A CHARACTERIZATION OF NORMED M-SPACES

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1. Introduction. Linear spaces on which both an order and a topology are defined and related in various ways have been studied for some time now. Given an order on a linear space it is sometimes possible to define a useful topology using the order and linear structure. In this note we focus on a special type of space called a linear lattice and determine those lattice properties which are both necessary and sufficient for the existence of a classical norm, called an M-norm, for the lattice. This result is a small step in a program to determine which intrinsic order properties of an ordered linear space are necessary and sufficient for the existence of various given types of topologies for the space. This study parallels, in a certain sense, the study of purely topological spaces to determine intrinsic properties of a topology which make it metrizable and the study of the relation between order and topology on spaces which have no algebraic structure, or algebraic structures other than a linear one.

2. Preliminairies. The positive reals are denoted $\mathbf{R}^+$. A reproducing cone in a real linear space V is a subset P of V such that

(i) $V = P - P$,

(ii) $aP + bP \subseteq P$ for any a and b in $\mathbf{R}^+$, and

(iii) $P \cap (-P) = \{0\}$.

If P is a reproducing cone in V an anti-symmetric order is defined on V by saying $x \leq y$ if $y - x$ is in P. A linear lattice is a real linear space V ordered by a reproducing cone P with the property that for each x and y in V the set $\{x, y\}$ has a least upper bound denoted $x \vee y$ and a greatest lower bound denoted $x \vee y$. If x is any element of a linear lattice we define $x^+ = x \vee 0$, $x^- = x \wedge 0$, and $|x| = x^+ \vee x^-$. The reader may consult Jameson [**3**, pp. 51–54] for a long list of basic relations which hold in a linear lattice. We will use the following two.

(A) For any x and y in V,

$$(x+y)^+ \leq x^+ + y^+ \quad \text{and} \quad (x+y)^- \leq x^- + y^-.$$

(B) If x and y are in V and t is in $\mathbf{R}^+$, then

$$t(x \vee y) = (tx) \vee (ty).$$

A norm, $\| \ \|$, on a linear lattice such that $|x| \geq |y|$ implies $\|x\| \geq \|y\|$ is called a lattice norm. Jameson [**3**, p. 69] shows that a norm is a lattice norm if and only if $0 \leq x \leq y$ implies that $\|x\| \leq \|y\|$ and $\| \, |x| \, \| = \|x\|$ for any x in V. A lattice norm which satisfies the condition $\|x \vee y\| \leq \|x\| \vee \|y\|$ for any x and y in P is called an M-norm and a linear lattice with an M-norm is called a normed M-space.

An upward directed set in a linear lattice is a set D with the property that for any two elements x and y in D there exists an element z in D such that $z - x$ and $z - y$ are in P.

If for any x in V and y in P, $y-nx \in P$ for all positive integers n implies that x is in $-P$, the lattice V is called archimedian. An element e in P is called an order unit if for every x in V there exists a t in $\mathbf{R}^+$ such that $te-|x|$ is in P.

It is well known that in an archimedian linear lattice V with order unit e the functional $\| \ \|$ defined on V by

$$\|x\| = \inf\{t \in \mathbf{R}^+ : te - |x| \in P\}$$

is an M-norm. (See Day [**2**, p. 101], for example.) The existence of an order unit is not a necessary condition for the existence of an M-norm, however. The space $C_0(\mathbf{R})$ of all continuous real valued functions on the real line vanishing at infinity with an order defined in a natural way by saying that $f \leq g$ for any f and g in $C_0(\mathbf{R})$ if $f(t) \leq g(t)$ for every t in $\mathbf{R}$ and with the topology induced by the standard norm,

$$\|x\| = \sup\{|x(t)| : t \in \boldsymbol{R}\}$$

is an M-space which contains no order unit.

3. Characterization. The following theorem gives lattice properties which are both necessary and sufficient for the existence of an M-norm for a linear lattice.

THEOREM. *An M-norm can be defined on a linear lattice V with positive cone P if an only if P contains an upward directed set D with the following properties.*

(1) *For every nonzero x in P there exists a nonzero t in $\mathbf{R}^+$ and an element d in D with $d-tx$ in P.*

(2) *For every nonzero x in P,*

$$\sup\{t \in \mathbf{R}^+ : d - tx \in P, d \in D\}$$

is finite.

