x)h_i(\pi_Z(x), \pi_H(x)+u)k_j(\pi_Z(x), \pi_H(x)+v)\, d\widetilde{\mu}(x)\, du\, dv \geq \mu(A)^3. \tag{32}$$

Let $\mathcal{W}_1$ be the σ-algebra generated by functions $f\in L^\infty(Z\times H)$, such that $f(z,x+y)=f(z,x)$ for every $y\in M_a$. Similarly, let $\mathcal{W}_2$ be the σ-algebra generated by functions $f\in L^\infty(Z\times H)$, such that $f(z,x+y)=f(z,x)$ for every $y\in M_b$. Then the left-hand side of (32) is equal to

$$\int_{\widetilde{X}} \widetilde{f}\cdot E(\widetilde{f}|\mathcal{W}_1\vee\mathcal{I}_a)\cdot E(\widetilde{f}|\mathcal{W}_2\vee\mathcal{I}_b)\, d\widetilde{\mu}. \tag{33}$$

By [13, Lemma 1.6], the quantity in (33) is bounded below by $\left(\int_{\widetilde{X}}\widetilde{f}\, d\widetilde{\mu}\right)^3=\mu(A)^3$, so (32) holds.

Now suppose $a,b\in\mathbb{Z}$ are arbitrary integers and write $a=a'\cdot d$ and $b=b'\cdot d$, where $d=\gcd(a,b)$ and a',b' are coprime. Since $(b-a)G$ has finite index in G, we deduce that so does dG. Therefore, we can find finitely many ergodic dG-invariant measures $\{\mu_i\}_{i=1}^{l}$, such that $\mu=\frac{1}{l}\sum_{i=1}^{l}\mu_i$ and all of the systems $\mathbf{X}_i=(X,\mathcal{X},\mu_i,dG)$ admit the same Kronecker factor. By the argument above, we can find a suitable η satisfying:

$$\mathrm{UC}\text{-}\lim_{g\in dG} \eta(\alpha_g)\mu_i(A\cap T_{a'g}^{-1}A\cap T_{b'g}^{-1}A) > \mu_i(A)^3-\varepsilon$$

for all $i=1,\dots,l$, and $\mathrm{UC}\text{-}\lim_{g\in dG}\eta(\alpha_g)=1$. Therefore, by Jensen's inequality, we have

$$\mathrm{UC}\text{-}\lim_{g\in dG} \eta(\alpha_g)\mu(A\cap T_{a'g}^{-1}A\cap T_{b'g}^{-1}A) > \mu(A)^3-\varepsilon.$$

As in the proof of Theorem 1.11, we conclude that

$$\{g\in dG : \mu(A\cap T_{a'g}^{-1}A\cap T_{b'g}^{-1}A) > \mu(A)^3-\varepsilon\}$$

is syndetic. Since dG has finite index in G, this implies that

$$\{g\in G : \mu(A\cap T_{ag}^{-1}A\cap T_{bg}^{-1}A) > \mu(A)^3-\varepsilon\}$$

is syndetic, as required. □

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6. Proof of Theorem 1.14

In this section, we prove Theorem 1.14, restated here for the convenience of the reader:

Theorem 6.1 (Theorem 1.14). *Let $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}$. Let $l \in \mathbb{N}$. There exists $P = P(l)$, such that, for any $a, b \in \mathbb{N}$ with $p \mid \gcd(a, b)$ for some prime $p \geq P$, there is an ergodic G-system $\left(X, \mathcal{X}, \mu, (T_g)_{g\in G}\right)$ and a set $A \in \mathcal{X}$ with $\mu(A) > 0$, such that*

$$\mu(A \cap T_{ag}^{-1} A \cap T_{bg}^{-1} A) \leq \mu(A)^l$$

for every $g \neq 0$.

Rather than constructing a $\bigoplus_{n=1}^{\infty} \mathbb{Z}$-system directly, we will instead construct a $\bigoplus_{n=1}^{\infty} \mathbb{Z}/p^2\mathbb{Z}$-system. Since $\bigoplus_{n=1}^{\infty} \mathbb{Z}/p^2\mathbb{Z}$ is a quotient of $\bigoplus_{n=1}^{\infty} \mathbb{Z}$, the system we construct can be lifted to an ergodic $\bigoplus_{n=1}^{\infty} \mathbb{Z}$-system. Hence, Theorem 1.14 follows from:

Theorem 6.2. *For any $a, b, l \in \mathbb{N}$, there exists a prime p (sufficiently large), an ergodic $\bigoplus_{n=1}^{\infty} \mathbb{Z}/p^2\mathbb{Z}$-system $\boldsymbol{X} = \left(X, \mathcal{X}, \mu, (T_g)_{g\in\bigoplus_{n=1}^{\infty}\mathbb{Z}/p^2\mathbb{Z}}\right)$ and a set $A \in \mathcal{X}$ with $\mu(A) > 0$, such that*

$$\mu(A \cap T_{pag}^{-1} A \cap T_{pbg}^{-1} A) \leq \mu(A)^l$$

for every $g \neq 0$.

The proof of Theorem 6.2 is based on the following result of Behrend [4].

Theorem 6.3. *Let $a, b \in \mathbb{N}$ be distinct and nonzero. There is an absolute constant $c > 0$, such that: for every $N \in \mathbb{N}$, there is a subset $B \subseteq \{0, 1, \ldots, N-1\}$, such that $|B| > N \cdot e^{-c\sqrt{\log(N)}}$ and B contains no configurations of the form $\{n, n + am, n + bm\}$ for $m \neq 0$.*

For every prime number p, let $C_p = \{z \in \mathbb{C} : z^p = 1\}$ denote the group of all roots of unity of order p and let $\omega_p = e^{2\pi i/p}$ be the first p-th root of unity in $\mathbb{C}$. The following is an immediate corollary of Behrend's theorem.

Lemma 6.4. *Let $a, b \in \mathbb{N}$ be distinct, then, for every l, there exists a sufficiently large prime p and a subset $B \subseteq C_p$ of size $|B| > p^{1-\frac{1}{l-1}}$ which contains no configurations of the form $\{y, y \cdot x^a, y \cdot x^b\}$ for $x \neq 1$.*

Throughout this section, we let $\mathcal{T}_p := C_p^{\mathbb{N}}$ and $G_p := \bigoplus_{i\in I} \mathbb{Z}/p\mathbb{Z}$.

We start by giving a proof that the large intersection property fails for nonergodic systems.

Lemma 6.5. *Let $a, b \in \mathbb{Z}$ be distinct and nonzero. For every $L \in \mathbb{N}$, there is a $P = P(L)$, such that for every prime $p \geq P$, there is a G_p-system $(X, \mathcal{X}, \mu, (T_g)_{g\in G_p})$, such that, for every $l \leq L$, there is a measurable set $A = A(l)$ with $\mu(A) > 0$ and*

$$\mu(A \cap T_{ag} A \cap T_{bg} A) \leq \mu(A)^l$$

for every $g \neq 0$.

This result was previously established in [2, Proposition 10.11], but we give a different proof that will be useful later on.

Proof. Let p be a prime number, and let $X_p = \mathcal{T}_p \times C_p$. We equip X_p with the Borel σ-algebra, the Haar measure μ and the action of G_p by

$$T_g(x, u) = (x, \prod_{i=1}^{\infty} x_i^{g_i} u).$$

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Now, fix a subset $B \subseteq C_p$ which avoids configurations of the form $\{y, y \cdot x^a, y \cdot x^b\}$ whenever $x \neq 1$, and let $A = \mathcal{T}_p \times B$. It is easy to see that $\mu(A) = \frac{|B|}{p}$ and we have

$$\mu(A \cap T_{ag}A \cap T_{bg}A) = \int_{\mathcal{T}_p^2} 1_B(y) 1_B\left(y \prod_{i \in I} x^{ag_i}\right) 1_B\left(y \prod_{i \in I} x^{bg_i}\right) dxdy =$$

$$\int_{\mathcal{T}_p^2} 1_B(y) 1_B\left(y \cdot \left(\prod_{\{i \,:\, g_i \neq 0\}} x_i\right)^a\right) 1_B\left(y \cdot \left(\prod_{\{i \,:\, g_i \neq 0\}} x_i\right)^b\right) dxdy =$$

$$\mu_{\mathcal{T}_p^2}\left(\left\{(y,x) \in \mathcal{T}_p^2 : \left\{y, y \cdot \left(\prod_{\{i \,:\, g_i \neq 0\}} x_i\right)^a, y \cdot \left(\prod_{\{i \,:\, g_i \neq 0\}} x_i\right)^b\right\} \subset B\right\}\right).$$

But, $\left\{y, y \cdot \left(\prod_{\{i \,:\, g_i \neq 0\}} x_i\right)^a, y \cdot \left(\prod_{\{i \,:\, g_i \neq 0\}} x_i\right)^b\right\} \subset B$ if and only if $\prod_{\{i \,:\, g_i \neq 0\}} x_i = 1$. Since $g \neq 0$, we deduce that $\mu(A \cap T_{ag}A \cap T_{bg}A) = \frac{|B|}{p^2} = \frac{p^{l-2}}{|B|^{l-1}} \mu(A)^l$. Now, choose P sufficiently large for which there exists a set B with $|B| > p^{1-\frac{1}{l-1}}$ (Lemma 6.4). Then, $\mu(A \cap T_{ag}A \cap T_{bg}A) < \mu(A)^l$ as required. □

Roughly speaking, the idea in this section is to construct an ergodic p-th root for the system above.

We fix some P sufficiently large as in Lemma 6.5, and let $p > P$ be a prime number. For convenience of notations, we let $\omega = e^{2\pi i/p}$ and $\eta = e^{2\pi i/p^2}$. We define an action of $G = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/p^2\mathbb{Z}$ on $\mathcal{T}$ by setting $S_g x = \zeta(g)x$, where $\zeta(g) = (\eta^{pg_i})_{i \in \mathbb{N}} = (\omega^{g_i})_{i \in \mathbb{N}}$. Since the image of ζ is dense in $\mathcal{T}$, the action is ergodic.

Now, we extend this action to the product space $X = \mathcal{T} \times C_{p^2}$. Let $\varphi : C_p \to C_{p^2}$ be the map

$$\varphi(e^{\frac{2\pi i x}{p}}) = e^{\frac{2\pi i |x|_p}{p^2}},$$

where $|x|_p = x \bmod p$. Then, φ is a cross-section of the canonical projection $C_{p^2} \to C_p$, and we have that $\varphi(x)^p = x$, and $\varphi(\omega) = \eta$. Our goal is to define an action $(T_g)_{g \in G}$ on X, such that $T_{pg}(t,u) = (t, \prod_{i \in \mathbb{N}} t_i^{pg_i} \cdot u)$.

We do so in two steps. We define an action T'_g on X which satisfies that $T'_{e_i}(t,u) = (S_{e_i}t, \varphi(t_i)u)$, for every $i \in \mathbb{N}$, where $e_i \in \bigoplus_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$ is the i-th unit vector. Writing $g = \sum_{i \in \mathbb{N}} g_i e_i$ and using the group law, we get the following action:

$$T'_g(t,u) = \left(S_g t, \prod_{j=1}^{\infty} \prod_{k=0}^{g_j - 1} \varphi(\omega^k t_j) \cdot u\right), \tag{34}$$

where an empty product $\prod_{k=0}^{-1} x_k$ is equal to 1.

Unfortunately, this action is not what we are looking for. Indeed,

$$(T'_{e_j})^p(t,u) = (t, \prod_{k=0}^{p-1} \varphi(\omega^k \cdot t_j) u) = (t, t_j \cdot \eta^{\binom{p}{2}} \cdot u).$$

To fix that, we let $\xi = \omega^{\frac{1-p}{2}}$ be a p-th root of $\overline{\eta}^{\binom{p}{2}}$ and change the action accordingly:

$$T_g(t,u) = \left(S_g t, \prod_{j=1}^{\infty} \left(\prod_{k=0}^{g_j - 1} \varphi(\omega^k t_j) \cdot \xi^{g_j}\right) \cdot u\right). \tag{35}$$

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Lemma 6.6. *For every $t \in \mathcal{T}$, $u \in C_{p^2}$ and $g \in G$, we have*

$$T_{pg}(t,u) = (t, t^{pg}u). \tag{36}$$

Proof. The proof is a direct computation. Indeed, it suffices to prove that (36) holds for $g = e_j$ for every $j \in \mathbb{N}$. Let $j \in \mathbb{N}$ be arbitrary. Since ω is of order p, $S_{pg}t = t$. As for the second coordinate, observe that

$$\prod_{k=0}^{p-1} \varphi(\omega^k t_j) \cdot \xi^p = \xi^p \cdot \eta^{\binom{p}{2}} \cdot t_j = t_j.$$

The first equality follows because the product is independent on t_j and always equals to $\varphi(\omega) \cdot \ldots \cdot \varphi(\omega^{p-1}) = \eta^{\binom{p}{2}}$, and the last equality follows from the definition of ξ. This completes the proof of the lemma. □

The main difficulty in the proof is showing that this action is ergodic.

Lemma 6.7. *The action in (35) on X is ergodic.*

Proof. We use Zimmer's criterion for ergodicity [31, Lemma 4.5]. Since the action of G on $\mathcal{T}$ is ergodic, it is enough to show that the cocycle $\sigma : G \times \mathcal{T} \to C_{p^2}, \sigma(g,t) = \prod_{i=1}^{\infty} \prod_{k=0}^{g_j-1} \varphi(\omega^k t_j)$ is minimal. Since C_p is the largest proper subgroup of C_{p^2}, it is enough to show that σ is not cohomologous to a cocycle taking values in C_p. Suppose, by contradiction, that there exists a cocycle $\tau : \mathcal{T} \to C_p$ cohomologous to σ. Since $\tau^p = 1$, we deduce that $\sigma(g,t)^p = \prod_{i=1}^{\infty} \omega^{\binom{g_i}{2}} t_i^{g_i} \xi^{pg_i}$ is a coboundary. Therefore, there exists $F : \mathcal{T} \to S^1$, such that

$$\sigma^p(g,t) = \frac{F(S_g t)}{F(t)} \tag{37}$$

for every $g \in G$ and $t \in T$. Observe that for every $g, h \in G$, $\Delta_h \sigma^p(g,t)$ is a constant in t. Therefore, by (37), $\Delta_{h_1}\Delta_{h_2}F$ is a constant for every $h_1, h_2 \in G$. Let $s \in \mathcal{T}$ and define $\Delta_s F(x) = \frac{F(sx)}{F(x)}$. We claim that $\Delta_s F(x)$ is an eigenfunction. Let $g_1, g_2 \in G$, then $\Delta_{g_1}\Delta_{g_2}\Delta_s F(x) = \Delta_s \Delta_{g_1}\Delta_{g_2} F(x) = 1$. Hence, by ergodicity, $\Delta_{g_2}\Delta_s F$ is constant and $\Delta_s F$ is an eigenfunction for every $s \in Z$. Recall that translations by $s \in Z$ are continuous with respect to the L^2-norm. In particular, there exists an open subgroup $U \le \mathcal{T}$, such that

$$\|\Delta_s F - 1\|_{L^2(\mu_{\mathcal{T}})} < \sqrt{2} \tag{38}$$

for all $s \in U$. By ergodicity, the multiplicity of each eigenvalue is 1. Since eigenfunctions with different eigenvalues are orthogonal, it follows that $\Delta_s F$ is a constant for all $s \in U$. Otherwise, $\Delta_s F$ is orthogonal to 1, and then

$$\|\Delta_s F - 1\|^2_{L^2(\mu_{\mathcal{T}})} = \|\Delta_s F\|^2_{L^2} + \|1\|^2_{L^2} = 2$$

which contradicts (38). Now, choose $g \in G$, such that $\prod_{i=1}^{\infty} \omega^{g_i} \in U$ (such g must exist by density). Then, if we take $s = \omega^g$, equation (37) implies that $\sigma^p(g,\cdot)$ is a constant. As $\sigma^p(g,t)$ clearly depends on t, this is a contradiction. □

We now complete the proof of Theorem 6.2. Let $B \subseteq C_p$ be as in Lemma 6.4. Let $\pi : C_{p^2} \to C_p$ be the map $\pi_i(x) = x_1^p$, and let $A = \mathcal{T} \times \widetilde{B}$, where $\widetilde{B} = \pi^{-1}(B)$. Then, $\mu_X(A) = \frac{|B|}{p}$, and, as in the proof of Lemma 6.5,

$$\mu_X(A \cap T_{apg}A \cap T_{bpg}A) = \frac{|B|}{p^2} = \frac{p^{l-2}}{|B|^{l-1}} \mu_X(A)^l < \mu_X(A)^l.$$

This completes the proof. □

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7. 3-point configurations in $\mathbb{Z}^2$

In this section, we establish ergodic popular difference densities for all 3-point matrix patterns in $\mathbb{Z}^2$. The results are summarised in Table 1 in the Introduction.

7.1. *Ergodic popular difference densities when* $r(M_1, M_2) = (2, 1, 1)$

The following theorem gives an affirmative answer to Question 1.12 for the group $G = \mathbb{Z}^2$:

Theorem 7.1. *Suppose M_1 and M_2 are 2×2 matrices, such that $r(M_1, M_2) = (2, 1, 1)$. Then, for any $\alpha \in (0, 1)$, $epdd_{M_1,M_2}(\alpha) = \alpha^3$.*

An example of the configurations handled by Theorem 7.1 is the class of all axis-aligned right triangles in $\mathbb{Z}^2$, $\{(a, b), (a + n, b), (a, b + m)\}$, which corresponds to the choice of matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof of Theorem 7.1. Without loss of generality, we may assume $\mathrm{rk}(M_1) = \mathrm{rk}(M_2) = 1$ and $\mathrm{rk}(M_2 - M_1) = 2$. Indeed, if $\mathrm{rk}(M_1) = 2$, we may rearrange the expression

$$\mu\left(A \cap T^{-1}_{M_1\vec{n}}A \cap T^{-1}_{M_2\vec{n}}A\right) = \mu\left(A \cap T^{-1}_{(M_1-M_2)\vec{n}}A \cap T^{-1}_{-M_2\vec{n}}A\right)$$

and the new matrices $N_1 = M_1 - M_2$ and $N_2 = -M_2$ satisfy the desired conditions.

We now break the proof into two cases depending on the diagonalisability of M_1 and M_2. Note that, since M_i has rank 1, its characteristic polynomial is of the form $x(x - a)$ for some $a \in \mathbb{Z}$. Hence, if M_i has a nonzero eigenvalue, then it has an integer eigenvalue (in this case, equal to a) and is diagonalisable.

Case 1: M_1 or M_2 has a nonzero eigenvalue.

Without loss of generality, we may assume that M_1 has a nonzero eigenvalue and is therefore diagonalisable. Hence, there is a nonsingular 2×2 integer matrix P, an integer $a \in \mathbb{Z}$ and a rank 1 matrix N_2 with integer entries, such that

$$M_1P = P\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_2P = PN_2 \qquad \text{and} \qquad \mathrm{rk}\left(N_2 - \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = 2.$$

It is straightforward to check that, in order to satisfy the constraints on rank, N_2 must be of the form

$$N_2 = \begin{pmatrix} cd & c \\ bd & b \end{pmatrix}$$

with $b \neq 0$. By changing to the basis $\binom{1}{-d}, \binom{0}{1}$, we may further assume $d = 0$.

Suppose $(X, \mathcal{X}, \mu, (T_{\vec{n}})_{\vec{n}\in\mathbb{Z}^2})$ is a measure-preserving $\mathbb{Z}^2$-system (we do not need to assume that the system is ergodic here), and let $A \in \mathcal{X}$ with $\mu(A) = \alpha$. Define a new $\mathbb{Z}^2$-action by $S_{\vec{n}} := T_{P\vec{n}}$. Then,

$$\text{UC-}\lim_{\vec{n}\in\mathbb{Z}^2} \mu\left(A \cap T^{-1}_{M_1P\vec{n}}A \cap T^{-1}_{M_2P\vec{n}}A\right) = \text{UC-}\lim_{\vec{n}\in\mathbb{Z}^2} \mu\left(A \cap S^{-1}_{(an_1,0)}A \cap S^{-1}_{(cn_2,bn_2)}A\right).$$

Now put $S_1 := S_{(a,0)}$ and $S_2 := S_{(c,b)}$. By Lemma 2.2 and the mean ergodic theorem, we have

$$\begin{aligned}
\text{UC-}\lim_{\vec{n}\in\mathbb{Z}^2} \mu\left(A \cap T^{-1}_{M_1P\vec{n}}A \cap T^{-1}_{M_2P\vec{n}}A\right) &= \text{UC-}\lim_{n_2\in\mathbb{Z}} \text{UC-}\lim_{n_1\in\mathbb{Z}} \mu\left(A \cap S_1^{-n_1}A \cap S_2^{-n_2}A\right) \\
&= \int_X \mathbf{1}_A \cdot E(\mathbf{1}_A \mid \mathcal{I}(S_1)) \cdot E(\mathbf{1}_A \mid \mathcal{I}(S_2)) \\
&\geq \alpha^3,
\end{aligned}$$

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where the inequality in the last line follows from [13, Lemma 1.6]. Therefore, for any $\varepsilon > 0$, the set

$$R_\varepsilon := \left\{\vec{n} \in \mathbb{Z}^2 : \mu\left(A \cap T^{-1}_{M_1 P\vec{n}} A \cap T^{-1}_{M_2 P\vec{n}} A\right) > \alpha^3 - \varepsilon\right\}$$

is syndetic. Noting that P is nonsingular, it follows that the set $P(R_\varepsilon)$ is also syndetic in $\mathbb{Z}^2$. But for any $\vec{m} \in P(R_\varepsilon)$, we have

$$\mu\left(A \cap T^{-1}_{M_1\vec{m}} A \cap T^{-1}_{M_2\vec{m}} A\right) > \alpha^3 - \varepsilon.$$

This shows $\mathrm{epdd}_{M_1,M_2}(\alpha) \geq \alpha^3$.

To see the upper bound $\mathrm{epdd}_{M_1,M_2}(\alpha) \leq \alpha^3$, let $(X, \mathcal{X}, \mu, (T_{\vec{n}})_{\vec{n}\in\mathbb{Z}^2})$ be mixing of order 3. Then, for any $A \in \mathcal{X}$, we have $\mu\left(A \cap T^{-1}_{\vec{n}} A \cap T^{-1}_{\vec{m}} A\right) \to \mu(A)^3$ as $\vec{n}, \vec{m}, \vec{m} - \vec{n} \to \infty$. Let P be a nonsingular 2×2 matrix with integer entries and $a, b, c \in \mathbb{Z}$ with $a, b \neq 0$, such that

$$PM_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} P \qquad \text{and} \qquad PM_2 = \begin{pmatrix} 0 & c \\ 0 & b \end{pmatrix} P.$$

The group of transformations $\widetilde{T}_{\vec{n}} := T_{P\vec{n}}$ is still mixing of order 3. Write $\vec{m} = P\vec{n}$ for $\vec{n} \in \mathbb{Z}^2$. If $m_1 \to \infty$ and $m_2 \to \infty$, then

$$\mu\left(A \cap \widetilde{T}^{-1}_{M_1\vec{n}} A \cap \widetilde{T}^{-1}_{M_2\vec{n}} A\right) = \mu\left(A \cap T^{-1}_{(am_1,0)} A \cap T^{-1}_{(cm_2,bm_2)} A\right) \to \mu(A)^3.$$

Hence, for any $\varepsilon > 0$, there is a finite set $F \subseteq \mathbb{Z}$, such that

$$\left\{\vec{n} \in \mathbb{Z}^2 : \mu\left(A \cap \widetilde{T}^{-1}_{M_1\vec{n}} A \cap \widetilde{T}^{-1}_{M_2\vec{n}} A\right) > \mu(A)^3 + \varepsilon\right\} \subseteq \left\{\vec{n} \in \mathbb{Z}^2 : P\vec{n} \in (F \times \mathbb{Z}) \cup (\mathbb{Z} \times F)\right\}.$$

A union of finitely many lines in $\mathbb{Z}^2$ is not syndetic, so

$$\text{synd-sup}_{\vec{n}\in\mathbb{Z}^2}\mu\left(A \cap \widetilde{T}^{-1}_{M_1\vec{n}} A \cap \widetilde{T}^{-1}_{M_2\vec{n}} A\right) \leq \mu(A)^3.$$

Case 2: M_1 and M_2 have no nonzero eigenvalues.

Since M_1 has rank 1, there is a nonsingular 2×2 integer matrix P, a nonzero integer $a \in \mathbb{Z}$ and a rank 1 matrix N_2 with integer entries and characteristic polynomial x^2, such that

$$M_1 P = P\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \qquad M_2 P = PN_2 \qquad \text{and} \qquad \mathrm{rk}\left(N_2 - \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = 2.$$

Write

$$N_2 = \begin{pmatrix} s & t \\ u & v \end{pmatrix}.$$

Since N_2 has characteristic polynomial x^2, we have $s + v = 0$ and $sv = tu$. Therefore, if $u = 0$, then $s = v = 0$. But then

$$N_2 - \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & t - a \\ 0 & 0 \end{pmatrix}$$

has rank at most 1. Thus, we must have $u \neq 0$. It follows that N_2 can be written in the form

$$N_2 = \begin{pmatrix} db & -d^2 b \\ b & -db \end{pmatrix}$$

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for some b, d with $b \neq 0$. Changing to the basis $\binom{1}{0}, \binom{d}{1}$, we may assume $d = 0$ so that

$$N_2 = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$

Given a $\mathbb{Z}^2$-system $(X, \mathcal{X}, \mu, (T_{\vec{n}})_{\vec{n}\in\mathbb{Z}^2})$, note that

$$\mu\left(A \cap T^{-1}_{N_1\vec{n}} A \cap T^{-1}_{N_2\vec{n}} A\right) = \mu\left(A \cap T^{-1}_{(an_2,0)} A \cap T^{-1}_{(0,bn_1)} A\right).$$

Hence, replacing (n_1, n_2) by (n_2, n_1), we reduce to Case 1. □

7.2. *Ergodic popular difference densities when* $r(M_1, M_2) = (1, 1, 1)$

For matrix configurations with $r(M_1, M_2) = (1, 1, 1)$, we must distinguish between several cases. First, when M_1 and M_2 commute, a construction based on Behrend's theorem shows that the ergodic popular difference density decays faster than any polynomial:

Theorem 7.2. *Suppose M_1 and M_2 are commuting 2×2 matrices, such that $r(M_1, M_2) = (1, 1, 1)$. Then, for any sufficiently small $\alpha \in (0, 1)$, $\mathrm{epdd}_{M_1,M_2}(\alpha) < \alpha^{c\log(1/\alpha)}$, where $c > 0$ is an absolute constant.*

Theorem 7.2 applies to collinear 3-point configurations up to scaling and translation.

Proof of Theorem 7.2. We first distinguish between two cases depending on diagonalisability of M_1 and M_2.

Case 1: M_1 or M_2 has a nonzero eigenvalue.

Without loss of generality, assume M_1 has a nonzero eigenvalue and is therefore diagonalisable. Since M_2 and $M_2 - M_1$ are also rank 1 and commute with M_1, there exists a nonsingular 2×2 matrix P with integer entries and $a, b \in \mathbb{Z}$ be distinct and nonzero, such that

$$PM_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} P \qquad \text{and} \qquad PM_2 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} P. \tag{39}$$

Case 2: M_1 and M_2 have no nonzero eigenvalues.

Using the condition $r(M_1, M_2) = (1, 1, 1)$, there is a nonsingular 2×2 integer matrix P, a nonzero integer $a \in \mathbb{Z}$ and a rank 1 matrix N_2 with integer entries and characteristic polynomial x^2, such that

$$M_1 P = P\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \qquad M_2 P = PN_2 \qquad \text{and} \qquad \mathrm{rk}\left(N_2 - \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = 1.$$

Moreover, N_2 commutes with the matrix $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. Write

$$N_2 = \begin{pmatrix} s & t \\ u & v \end{pmatrix}.$$

Note that

$$\left[\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} s & t \\ u & v \end{pmatrix}\right] = \begin{pmatrix} au & a(v-s) \\ 0 & au \end{pmatrix},$$

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so $u = 0$ and $v = s$. On the other hand, since N_2 has characteristic polynomial x^2, we have $s + v = 0$ and $sv = tu$. Hence, $s = v = 0$, and N_2 is of the form

$$N_2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

with $b \notin \{0, a\}$.

Now, replacing $(n_1, n_2) \in \mathbb{Z}^2$ by $(n_2, n_1) \in \mathbb{Z}^2$ and using the identity

$$\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\begin{pmatrix} n_2 \\ n_1 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

for $c \in \mathbb{Z}$, we can reduce Case 2 to Case 1.

Without loss of generality, let P be a nonsingular 2×2 matrix with integer entries and $a, b \in \mathbb{Z}$ distinct and nonzero, such that (39) holds. Put $d := |\det(P)| \in \mathbb{N}$.

Define $S : \mathbb{T}^2 \to \mathbb{T}^2$ by $S(x, y) := (x, y + x)$. Let $R : \mathbb{T}^2 \to \mathbb{T}^2$ be the transformation $R(x, y) = (2x, 2y+x)$. Both S and R preserve the Haar probability measure μ on $\mathbb{T}^2$. We claim that the $(\mathbb{Z}_{\geq 0})^2$-action generated by S and R is ergodic (with respect to μ). To see this, suppose $f \in L^2(\mathbb{T}^2)$ is simultaneously S- and R-invariant, and expand f as a Fourier series

$$f(x, y) = \sum_{n,m} c_{n,m} e(nx + my),$$

where $e(t) := e(2\pi i t)$. Then

$$(Sf)(x, y) = \sum_{n,m} c_{n,m} e((n + m)x + my) = \sum_{n,m} c_{n-m,m} e(nx + my).$$

Therefore, since $Sf = f$, we have $c_{n,m} = c_{n-m,m}$ for all $n, m \in \mathbb{Z}$. By Parseval's identity, $\sum_{n,m} |c_{n,m}|^2 = \|f\|_2^2 < \infty$, so $c_{n,m} = 0$ whenever $m \neq 0$. That is, $f(x, y) = \sum_n c_{n,0} e(nx)$. Now,

$$(Rf)(x, y) = \sum_n c_{n,0} e(2nx).$$

Hence, since $Rf = f$, we have $c_{2n,0} = c_{n,0}$ for every $n \in \mathbb{Z}$. Applying Parseval's identity once again, we conclude that $c_{n,0} = 0$ for $n \neq 0$. Thus, $f(x, y) = c_{0,0}$ is a constant function.

Fix $\alpha \in (0, 1)$. By [8, Theorem 1.3], there exists a set $A \subseteq \mathbb{T}^2$ with $\mu(A) = \alpha$, such that $\mu\big(A \cap S^{-an} A \cap S^{-bn} A\big) < \alpha^{c \log(1/\alpha)}$ for $n \neq 0$, where $c > 0$ is an absolute constant.[4]

Let $(X, \mathcal{X}, \nu, (T_{\vec{n}})_{\vec{n} \in \mathbb{Z}^2})$ be an ergodic $\mathbb{Z}^2$-system and $B \in \mathcal{X}$ with $\nu(B) = \alpha$, such that

$$\nu\Big(B \cap T_{\vec{n}}^{-1} B \cap T_{\vec{m}}^{-1} B\Big) = \mu(A \cap S^{-n_1} R^{-n_2} A \cap S^{-m_1} R^{-m_2} A)$$

for every $\vec{n}, \vec{m} \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ (note that, because R is noninvertible, we cannot simply take $X = \mathbb{T}^2$, $\nu = \mu$, $B = A$ and $T_{\vec{n}} = S^{n_1} R^{n_2}$). Then, let $\widetilde{T}_{\vec{n}} := T_{P\vec{n}}$ for $\vec{n} \in \mathbb{Z}^2$.

Since $[\mathbb{Z}^2 : P(\mathbb{Z}^2)] = |\det(P)| = d < \infty$, the system $\Big(X, \mathcal{X}, \nu, (\widetilde{T}_{\vec{n}})_{\vec{n} \in \mathbb{Z}^2}\Big)$ has at most d ergodic components. Hence, we may write the ergodic decomposition as $\nu = \frac{1}{k} \sum_{i=1}^{k} \nu_i$ for some $k \leq d$ and some measure ν_i. For some $1 \leq i \leq k$, we must have $\nu_i(B) \geq \alpha$. Without loss of generality, we may therefore assume $\nu_1(B) \geq \alpha$.

[4]The statement of [8, Theorem 1.3] only gives a bound of the form α^l rather than $\alpha^{c \log(1/\alpha)}$. However, as noted in [8] immediately after the statement, the construction of the set A gives this stronger bound via Behrend's theorem on sets without 3-term arithmetic progressions [4]. Additionally, [8, Theorem 1.3] is only stated for the case $a = 1, b = 2$, but the same method works for general a, b (see, e.g. [2, Section 11]).

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Let $\vec{n} \in \mathbb{Z}^2 \setminus \{0\}$. Let $\vec{m} = P\vec{n} \in \mathbb{Z}^2$. Then

$$\begin{aligned}\nu_1\left(B \cap \widetilde{T}^{-1}_{M_1\vec{n}} B \cap \widetilde{T}^{-1}_{M_2\vec{n}} B\right) &= \nu_1\left(B \cap T^{-1}_{(am_1,0)} B \cap T^{-1}_{(bm_1,0)} B\right) \\ &\leq d \cdot \mu\left(A \cap S^{-am_1} A \cap S^{-bm_1} A\right).\end{aligned}$$

Hence, if $\nu_1\left(B \cap \widetilde{T}^{-1}_{M_1\vec{n}} B \cap \widetilde{T}^{-1}_{M_2\vec{n}} B\right) > d \cdot \alpha^{c\log(1/\alpha)}$, then $m_1 = 0$. But since P is nonsingular,

$$\left\{\vec{n} \in \mathbb{Z}^2 : P\vec{n} \in \{0\} \times \mathbb{Z}\right\} = \mathbb{Q}\vec{v} \cap \mathbb{Z}^2,$$

where $\vec{v}$ is the vector $P^{-1}\binom{0}{1} \in \mathbb{Q}^2$. Such a set is never syndetic, so $\mathrm{epdd}_{M_1,M_2}(\alpha) \leq d \cdot \alpha^{c\log(1/\alpha)}$. For $c' < c$ and α sufficiently small, one has $d \cdot \alpha^{c\log(1/\alpha)} < \alpha^{c'\log(1/\alpha)}$, so this completes the proof. □

Now suppose $r(M_1, M_2) = (1, 1, 1)$, and M_1 and M_2 do not commute. In this case, M_1 or M_2 must be diagonalisable,[5] so we assume without loss of generality that M_1 is diagonalisable. We then distinguish between two cases, depending on the form of M_2 when M_1 is diagonalised. Call the pair of matrices (M_1, M_2) *row-like* if there is a nonsingular 2×2 matrix P with rational entries and rational numbers $a, b, c \in \mathbb{Q}$ with $a, b \neq 0$, such that

$$PM_1P^{-1} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad PM_2P^{-1} = \begin{pmatrix} c & b \\ 0 & 0 \end{pmatrix}.$$

Similarly, call the pair (M_1, M_2) *column-like* if there is a nonsingular 2×2 matrix P with rational entries and rational numbers $a, b, c \in \mathbb{Q}$ with $a, b \neq 0$, such that

$$PM_1P^{-1} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad PM_2P^{-1} = \begin{pmatrix} c & 0 \\ b & 0 \end{pmatrix}.$$

For row-like configurations, we can use the 'Fubini' property of uniform Cesàro limits (Lemma 2.2) to show $\mathrm{epdd}(\alpha) = \alpha^3$:

Theorem 7.3. *Suppose M_1 and M_2 are 2×2 matrices with $r(M_1, M_2) = (1, 1, 1)$, such that (M_1, M_2) is row-like. Then, for any $\alpha \in (0, 1)$, $\mathrm{epdd}_{M_1,M_2}(\alpha) = \alpha^3$.*

Proof. Let P be a nonsingular 2×2 matrix with integer entries, such that

$$M_1P = P\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad M_2P = P\begin{pmatrix} c & b \\ 0 & 0 \end{pmatrix}.$$

By changing to the basis $\binom{b}{-c}, \binom{0}{1}$, we may assume $c = 0$.

Let $(X, \mathcal{X}, \mu, (T_{\vec{n}})_{\vec{n}\in\mathbb{Z}^2})$ be a measure-preserving system, and let $A \in \mathcal{X}$ with $\mu(A) = \alpha > 0$. Define a new $\mathbb{Z}^2$-action by $\widetilde{T}_{\vec{n}} := T_{P\vec{n}}$, and let $S := \widetilde{T}_{(1,0)}$. Then

$$\mu\left(A \cap T^{-1}_{M_1P\vec{n}} A \cap T^{-1}_{M_2P\vec{n}} A\right) = \mu\left(A \cap S^{-an_1} A \cap S^{-bn_2} A\right).$$

[5] If neither M_1 nor M_2 are diagonalisable, then they both have characteristic polynomial x^2. By a change of basis, we may assume M_1 is in its Jordan form $M_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Write $M_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The condition $\mathrm{rk}(M_2) = \mathrm{rk}(M_2 - M_1) = 1$ implies that $ad - bc = ad - (b - 1)c = 0$, so $c = 0$ and $ad = 0$. Moreover, since M_2 has characteristic polynomial x^2, we have $a + d = 0$. Hence, $M_2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. But then M_2 commutes with M_1.

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Thus, by Lemma 2.2, we have

$$\text{UC-}\lim_{\vec{n}\in\mathbb{Z}^2} \mu\left(A \cap T^{-1}_{M_1 P\vec{n}} A \cap T^{-1}_{M_2 P\vec{n}} A\right) \geq \alpha^3.$$

Since P is nonsingular, it follows that

$$\text{synd-sup}_{\vec{n}\in\mathbb{Z}^2} \mu\left(A \cap T^{-1}_{M_1 \vec{n}} A \cap T^{-1}_{M_2 \vec{n}} A\right) \geq \alpha^3.$$

Now we will show $\text{epdd}_{M_1,M_2}(\alpha) \leq \alpha^3$. Let P be a nonsingular 2×2 matrix with integer entries and $a, b, c \in \mathbb{Z}$ with $a, b \neq 0$, such that

$$PM_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} P \qquad \text{and} \qquad PM_2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} P.$$

Let $(X, \mathcal{X}, \mu, S, R)$ be an ergodic $\mathbb{Z}^2$-system, such that S is mixing of order 3. Define $T_{\vec{n}} := S^{n_1} R^{n_2}$ and $\widetilde{T}_{\vec{n}} := T_{P\vec{n}}$ for $\vec{n} \in \mathbb{Z}^2$. Then, for $A \in \mathcal{X}$ and $\vec{m} = P\vec{n} \in \mathbb{Z}^2$, we have

$$\mu\left(A \cap \widetilde{T}^{-1}_{M_1 \vec{n}} A \cap \widetilde{T}^{-1}_{M_2 \vec{n}} A\right) = \mu\left(A \cap S^{-am_1} A \cap S^{-bm_2} A\right).$$

Since S is mixing of order 3, given $\varepsilon > 0$, there exists a finite set $F \subseteq \mathbb{Z}$, such that

$$\begin{aligned} &\left\{\vec{n} \in \mathbb{Z}^2 : \mu\left(A \cap \widetilde{T}^{-1}_{M_1 \vec{n}} A \cap \widetilde{T}^{-1}_{M_2 \vec{n}} A\right) > \mu(A)^3 + \varepsilon\right\} \\ &\quad \subseteq P^{-1}\left(\left\{\vec{m} \in \mathbb{Z}^2 : m_1 \in F, m_2 \in F, \text{ or } bm_2 - am_1 \in F\right\}\right). \end{aligned}$$

This set is a union of finitely many lines in $\mathbb{Z}^2$, so it is not syndetic. Hence,

$$\text{synd-sup}_{\vec{n}\in\mathbb{Z}^2} \mu\left(A \cap \widetilde{T}^{-1}_{M_1 \vec{n}} A \cap \widetilde{T}^{-1}_{M_2 \vec{n}} A\right) \leq \mu(A)^3.$$

□

The prototypical column-like configuration is the class of axis-aligned isosceles right triangles, for which it is known by previous work of Chu [13] and Donoso and Sun [16] that $\alpha^4 \leq \text{epdd}(\alpha) \leq \alpha^{4-o(1)}$, where the $o(1)$ term refers to a small positive value tending to 0 as $\alpha \to 0$. We prove that these bounds extend to all column-like configurations:

Theorem 7.4. *Suppose M_1 and M_2 are 2×2 matrices with $r(M_1, M_2) = (1, 1, 1)$, such that (M_1, M_2) is column-like. Then, for any $\alpha \in (0, 1)$, $\text{epdd}_{M_1,M_2}(\alpha) \geq \alpha^4$. Moreover, for any $l < 4$ and all sufficiently small α (depending on l), one has $\text{epdd}_{M_1,M_2}(\alpha) \leq \alpha^l$.*

Proof. Let $(X, \mathcal{X}, \mu, (T_{\vec{n}})_{\vec{n}\in\mathbb{Z}^2})$ be an ergodic $\mathbb{Z}^2$-system. Since the pair (M_1, M_2) is column-like, there exists a nonsingular 2×2 matrix P with integer entries and integers $a, b, c \in \mathbb{Z}$ with $a, b \neq 0$, such that

$$M_1 P = P\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad M_2 P = P\begin{pmatrix} c & 0 \\ b & 0 \end{pmatrix}.$$

Then, for any $\vec{n} \in \mathbb{Z}^2$, we have

$$\mu\left(A \cap T^{-1}_{M_1 P\vec{n}} A \cap T^{-1}_{M_2 P\vec{n}} A\right) = \mu\left(A \cap T^{-1}_{P(an_1,0)} A \cap T^{-1}_{P(cn_1,bn_1)} A\right).$$

Letting $S := T_{P(a,0)}$ and $R := T_{P(c,b)}$, we therefore have the identity

$$\mu\left(A \cap T^{-1}_{M_1 P\vec{n}} A \cap T^{-1}_{M_2 P\vec{n}} A\right) = \mu(A \cap S^{-n_1} A \cap R^{-n_1} A).$$

Now, since T is ergodic and P is nonsingular, the $\mathbb{Z}^2$-action generated by S and R has finitely many ergodic components. Thus, by [13, Theorem 1.1],

$$\{n \in \mathbb{Z} : \mu(A \cap S^{-n}A \cap R^{-n}A) \geq \mu(A)^4\}$$

is syndetic in $\mathbb{Z}$.[6] It follows that

$$\left\{\vec{n} \in \mathbb{Z}^2 : \mu\left(A \cap T^{-1}_{M_1\vec{n}}A \cap T^{-1}_{M_2\vec{n}}A\right) \geq \mu(A)^4\right\}$$

is syndetic in $\mathbb{Z}^2$. Hence, $\mathrm{epdd}_{M_1,M_2}(\alpha) \geq \alpha^4$.

Let $l < 4$. By [16, Theorem 1.2], there exists an ergodic $\mathbb{Z}^2$-system $(X, \mathcal{X}, \mu, S, R)$ and a set $A \in \mathcal{X}$, such that $\mu(A \cap S^{-n}A \cap R^{-n}A) < \mu(A)^l$ for every $n \neq 0$. Since the pair (M_1, M_2) is column-like, there is a nonsingular 2×2 matrix P with integer entries and integers $a, b, c \in \mathbb{Z}$ with $a, b \neq 0$, such that

$$PM_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} P \qquad \text{and} \qquad PM_2 = \begin{pmatrix} c & 0 \\ b & 0 \end{pmatrix} P.$$

Define $T_{\vec{n}} := S^{bn_1}(R^a S^{-c})^{n_2}$, and let $\widetilde{T}_{\vec{n}} := T_{P\vec{n}}$ for $n \in \mathbb{Z}^2$. Note that $\left(X, \mathcal{X}, \mu, (\widetilde{T}_{\vec{n}})_{\vec{n}\in\mathbb{Z}^2}\right)$ has finitely many ergodic components. To be more precise, the ergodic decomposition has the form $\mu = \frac{1}{k}\sum_{i=1}^k \mu_i$ with $k \leq d := |ab\det(P)|$. Without loss of generality, we may assume $\mu_1(A) \geq \mu(A)$.

Now, for any $\vec{n} \neq 0$, we have

$$\mu_1\left(A \cap \widetilde{T}^{-1}_{M_1\vec{n}}A \cap \widetilde{T}^{-1}_{M_2\vec{n}}A\right) \leq d \cdot \mu\left(A \cap S^{-abm_1}A \cap R^{-abm_1}A\right),$$

where $\vec{m} = P\vec{n} \in \mathbb{Z}^2$. Therefore,

$$\left\{\vec{n} \in \mathbb{Z}^2 : \mu_1\left(A \cap \widetilde{T}^{-1}_{M_1\vec{n}}A \cap \widetilde{T}^{-1}_{M_2\vec{n}}A\right) \geq d \cdot \mu_1(A)^l\right\} \subseteq \left\{\vec{n} \in \mathbb{Z}^2 : P\vec{n} \in \{0\} \times \mathbb{Z}\right\} \subseteq \mathbb{Q}\vec{v} \cap \mathbb{Z}^2,$$

where $\vec{v} = P^{-1}\binom{0}{1} \in \mathbb{Q}^2$. The set $\mathbb{Q}\vec{v} \cap \mathbb{Z}^2$ is not syndetic, so this shows $\mathrm{epdd}_{M_1,M_2}(\alpha) \leq d \cdot \alpha^l$ for $\alpha = \mu(A)$. Moreover, for any $l' < l$, we have the inequality $d \cdot \alpha^l < \alpha^{l'}$ for all $\alpha > 0$ sufficiently small. □

7.3. Finitary combinatorial consequences and open questions

There are two cases in which our ergodic-theoretic results directly imply finitary combinatorial analogues. Namely, when $r(M_1, M_2) = (2, 1, 1)$ and when (M_1, M_2) is a row-like pair of noncommuting matrices with $r(M_1, M_2) = (1, 1, 1)$, we establish the bound $\mathrm{epdd}_{M_1,M_2}(\alpha) \geq \alpha^3$ with the help of the 'Fubini' property for uniform Cesàro limits (Lemma 2.2), and this allows us to avoid assuming that the underlying $\mathbb{Z}^2$-system is ergodic. For this reason, we can obtain the following combinatorial result:

Theorem 7.5. *Let M_1, M_2 be 2×2 matrices with integer entries. Suppose that either*

(i) $r(M_1, M_2) = (2, 1, 1)$*, or*
(ii) $r(M_1, M_2) = (1, 1, 1)$*, M_1 and M_2 do not commute, and (M_1, M_2) is row-like.*

Then, for any $\alpha, \varepsilon > 0$, there exists $N_0 = N_0(\alpha, \varepsilon) \in \mathbb{N}$, such that, if $N \geq N_0$ and $A \subseteq \{1, \ldots, N\}^2$ has $|A| \geq \alpha N^2$, then there exists $\vec{n} \in \mathbb{Z}^2$ with $M_1\vec{n}, M_2\vec{n}, (M_2 - M_1)\vec{n} \neq 0$, such that

$$\left|\left\{\vec{x} \in \mathbb{Z}^2 : \{\vec{x}, \vec{x} + M_1\vec{n}, \vec{x} + M_2\vec{n}\} \subseteq A\right\}\right| > (\alpha^3 - \varepsilon)N^2.$$

[6] In [13], it is assumed that the system $(X, \mathcal{X}, \mu, S, R)$ is ergodic. However, the proof easily extends to the case that the system has finitely many ergodic components by noting that all of the ergodic components will have the same Kronecker factor.

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Proof. Let $\alpha, \varepsilon > 0$, and suppose no such N_0 exists. Then, there is an increasing sequence $(N_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and sets $A_k \subseteq \{1, \ldots, N_k\}^2$ with $|A_k| \geq \alpha N_k^2$, such that

$$|A_k \cap (A_k - M_1\vec{n}) \cap (A_k - M_2\vec{n})| \leq (\alpha^3 - \varepsilon)N_k^2$$

whenever $M_1\vec{n}, M_2\vec{n}, (M_2 - M_1)\vec{n} \neq 0$.

For notational convenience, let $A_{k,0} := \mathbb{Z}^2 \setminus A_k$ and $A_{k,1} := A_k$. By passing to a subsequence if necessary, we may assume without loss of generality that

$$\lim_{k\to\infty} \frac{\left|\left(A_{k,i_1} - \vec{n}_1\right) \cap \cdots \cap \left(A_{k,i_r} - \vec{n}_r\right) \cap \{1, \ldots, N_k\}^2\right|}{N_k^2} \tag{40}$$

exists for all $r \in \mathbb{N}$, $\vec{n}_1, \ldots, \vec{n}_r \in \mathbb{Z}^2$ and $i_1, \ldots, i_r \in \{0, 1\}$. Hence, we may define a measure μ on the sequence space $\{0, 1\}^{\mathbb{Z}^2}$ by setting

$$\mu(\{x \in X : x(\vec{n}_1) = i_1, \ldots, x(\vec{n}_r) = i_r\})$$

equal to the limit in (40) and extending with the use of Kolmogorov's extension theorem. Since $\left(\{1, \ldots, N_k\}^2\right)_{k\in\mathbb{N}}$ is a Følner sequence in $\mathbb{Z}^2$, the measure μ is invariant under the shift transformations $(T_{\vec{n}}x)(\vec{m}) := x(\vec{m} + \vec{n})$.

Let $A := \{x \in X : x(\vec{0}) = 1\}$. Then $\mu(A) = \lim_{k\to\infty} \frac{|A_k|}{N_k^2} \geq \alpha$. On the other hand, if $M_1\vec{n}, M_2\vec{n}, (M_2 - M_1)\vec{n} \neq 0$, then

$$\begin{aligned}
\mu\left(A \cap T_{M_1\vec{n}}^{-1}A \cap T_{M_2\vec{n}}^{-1}A\right) &= \mu\left(\{x \in X : x(\vec{0}) = x(M_1\vec{n}) = x(M_2\vec{n}) = 1\}\right) \\
&= \lim_{k\to\infty} \frac{|A_k \cap (A_k - M_1\vec{n}) \cap (A_k - M_2\vec{n})|}{N_k^2} \\
&\leq \alpha^3 - \varepsilon.
\end{aligned}$$

Hence,

$$\begin{aligned}
R_\varepsilon := &\left\{\vec{n} \in \mathbb{Z}^2 : \mu\left(A \cap T_{M_1\vec{n}}^{-1}A \cap T_{M_2\vec{n}}^{-1}A\right) > \mu(A)^3 - \varepsilon\right\} \\
&\subseteq \ker(M_1) \cup \ker(M_2) \cup \ker(M_2 - M_1).
\end{aligned}$$

But by the proofs of Theorems 7.1 and 7.3, R_ε is a syndetic subset of $\mathbb{Z}^2$, so this is a contradiction. □

For general 3-point matrix patterns in $\mathbb{Z}^2$, it remains an open problem to fully determine (finitary combinatorial) popular difference densities. One particularly attractive case, which can be seen as a finitary version of Question 1.12 for the group $G = \mathbb{Z}^2$, is the following:

Conjecture 7.6. *Let M_1 and M_2 be 2×2 matrices with integer entries, such that $M_2 - M_1$ has full rank. Then, for any $\alpha, \varepsilon > 0$, there exists $N_0 = N_0(\alpha, \varepsilon) \in \mathbb{N}$, such that, if $N \geq N_0$ and $A \subseteq \{1, \ldots, N\}^2$ has cardinality $|A| \geq \alpha N^2$, then there exists $\vec{n} \in \mathbb{Z}^2$ with $M_1\vec{n}, M_2\vec{n} \neq 0$, such that*

$$\left|\left\{\vec{x} \in \mathbb{Z}^2 : \{\vec{x}, \vec{x} + M_1\vec{n}, \vec{x} + M_2\vec{n}\} \subseteq A\right\}\right| > (\alpha^3 - \varepsilon)N^2.$$

The special case when M_1, M_2 and $M_2 - M_1$ are all invertible, Conjecture 7.6 was verified by [12, Theorem 1.1]. Moreover, Theorem 7.5 shows that Conjecture 7.6 holds when M_1 and M_2 are both rank 1 matrices. The most interesting remaining case is when M_1 has full rank and M_2 is a rank 1 matrix.

Finally, the column-like family of configurations $\{(a, b), (a+n, b), (a, b+n)\}$, known as *corners*, has been well studied from the perspective of popular differences in finitary combinatorics. In particular, it is known that the popular difference density for corners is of the form $\alpha^{4-o(1)}$ (see [11] and also [17, 25]

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for an analogous result in a finite characteristic setting). To the authors' knowledge, such results are not known for general column-like matrix patterns, but we anticipate that techniques for handling corners should apply in this generality with only minor modifications needed.

8. Khintchine-type recurrence for actions of semigroups

As a consequence of Theorem 1.13, we obtain the following combinatorial result. For any set $E \subseteq \mathbb{Q}_{>0}$ of positive multiplicative upper Banach density $d^*_{mult}(E) > 0$ and any $\varepsilon > 0$, there exists $q \in \mathbb{Q}_{>0} \setminus \{1\}$, such that

$$d^*_{mult}\left(E \cap q^{-1}E \cap q^{-2}E\right) > d^*_{mult}(E)^3 - \varepsilon$$

(in fact, the set of such q is multiplicatively syndetic). More generally, for any countable field $\mathbb{F}$, any set $E \subseteq \mathbb{F}^\times$ of positive multiplicative upper Banach density $d^*_{mult}(E) > 0$ and any $\varepsilon > 0$, the set of $x \in \mathbb{F}^\times$, such that

$$d^*_{mult}\left(E \cap x^{-1}E \cap x^{-2}E\right) > d^*_{mult}(E)^3 - \varepsilon$$

is multiplicatively syndetic.[7] This is suggestive of the following problem. Let R be an integral domain (for example, R can be the ring $\mathbb{Z}$, the ring of integers of a number field or the polynomial ring $\mathbb{F}[t]$ over a finite field $\mathbb{F}$). Given a set $E \subseteq R^\times$ of positive multiplicative upper Banach density $d^*_{R,mult}(E) > 0$ and $\varepsilon > 0$, does there exist $r \in R \setminus \{1\}$, such that

$$d^*_{R,mult}\left(E \cap E/r \cap E/r^2\right) > d^*_{R,mult}(E)^3 - \varepsilon,$$

where $E/r := \{t \in R : rt \in E\}$ for $r \in R$? The goal of this section is to transfer our results into the setting of cancellative abelian semigroups in order to answer this question affirmatively.

8.1. *The group generated by a cancellative abelian semigroup*

Let $(S, +)$ be a countable cancellative abelian semigroup. That is, S is a countable set equipped with a commutative and associative binary operation $+$, such that if $s + t = s + r$ for some $r, s, t \in S$, then $t = r$.

We can define a group G_S as the set of formal differences $\{s - t : s, t \in S\}$ where we identify $s - t$ and $s' - t'$ if $s + t' = s' + t$. More formally, we may define an equivalence relation $\sim$ on S^2 by $(s, t) \sim (s', t')$ if $s + t' = s' + t$. Then G_S is the set of equivalence classes $S^2/\sim$ with the operation $[(s, t)] + [(s', t')] := [(s + s', t + t')]$. It is easy to check that this operation is well defined because S is cancellative. Moreover, G_S has an identity $0 := [(s, s)]$, and for any $s, t \in S$, we have $[(s, t)] + [(t, s)] = 0$. Thus, G_S is a group. Note that there is a natural embedding $S \to G_S$ given by $s \mapsto [(s + s, s)]$.

8.2. *Notions of largeness*

For a set $E \subseteq S$ and an element $t \in S$, let $E - t := \{s \in S : s + t \in E\}$ and $E + t := \{s + t : s \in S\}$. The following definition summarises combinatorial notions of largeness that we will use, some of which are defined above in the setting of abelian groups.

Definition 8.1. Let $(S, +)$ be a countable cancellative abelian semigroup.

- A set $E \subseteq S$ is *syndetic* if there are finitely many elements $t_1, \ldots, t_k \in S$, such that $\bigcup_{i=1}^k (E - t_i) = S$.
- A set $T \subseteq S$ is *thick* if for any finite set $F \subseteq S$, there exists $t \in S$, such that $F + t \subseteq T$.

[7] In fact, our results show that for any $k \in \mathbb{N}$, $d^*_{\mathrm{mult}}\left(E \cap x^{-k}E \cap x^{-(k+1)}E\right)$ and $d^*_{\mathrm{mult}}\left(E \cap x^{-1}E \cap x^{-k}E\right)$ can be made arbitrarily close to $d^*_{\mathrm{mult}}(E)^3$ for a multiplicatively syndetic set of $x \in \mathbb{F}^\times$. On the other hand, by Theorem 1.14, there are $n, m \in \mathbb{N}$, such that $d^*_{\mathrm{mult}}(E \cap x^{-n}E \cap x^{-m}E)$ is much smaller than $d^*_{\mathrm{mult}}(E)^3$ for all $x \neq 1$.

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- A set $P \subseteq S$ is *piecewise syndetic* if there is a syndetic set $E \subseteq S$ and a thick set $T \subseteq S$, such that $P = E \cap T$.
- A sequence $(F_N)_{N\in\mathbb{N}}$ of finite subsets of S is a *Følner sequence* if, for any $t \in S$,

$$\frac{|(F_N + t)\triangle F_N|}{|F_N|} \to 0.$$

- The *lower Banach density* of a set $E \subseteq S$ is the quantity

$$d_*(E) := \inf\left\{\liminf_{N\to\infty} \frac{|E \cap F_N|}{|F_N|} : (F_N)_{N\in\mathbb{N}} \text{ is a Følner sequence in } S\right\}.$$

- The *upper Banach density* of a set $E \subseteq S$ is the quantity

$$d^*(E) := \sup\left\{\limsup_{N\to\infty} \frac{|E \cap F_N|}{|F_N|} : (F_N)_{N\in\mathbb{N}} \text{ is a Følner sequence in } S\right\}.$$

The following is a standard characterisation of syndetic and thick sets (see, e.g. [7, Section 2]).

Proposition 8.2. *Let $(S, +)$ be a countable cancellative abelian semigroup.*

1. *E is syndetic if and only if $d_*(E) > 0$ if and only if $E \cap T \neq \emptyset$ for any thick set $T \subseteq S$;*
2. *T is thick if and only if $d^*(T) = 1$ if and only if $T \cap E \neq \emptyset$ for any syndetic set $E \subseteq S$.*

Lemma 8.3. *Let $(S, +)$ be a countable cancellative abelian semigroup. Then S is thick in G_S.*

Proof. Let $F \subseteq G_S$ be a finite set. Write $F = \{s_i - t_i : 1 \leq i \leq k\}$, where $s_i, t_i \in S$. Put $t = \sum_{i=1}^{k} t_i \in S$. Then

$$F + t = \left\{s_i + \sum_{j\neq i} t_j : 1 \leq i \leq k\right\} \subseteq S.$$

□

The fact that S is thick in G_S is closely related to the fact that any Følner sequence in S is also a Følner sequence in G_S, from which we deduce the following density result:

Proposition 8.4. *Let $E \subseteq S$. Then $d^*_S(E) = d^*_{G_S}(E)$.*

Proof. To show the inequality $d^*_{G_S}(E) \geq d^*_S(E)$, it suffices to show that any Følner sequence in S is a Følner sequence in G_S. Let $(F_N)_{N\in\mathbb{N}}$ be a Følner sequence in S, and let $x \in G_S$. We want to show

$$\frac{|(F_N + x)\triangle F_N|}{|F_N|} \to 0.$$

Write $x = s - t$ with $s, t \in S$. Then

$$\frac{|(F_N + x)\triangle F_N|}{|F_N|} = \frac{|(F_N + s)\triangle(F_N + t)|}{|F_N|} \leq \frac{|(F_N + s)\triangle F_N|}{|F_N|} + \frac{|F_N \triangle (F_N + t)|}{|F_N|} \to 0.$$

Hence, $(F_N)_{N\in\mathbb{N}}$ is a Følner sequence in G_S as claimed.

Now we show the reverse inequality $d^*_S(E) \geq d^*_{G_S}(E)$. If $d^*_{G_S}(E) = 0$, there is nothing to show, so assume $d^*_{G_S}(E) > 0$. Let m be an invariant mean on G_S, such that $m(E) = d^*_{G_S}(E)$. Put $c = m(S) \geq m(E) > 0$. Then, $\widetilde{m} := \frac{1}{c} m$ is an invariant mean on S. Moreover, $\widetilde{m}(E) = \frac{1}{c} m(E) \geq m(E) = d^*_{G_S}(E)$. Therefore, $d^*_S(E) \geq \widetilde{m}(E) \geq d^*_{G_S}(E)$. □

Lemma 8.5. *Suppose $E \subseteq G_S$ is syndetic in G_S. Then, $E \cap S$ is syndetic in S.*

Proof. Let $x_1, \ldots, x_k \in G_S$, such that $\bigcup_{i=1}^k (E - x_i) = G_S$. By Lemma 8.3, S is thick, so we may assume $x_i \in S$ for each $i = 1, \ldots, k$. We claim

$$\bigcup_{i=1}^{k} ((E \cap S) - x_i) \supseteq S.$$

It suffices to check $(E \cap S) - x_i \supseteq (E - x_i) \cap S$ for each $i = 1, \ldots, k$. Suppose $y \in (E - x_i) \cap S$, and let $t \in E$, such that $t - x_i = y$. Then, $t = y + x_i \in S + S \subseteq S$. Hence, $y \in (E \cap S) - x_i$ as desired. □

8.3. *Extending main results to actions of cancellative abelian semigroups*

Any homomorphism $\varphi : S \to S$ extends uniquely to a homomorphism $\widetilde{\varphi} : G_S \to G_S$ via $\widetilde{\varphi}(s - t) = \varphi(s) - \varphi(t)$. To extend our Khintchine-type results to the semigroup setting, we need a condition on φ characterising when $\widetilde{\varphi}(G_S)$ has finite index in G_S.

Proposition 8.6. *Let $(S, +)$ be a countable cancellative abelian semigroup. Let $\varphi : S \to S$ be a homomorphism, and let $\widetilde{\varphi} : G_S \to G_S$ be the group homomorphism $\widetilde{\varphi}(s - t) := \varphi(s) - \varphi(t)$. The following are equivalent:*

(i) *$\varphi(S)$ is a piecewise syndetic subset of S;*
(ii) *$\widetilde{\varphi}(G_S)$ has finite index in G_S.*

Proof. Let $T := \varphi(S)$, and let $H := \widetilde{\varphi}(G_S)$. Note that $H = T - T = G_T$.

(i) $\Longrightarrow$ (ii). Suppose T is piecewise syndetic in S. Then, $d^*_S(T) > 0$. Thus, by Proposition 8.4, $d^*_{G_S}(H) \geq d^*_{G_S}(T) = d^*_S(T) > 0$. But in the group G_S, we have the identity

$$d^*_{G_S}(H) = \frac{1}{[G_S : H]},$$

so $[G_S : H] < \infty$.

(ii) $\Longrightarrow$ (i). Suppose H has finite index in G_S. Then, H is a syndetic subset of G_S, so $H \cap S$ is syndetic in S by Lemma 8.5. Moreover, by Lemma 8.3, T is a thick subset of H. Let $\widetilde{T} := T \cup (S \setminus H)$ so that $T = \widetilde{T} \cap (H \cap S)$. We claim that $\widetilde{T}$ is thick in S.

Let $F \subseteq S$ be a finite set. Put $F_1 = F \cap H$ and $F_2 = F \setminus H$. Since T is a thick subset of H, there exists $x \in H$, such that $F_1 + x \subseteq T$. Write $x = s - t$ with $s, t \in T \subseteq H \cap S$. Then, $F_1 + s = F_1 + x + t \subseteq T + t \subseteq T$. Now, since $s \in H \cap S$ and H is a group, we have $F_2 + s \subseteq S \setminus H$. Thus, $F + s = (F_1 + s) \cup (F_2 + s) \subseteq T \cup (S \setminus H) = \widetilde{T}$.

This shows that $\widetilde{T}$ is a thick subset of S, so $T = \widetilde{T} \cap (H \cap S)$ is piecewise syndetic in S. □

Now we can extend Theorems 1.11 and 1.13 to the semigroup setting:

Theorem 8.7. *Let $(S, +)$ be a countable cancellative abelian semigroup. Let $\varphi, \psi : S \to S$ be homomorphisms. If at least two of the three subsemigroups $\varphi(S)$, $\psi(S)$ and $(\varphi + \psi)(S)$ are piecewise syndetic in S, then, for any set $E \subseteq S$ with positive upper Banach density $d^*_S(E) > 0$ and any $\varepsilon > 0$, the set*

$$\left\{ s \in S : d^*_S(E \cap (E - \varphi(s)) \cap (E - (\varphi + \psi)(s))) > d^*_S(E)^3 - \varepsilon \right\}$$

is syndetic in S.

Remark 8.8. We use the pair $\{\varphi, \varphi + \psi\}$ rather than $\{\varphi, \psi\}$ since the difference $\psi - \varphi$ is not necessarily defined as a map into S.

Proof. By Proposition 8.4, we have $\delta := d^*_{G_S}(E) = d^*_S(E) > 0$. Let $\widetilde{\varphi}$ and $\widetilde{\psi}$ be the extensions of φ and ψ to G_S. By Proposition 8.6, at least two of the subgroups $\widetilde{\varphi}(G_S)$, $\widetilde{\psi}(G_S)$ and $\left(\widetilde{\varphi} + \widetilde{\psi}\right)(G_S)$ have finite

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index in G_S. Hence, by Theorem 1.11, the set

$$R := \left\{g \in G_S : d^*_{G_S}\left(E \cap (E - \widetilde{\varphi}(g)) \cap \left(E - \left(\widetilde{\varphi} + \widetilde{\psi}\right)(g)\right)\right) > \delta^3 - \varepsilon\right\}$$

is syndetic in G_S.

By Lemma 8.5, the set $R \cap S$ is syndetic in S. But

$$R \cap S = \left\{s \in S : d^*_S(E \cap (E - \varphi(s)) \cap (E - (\varphi + \psi)(s))) > \delta^3 - \varepsilon\right\},$$

so this completes the proof. □

Theorem 8.9. *Let $(S, +)$ be a countable cancellative abelian semigroup. Let $a, b \in \mathbb{N}$. If at least one of the three subsemigroups aS, bS or $(a + b)S$ is piecewise syndetic in S, then, for any set $E \subseteq S$ with positive upper Banach density $d^*_S(E) > 0$ and any $\varepsilon > 0$, the set*

$$\left\{s \in S : d^*_S(E \cap (E - as) \cap (E - (a + b)s)) > d^*_S(E)^3 - \varepsilon\right\}$$

is syndetic in S.

Proof. The proof is identical to the proof of Theorem 8.7, except one must use Theorem 1.13 in place of Theorem 1.11. □

8.4. *Two combinatorial questions*

Applying Theorem 8.9 in the semigroup $(\mathbb{N}, \cdot)$, for any $E \subseteq \mathbb{N}$ with positive multiplicative upper Banach density $d^*_{mult}(E) > 0$, any $k \in \mathbb{N}$ and any $\varepsilon > 0$, the set of $m \in \mathbb{N}$, such that

$$d^*_{mult}\left(E \cap E/m^k \cap E/m^{k+1}\right) > d^*_{mult}(E)^3 - \varepsilon$$

is multiplicatively syndetic in $\mathbb{N}$. It is natural to ask if a finitary variant of this result holds.

Question 8.10. Let $p_1, p_2, \ldots$ be an enumeration of the positive prime numbers. Let $\delta, \varepsilon > 0$, and let $k \in \mathbb{N}$. Does there exists $N = N(k, \delta, \varepsilon) \in \mathbb{N}$, such that the following holds: for any $n \geq N$ and any set $A \subseteq \left\{p_1^{r_1} \ldots p_n^{r_n} : 0 \leq r_i \leq n\right\}$ with $|A| \geq \delta n^n$, there exists $y \in \mathbb{N} \setminus \{1\}$, such that

$$\left|\left\{x \in \mathbb{N} : \{x, xy^k, xy^{k+1}\} \subseteq A\right\}\right| > \left(\delta^3 - \varepsilon\right)n^n.$$

Now, we describe an application of Theorem 8.7. Let $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ be enumerations of the positive prime numbers. The map $\varphi : \mathbb{N} \to \mathbb{N}$ defined by $\varphi\left(\prod_{i=1}^n p_i^{r_i}\right) := \prod_{i=1}^n q_i^{r_i}$ is an automorphism of the semigroup $(\mathbb{N}, \cdot)$. Hence, by Theorem 8.7, if $E \subseteq \mathbb{N}$ has positive multiplicative upper Banach density $d^*_{mult}(E) > 0$ and $\varepsilon > 0$, then there is a multiplicatively syndetic set of numbers $y = \prod_{i=1}^n p_i^{r_i} \in \mathbb{N}$, such that

$$d^*_{mult}\left(\left\{x \in \mathbb{N} : \left\{x, x\prod_{i=1}^n p_i^{r_i}, x\prod_{i=1}^n q_i^{r_i}\right\} \subseteq E\right\}\right) > d^*_{mult}(E)^3 - \varepsilon. \tag{41}$$

The IP Szemerédi theorem of Furstenberg and Katznelson [20] implies that, for any $k \in \mathbb{N}$ and any multiplicative automorphisms $\varphi_1, \ldots, \varphi_k : \mathbb{N} \to \mathbb{N}$, the set of $m \in \mathbb{N}$, such that

$$d^*_{mult}(E \cap E/\varphi_1(m) \cap \cdots \cap E/\varphi_k(m)) > 0$$

is a multiplicative IP* set and, hence, multiplicatively syndetic. It is therefore natural to ask if a large intersections variant holds for families of more than two multiplicative automorphisms:

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Question 8.11. Let $p_1, p_2, \ldots$ be the enumeration of the positive prime numbers in increasing order. For each $j \in \mathbb{N}$, let $q_{j,1}, q_{j,2}, \ldots$ be a distinct enumeration of the positive prime numbers. For which $k \in \mathbb{N}$ does the following hold: for any $E \subseteq \mathbb{N}$ with $d^*_{mult}(E) > 0$ and any $\varepsilon > 0$, there exists $y = \prod_{i=1}^n p_i^{r_i} \in \mathbb{N} \setminus \{1\}$, such that

$$d^*_{mult}\left(\left\{x \in \mathbb{N} : \left\{x, x\prod_{i=1}^n q_{1,i}^{r_i}, x\prod_{i=1}^n q_{2,i}^{r_i}, \ldots, x\prod_{i=1}^n q_{k,i}^{r_i}\right\} \subseteq E\right\}\right) > d^*_{mult}(E)^{k+1} - \varepsilon. \tag{42}$$

Note that (42) holds for $k \leq 2$ (see (41) and the discussion above).

A. Proof of Lemma 3.6

In this section we prove Lemma 3.6, restated here for the convenience of the reader:

Lemma A.1 (Lemma 3.5). *Let $(X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a G-system, and let $H \leq G$ be a subgroup of finite index. Then, for every $k \geq 1$, one has $\mathcal{Z}^k_H(X) = \mathcal{Z}^k_G(X)$.*

We follow the arguments in [5, Appendix A] and generalise them to arbitrary countable discrete abelian groups. We start with some background related to the Host–Kra parallelepipeds construction.

Definition A.2. Let G be a countable discrete abelian group, and let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g))$ be a G-system. For every $k \geq 0$, we define a G-system $\mathbf{X}_G^{[k]} = (X^{[k]}, \mathcal{X}^{[k]}, \mu^{[k]}, (T_g^{[k]})_{g\in G})$ inductively by setting $X_G^{[0]} = X$, and $X_G^{[k+1]} = X_G^{[k]} \times_{\mathcal{I}(X_G^{[k]})} X_G^{[k]}$, where $\mathcal{I}(X_G^{[k]})$ is the σ-algebra of $(T_g^{[k]})_{g\in G}$-invariant functions.

Host and Kra [23] proved the following result for $\mathbb{Z}$-systems, but the same proof works for arbitrary countable discrete abelian groups.

Theorem A.3 ([23], Proposition 4.7). *$\mathcal{Z}^k_G(X)$ is the minimal σ-algebra with the property that $\mathcal{I}(X^{[k]})$ is a sub-σ-algebra of $(\mathcal{Z}^k_G(X))^{[k]}$.*

Let $X = \bigcup_{\alpha\in J} X_\alpha$ be a partition of X to G-invariant sets. Then, $X_G^{[k]} = \bigcup_{\alpha\in J} X_\alpha^{[k]}$, $\mathcal{I}(X^{[k]}) = \bigvee_{\alpha\in J} \mathcal{I}(X_\alpha^{[k]})$ and $\mathcal{Z}^k_G(X) = \bigvee_{\alpha\in J} \mathcal{Z}^k_G(X_\alpha)$. Therefore, by the ergodic decomposition, it is enough to prove Lemma 3.6 in the case where the G-action is ergodic.

The following lemma gives the easy inclusion in Lemma 3.6.

Lemma A.4. *In the setting of Lemma 3.6, $\mathcal{Z}^k_G(X) \leq \mathcal{Z}^k_H(X)$.*

Proof. The proof is immediate by Theorem A.3 and since any $(T_g^{[k]})_{g\in G}$-invariant function is also a $(T_h^{[k]})_{h\in H}$-invariant function. □

We need the following observation.

Lemma A.5. *Let G be a countable discrete abelian group, let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic measure preserving G-system and let $H \leq G$ be a subgroup of finite index. Then, $\mathcal{I}_H(X) \leq \mathcal{Z}_G(X)$.*

Proof. The group G/H acts ergodically by unitary transformations on $\mathcal{H} = L^2(X, \mathcal{I}_H, \mu|_{\mathcal{I}_H})$. Since G/H is a finite abelian group, the unitary representation splits into a direct sum of one-dimensional irreducible representations. In other words, $\mathcal{H}$ is generated by eigenfunctions of the action of G/H, which are measurable with respect to $\mathcal{Z}_G(X)$. This completes the proof. □

Now, we prove the $k = 1$ case of Lemma 3.6 under the additional assumption that the action of H is ergodic.

Lemma A.6. *Let G be countable discrete abelian groups, and let $H \leq G$ be a finite index subgroup. Let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be an ergodic G-system, and suppose the action of H is ergodic. Then, $\mathcal{Z}_H(X) = \mathcal{Z}_G(X)$.*

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Proof. The group G/H is finite, and therefore it is a direct product of finite cyclic groups. In particular, we can find $d \in \mathbb{N}$ and a sequence of subgroups $H_0 = H \le H_1 \le \cdots \le H_d \le G$, such that G/H_d and H_i/H_{i-1}, $1 \le i \le d$, are cyclic groups of prime order. Using a proof by induction on d, we may assume without loss of generality that G/H is cyclic and of prime order. Let $g_0 \in G$ be a representative of a generator of G/H and $l := [G : H]$ be a prime number. By the ergodicity of H, the σ-algebra $\mathcal{Z}_H(X)$ is generated by H-eigenfunctions. Hence, it is enough to show that every H-eigenfunction f is a linear combination of G-eigenfunctions. Let $\lambda : H \to S^1$ be the eigenvalue of f, and observe that for any l-th root $\omega \in S^1$ of $\lambda(lg_0)$, the function

$$f + \omega \cdot T_{g_0} f + \ldots + \omega^{l-1} \cdot T_{(l-1)g_0} f$$

is a G-eigenfunction. Now, since

$$f = \sum_{\omega \in S^1\,:\,\omega^l = \lambda(lg_0)} f + \omega \cdot T_{g_0} f + \ldots + \omega^{l-1} \cdot T_{(l-1)g_0} f,$$

f is measurable with respect to $\mathcal{Z}_G(X)$, and this completes the proof. □

Let G be a countable discrete abelian group, and let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ be a G-system. If the system $\mathbf{X}$ is ergodic, it follows from the definition that $X_G^{[1]}$ is the Cartesian product of X with itself, and the measure is the product measure. As a consequence of Lemma A.6, we have:

Lemma A.7. *If the action of H on X is ergodic, then*

$$\mathcal{I}(X_H^{[1]}) = \mathcal{I}(X_G^{[1]}).$$

Proof. The inclusion $\mathcal{I}(X_G^{[1]}) \le \mathcal{I}(X_H^{[1]}))$ is trivial. Now, let $f : X\times X \to \mathbb{C}$ be a $(T_h\times T_h)_{h\in H}$ invariant function. By Lemma A.6, we can find an orthonormal basis of G-eigenfunctions $\{f_i\}_{i\in\mathbb{N}}$ for $\mathcal{Z}_H(X)$. By Lemma 4.6, there exist constants $a_{i,j} \in \mathbb{C}$ for all $i, j \in \mathbb{N}$, such that

$$f(x, y) = \sum_{i=1}^{\infty} a_{i,j} f_i(x)\overline{f_j}(y).$$

Applying the H-action and using the uniqueness of the decomposition, we see that $a_{i,j} = 0$ unless $i = j$. In particular, f is spanned by the G-invariant functions $f_i \otimes \overline{f_i}$. Thus, f is measurable with respect to $\mathcal{I}(X_G^2)$ and the claim follows. □

We use Lemma A.7 to prove the following:

Proposition A.8. *If the action of H on X is ergodic, then for $k \ge 0$, one has*

$$\mathcal{I}(X_H^{[k]}) = \mathcal{I}(X_G^{[k]}) \qquad \textit{and} \qquad \mu_G^{[k]} = \mu_H^{[k]}.$$

Proof. We prove the claim by induction on k. The case $k = 0$ is trivial.

Assume that for some $k \ge 0$, $\mathcal{I}(X_H^{[k]}) = \mathcal{I}(X_G^{[k]})$ and $\mu_G^{[k]} = \mu_H^{[k]}$. It is immediate that

$$\mu_G^{[k+1]} = \mu_G^{[k]} \times_{\mathcal{I}(X_G^{[k]})} \mu_G^{[k]} = \mu_H^{[k]} \times_{\mathcal{I}(X_H^{[k]})} \mu_H^{[k]} = \mu_H^{[k+1]}.$$

By the ergodic decomposition theorem, applied with respect to the σ-algebra $\mathcal{I}(X_G^{[k]})$, we can find a partition $X_G^{[k]} = \bigcup_{\alpha\in J} X_\alpha$ of $X_G^{[k]}$ to $(T_g^{[k]})_{g\in G}$ invariant sets. Let S_g^α be the restriction of $T_g^{[k]}$ to the set X_α. By the induction hypothesis, the action of $(S_h^\alpha)_{h\in H}$ on X_α is ergodic. Hence, by Lemma A.7,

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we have

$$\mathcal{I}(X_H^{[k+1]}) = \bigcup_{\alpha\in J} \mathcal{I}_H(X_\alpha^{[1]})) = \bigcup_{\alpha\in J} \mathcal{I}_G(X_\alpha^{[1]}) = \mathcal{I}(X_G^{[k+1]}),$$

as required. □

Proposition A.8 establishes Lemma 3.6 in the case where the action of H is ergodic. Now, we assume that the H-action is nonergodic. As in the proof of Lemma A.6, we may assume without loss of generality that G/H is cyclic of order l for some prime l. In particular, there exists a partition $X = \bigcup_{i\in\mathbb{Z}/l\mathbb{Z}} X_i$ into H-invariant sets and some $g_0 \in G$, such that $T_{g_0}X_i = X_{i+1}$, $i \in \mathbb{Z}/l\mathbb{Z}$.

We need the following technical lemma.

Lemma A.9. *Let G be a countable discrete abelian group, and let* $\mathbf{Y} = (Y, \mathcal{Y}, \nu, (T_g)_{g\in G})$ *be an ergodic G-system. Suppose that there exists some* $g_0 \in G$ *and H-invariant subsets* Y_i*, such that* $Y = \bigcup_{i\in\mathbb{Z}/l\mathbb{Z}} Y_i$ *and* $T_{g_0}Y_i = Y_{i+1}$ *for* $i \in \mathbb{Z}/l\mathbb{Z}$*. Then,* $Y \times_{\mathcal{I}_G(Y)} Y = \bigcup_{i,j\in\mathbb{Z}/l\mathbb{Z}} Y_{i,j}$ *where* $Y_{i,i} = Y_i \times_{\mathcal{I}_H(Y_i)} Y_i$ *and* $T_{sg_0}\times T_{tg_0}$ *is an isomorphism between* $Y_{i,i}$ *and* $Y_{i+s,i+t}$*,* $i \in \mathbb{Z}/l\mathbb{Z}$*.*

Proof. Let $A \in \mathcal{I}_G(Y)$ be a measurable G-invariant subset of Y. For each $0 \le i \le l-1$, $A_i = A \cap Y_i$ is an H-invariant set. In particular, A_0 is H-invariant and $A_i = T_{ig_0}A_0$. We deduce that the mapping $A \mapsto A\cap Y_0$ is an isomorphism between $\mathcal{I}_G(Y)$ and $\mathcal{I}_H(Y_0)$. Using the ergodic decomposition, we can find a partition

$$Y_0 = \bigcup_{\alpha\in I} Y_{0,\alpha}$$

of Y_0 to H-invariant sets. For every $\alpha \in I$, and $i \ne 0$, let $Y_{i,\alpha} = T_{ig_0}Y_{0,\alpha}$ and $Y_\alpha = \bigcup_{i\in\mathbb{Z}/l\mathbb{Z}} Y_{i,\alpha}$. Then, $Y = \bigcup_{\alpha\in I} Y_\alpha$ is the ergodic decomposition of Y with respect to the factor $\mathcal{I}_G(Y)$. Thus, if we let $Y_{i,j} = \bigcup_{\alpha\in I} Y_{i,\alpha}\times Y_{j,\alpha}$, we have,

$$Y_G^{[1]} = \bigcup_{\alpha\in I}(Y_\alpha \times_{\mathcal{I}_G(Y_\alpha)} Y_\alpha) = \bigcup_{\alpha\in I}\bigcup_{i,j\in\mathbb{Z}/l\mathbb{Z}}(Y_{i,\alpha}\times Y_{j,\alpha}) = \bigcup_{i,j\in\mathbb{Z}/l\mathbb{Z}}\bigcup_{\alpha\in I}(Y_{i,\alpha}\times Y_{j,\alpha}) = \bigcup_{i,j\in\mathbb{Z}/l\mathbb{Z}} Y_{i,j}.$$

In particular, $Y_{i,i} = \bigcup_{\alpha\in I}(Y_{i,\alpha}\times Y_{i,\alpha}) = Y_i \times Y_i$, as required. □

Recall that $G = \bigcup_{i=0}^{l-1} ig_0 + H$. It follows from Lemma A.9 that for $i, j \in \mathbb{Z}/l\mathbb{Z}$,

$$(T_{g_0}\times T_{g_0})(Y_i \times_{\mathcal{I}_H(Y)} Y_j) = Y_{i+1,j+1}.$$

Therefore, the subsets $V_i = \bigcup_{j\in\mathbb{Z}/l\mathbb{Z}} Y_{j,j+i}$, $i \in \mathbb{Z}/l\mathbb{Z}$ form a partition of $Y \times_{\mathcal{I}_G(Y)} Y$ into $(T_g\times T_g)_{g\in G}$-invariant sets. Furthermore, $\mathrm{Id}\times T_{ig_0}$ is an isomorphism between V_0 and V_i.

We use Lemma A.9 to show the following:

Lemma A.10. *Let* $\mathbf{X} = (X, \mathcal{X}, \mu, (T_g)_{g\in G})$ *be an ergodic G-system. Let* $X = \bigcup_{i\in\mathbb{Z}/l\mathbb{Z}} X_i$ *be a partition into H-invariant sets and let* $g_0 \in G$ *be as above. Then, for any* $k \ge 0$*, there exists a partition* $X_G^{[k]} = \bigcup_{j\in(\mathbb{Z}/l\mathbb{Z})^k} W_j$*, into* $(T_g^{[k]})_{g\in G}$*-invariant sets, such that* $W_0 = \bigcup_{i\in\mathbb{Z}/l\mathbb{Z}}(X_i)_H^{[k]}$ *and* $T_{g_0}^{[k]}\left((X_i)_H^{[k]}\right) = (X_{i+1})_H^{[k]}$*. Furthermore, for every* $j \in (\mathbb{Z}/l\mathbb{Z})^k$*, there exists an isomorphism of measure spaces* $\tau_j : W_0 \to W_j$*, which in every coordinate of* $X^{[k]}$ *is a power of* T_{g_0}*.*

Proof. We induct on k. The case $k = 0$ is trivial.

Assume that the claim holds for some $k \ge 0$. Then

$$X_G^{[k+1]} = X_G^{[k]} \times_{\mathcal{I}(X_G^{[k]})} X_G^{[k]} = \bigcup_{j\in(\mathbb{Z}/l\mathbb{Z})^k}(W_j \times_{\mathcal{I}(W_j)} W_j).$$

Fix $j \in (\mathbb{Z}/l\mathbb{Z})^k$. Since the isomorphism $\tau_j : W_0 \to W_j$ commutes with $(T_g^{[k]})_{g\in G}$, it induces an isomorphism $\tau_j \times \tau_j : W_0 \times_{\mathcal{I}(W_0)} W_0 \to W_j \times_{\mathcal{I}(W_j)} W_j$. By assumption, $W_0 = \bigcup_{i\in\mathbb{Z}/l\mathbb{Z}} (X_i)_H^{[k]}$, and by Lemma A.9, $W_0 \times_{\mathcal{I}(W_0)} W_0$ can be partitioned into $(T_g^{[k+1]})_{g\in G}$-invariant sets $\{V_i\}_{i\in\mathbb{Z}/l\mathbb{Z}}$, such that

$$V_0 = \bigcup_{i\in\mathbb{Z}/l\mathbb{Z}} \left((X_i)_H^{[k]} \times_{\mathcal{I}\left((X_i)_H^{[k]}\right)} (X_i)_H^{[k]} \right) = \bigcup_{i\in\mathbb{Z}/l\mathbb{Z}} (X_i)_H^{[k+1]}.$$

Moreover, V_0 is isomorphic to V_j via an isomorphism whose projections are powers of $T_{g_0}^{[k]}$. Since W_0 is isomorphic to W_j, this completes the proof. □

We recall that it suffices to establish the proof of Lemma 3.6 in the case where the G-action is ergodic and G/H is a cyclic group of order l for some $l > 0$. As before, we find a partition $X = \bigcup_{i\in\mathbb{Z}/l\mathbb{Z}} X_i$ of X into H-invariant sets and some $g_0 \in G$, such that $T_{g_0}(X_i) = X_{i+1}$ for $i \in \mathbb{Z}/l\mathbb{Z}$.

Proof of Lemma 3.6. Let $k \geq 0$, and let $\{W_i\}_{i\in(\mathbb{Z}/l\mathbb{Z})^k}$ be as in Lemma A.10. Since $X_0, \ldots, X_{l-1}$ are disjoint $(T_h)_{h\in H}$-invariant subsets of X, we have $\mathcal{I}(X_H^{[k]}) = \prod_{i\in\mathbb{Z}/l\mathbb{Z}} \mathcal{I}\left((X_i)_H^{[k]}\right)$ and $Z_H^k(X) = \prod_{i\in\mathbb{Z}/l\mathbb{Z}} Z_H^k(X_i)$. Let B be a $(T_h^{[k]})_{h\in H}$-invariant subset of $(X_i)_H^{[k]}$. For every $j \in \mathbb{Z}/l\mathbb{Z}$, let $A_j = (T_{(j-i)g_0}^{[k]})(B)$ and $A = \bigcup_{j\in\mathbb{Z}/l\mathbb{Z}} A_j$. By definition, $A \subseteq W_0$ is a $(T_g^{[k]})_{g\in G}$-invariant set. Therefore, by Theorem A.3, $A \in \left(\mathcal{Z}_G^k(X)\right)^{[k]}$. Since X_i is $(T_h^{[k]})$-invariant, by Lemma A.5, $X_i \in \mathcal{Z}_G^1(X)$. Therefore, $B = A_i = A \cap (X_i)_H^{[k]}$ is an element of $\left(\mathcal{Z}_G^k(X)\right)^{[k]}$. Since B is arbitrary, and this holds for all $i \in \mathbb{Z}/l\mathbb{Z}$, we deduce that $\mathcal{I}(X_H^{[k]}) \leq \mathcal{Z}_G^k(X)$. By Theorem A.3, we have $\mathcal{Z}_H^k(X) \leq \mathcal{Z}_G^k(X)$. Lemma A.4 provides the other inclusion, and this completes the proof. □

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References

[1] E. Ackelsberg and V. Bergelson, 'Popular differences for polynomial patterns in rings of integers', Preprint, 2021, arXiv:2107.07626.
[2] E. Ackelsberg, V. Bergelson and A. Best, 'Multiple recurrence and large intersections for abelian group actions', *Discrete Anal.* (2021), paper 18, 91 pp.
[3] T. Austin, 'Non-conventional ergodic averages for several commuting actions of an amenable group', *J. Anal. Math.* **130** (2016), 243–274.
[4] F. A. Behrend, 'On sets of integers which contain no three terms in arithmetical progression', *Proc. Nat. Acad. Sci. U.S.A.* **32** (1946), 331–332.
[5] V. Bergelson, 'Combinatorial and Diophantine applications of ergodic theory', Appendix A by A. Leibman and Appendix B by A. Quas and M. Wierdl, in *Handbook of Dynamical Systems* (Elsevier B. V., Amsterdam, 2006), pp. 745–869.
[6] V. Bergelson and A. Ferré Moragues, 'An ergodic correspondence principle, invariant means and applications', *Israel J. Math.* **245** (2021), 921–962.
[7] V. Bergelson, N. Hindman and R. McCutcheon, 'Notions of size and combinatorial properties of quotient sets in semigroups', *Topology Proc.* **23** (1998), 23–60.
[8] V. Bergelson, B. Host and B. Kra, 'Multiple recurrence and nilsequences', Appendix by Imre Ruzsa, *Invent. Math.* **160** (2005), 261–303.
[9] V. Bergelson and A. Leibman, 'Cubic averages and large intersections', in *Recent Trends in Ergodic Theory and Dynamical Systems* (Amer. Math. Soc. , Providence, RI, 2015), pp. 5–19.
[10] V. Bergelson, T. Tao and T. Ziegler, 'An inverse theorem for the uniformity seminorms associated with the action of F_p^∞', *Geom. Funct. Anal.* **19** (2010), 1539–1596.
[11] A. Berger, 'Popular differences for corners in Abelian groups', *Math. Proc. Cambridge Philos. Soc.* **171** (2021), 207–225.
[12] A. Berger, A. Sah, M. Sawhney and J. Tidor, 'Popular differences for matrix patterns', *Trans. Amer. Math. Soc.* **375** (2022), 2677–2704.

[13] Q. Chu, ‘Multiple recurrence for two commuting transformations’, *Ergodic Theory Dynam. Systems* **31** (2011), 771–792.

[14] Q. Chu, N. Frantzikinakis and B. Host, ‘Commuting averages with polynomial iterates of distinct degrees’, *Proc. Lond. Math. Soc. (3)* **102** (2011), 801–842.

[15] S. Donoso, A. Le, J. Moreira and W. Sun, ‘Optimal lower bounds for multiple recurrence’, *Ergodic Theory Dynam. Systems* **41** (2021), 379–407.

[16] S. Donoso and W. Sun, ‘Quantitative multiple recurrence for two and three transformations’, *Israel J. Math.* **226** (2018), 71–85.

[17] J. Fox, A. Sah, M. Sawhney, D. Stoner and Y. Zhao, ‘Triforce and corners’, *Math. Proc. Cambridge Philos. Soc.* **169** (2020), 209–223.

[18] N. Frantzikinakis, ‘Multiple ergodic averages for three polynomials and applications’, *Trans. Amer. Math. Soc.* **360** (2008), 5435–5475.

[19] N. Frantzikinakis and B. Host, ‘Weighted multiple ergodic averages and correlation sequences’, *Ergodic Theory Dynam. Systems* **38** (2018), 81–142.

[20] H. Furstenberg and Y. Katznelson, ‘An ergodic Szemerédi theorem for IP-systems and combinatorial theory’, *J. Analyse Math.* **45** (1985), 117–168.

[21] H. Furstenberg and B. Weiss, ‘A mean ergodic theorem for $\frac{1}{N}\sum_{n=1}^{N} f(T^n(x))g\left(T^{n^2}(x)\right)$’, in *Convergence in Ergodic Theory and Probability*, *Columbus, OH 1993 (Bergelson, March, and Rosenblatt, eds.)* **5** (Ohio State Univ. Math. Res. Inst. Publ., de Gruyter, Berlin, 1996), 193–227.

[22] T. Gowers, ‘A new proof of Szemeredi’s theorem’, *Geom. Func. Anal.* **11** (2001), 465–588.

[23] B. Host and B. Kra, ‘Nonconventional ergodic averages and nilmanifolds’, *Ann. Math.* **161** (2005), 397–488.

[24] A. Khintchine, ‘Eine Verschärfung des Poincaréschen “Wiederkehrsatzes”’, *Compositio Math.* **1** (1935), 177–179.

[25] M. Mandache, ‘A variant of the Corners theorem’, *Math. Proc. Cambridge Philos. Soc.* **171** (2021), 607–621.

[26] A. Sah, M. Sawhney and Y. Zhao, ‘Patterns without a popular difference’, *Discrete Anal.* (2021), paper 8, 30pp.

[27] O. Shalom, ‘Multiple ergodic averages in abelian groups and Khintchine type recurrence’, *Trans. Amer. Math. Soc.* **375** (2022), 2729–2761.

[28] S. M. Srivastava, *A Course on Borel Sets* (Springer-Verlag, New York, 1998).

[29] T. Tao and T. Ziegler, ‘Concatenation theorems for anti-Gowers-uniform functions and Host–Kra characteristic factors’, *Discrete Anal.* (2016), paper 13, 60.

[30] T. Ziegler, ‘Universal characteristic factors and Furstenberg averages’, *J. Amer. Math. Soc.* **20** (2007), 53–97.

[31] R. Zimmer, ‘Extensions of ergodic group actions’, *Illinois J. Math.* **20** (1976), 373–409.

[32] P. Zorin-Kranich, ‘Norm convergence of multiple ergodic averages on amenable groups’, *J. Anal. Math.* **130** (2016), 219–241.

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