Proof. To show the conditions are necessary, let $\| \ \|$ be an M-norm for V and let D be all elements of P of norm one. If x and y are in D, then

$$\|x \vee y\| \leq \|x\| \vee \|y\| = 1,$$

so $x \vee y$ is in D and thus D is directed upwards. If x is a nonzero element of P, then $(x/\|x\|)$ is in D so (1) holds. If t is in $\mathbf{R}^+$ and $d-tx$ is in P, then $t\|x\| \leq \|d\| = 1$ so $t \leq (1/\|x\|)$ and (2) holds.

To prove the conditions are sufficient first define a real valued function r on the nonzero elements of P by

$$r(x) = \sup\{t \in \mathbf{R}^+ : d - tx \in P \text{ and } d \in D\}.$$

Then for a nonzero element x of P define $\|x\|$ by $\|x\| = [r(x)]^{-1}$ and let $\|0\| = 0$. If x is an arbitrary element of V define $\|x\|$ by $\|x\| = \|x^+\| \vee \|x^-\|$.

If x is a nonzero element of P, property (1) implies that $r(x) > 0$. If x is a nonzero element of V, then either x^+ or x^- is nonzero and hence either $r(x^+)$ or $r(x^-)$ is nonzero so that $\|x\|$ is nonzero.

If x and y and $x-y$ are in P, then

$$\{t\in\mathbf{R}^+ : d-tx\in P\}\subseteq\{t\in\mathbf{R}^+ : d-ty\in P\}$$

for each d in D so that $r(x)\le r(y)$ and $\|x\|\ge\|y\|$; that is $\|\ \|$ is monotone on P.

Let x and y be nonzero elements of P. Then given any positive scalar t there exist elements w and u in D and positive scalars a and b such that $a\ge r(x)-t$, $b\ge r(y)-t$ and $w-ax$ and $u-by$ are in P. Take z in D with $z-u$ and $z-w$ in P. Then $a^{-1}z-x$ and $b^{-1}z-y$ and thus

$$z-ab(a+b)^{-1}(x+y)$$

are in P. Hence

$$[r(x+y)]^{-1}\le a^{-1}+b^{-1}\le(r(x)-t)^{-1}+(r(y)-t)^{-1}.$$

Since t is an arbitrary positive scalar this establishes the triangle inequality

$$\|x+y\|\le\|x\|+\|y\|$$

for elements of P.

In addition, if $r(x)\ge r(y)$ then one may take $a\ge b$ and, using (B), we find that

$$b(x\vee y)\le b[(ab^{-1}x)\vee y]\le z.$$

Therefore,

$$r(x\vee y)\ge b\ge r(y)-t$$

so that $r(x\vee y)\ge r(y)$. Similarly, if $r(y)\ge r(x)$ then $r(x\vee y)\ge r(x)$. Thus,

$$\|x\vee y\|=|r(x\vee y)|^{-1}\le|\min\{r(x),r(y)\}|^{-1}=\|x\|\vee\|y\|.$$

To establish the triangle inequality for arbitrary elements x and y in V observe that (A), the monotonicity of $\|\ \|$ on P, and the fact that the triangle inequality holds on P implies that

$$\|(x+y)^+\|\le[\|x^+\|\vee\|x^-\|]+[\|y^+\|\vee\|y^-\|]=\|x\|\vee\|y\|$$

and that

$$\|(x+y)^-\|\le\|x\|+\|y\|.$$

Thus,

$$\|x+y\|\le\|x\|+\|y\|$$

for any x and y in V.

If a is a positive scalar and x is in P, then $r(ax)=a^{-1}r(x)$ and $\|ax\|=|a|\,\|x\|$. If a is a positive scalar and x is an arbitrary element of V the relation (B) shows that

$$\|ax\|=\|(ax)^+\|\vee\|(ax)^-\|=|a|\,\|x\|.$$

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Now $(-x)^+ = x^-$ and $(-x)^- = x^+$ so that $\|-x\| = \|x\|$. If a is a negative scalar, then

$$\|ax\| = \| \, |a| \, (-x)\| = |a| \, \|x\|.$$

It remains to show that $\||x|\| = \|x\|$. However, $|x| - x^+$ and $|x| - x^-$ are in P so that

$$\|x\| = \|x^+\| \vee \|x^-\| \le \||x|\| = \|x^+ \vee x^-\| \le \|x^+\| \vee \|x^-\| = \|x\|.$$

The space ℓ_1 with the usual norm $\|x\| = \sum_{i=1}^{\infty} |x_i|$ is not a normed M-space, because if x and y are sequences with a one in their first and second coordinates respectively and zeros elsewhere, then $\|x \vee y\| = 2$ while $\|x\| = 1$ and $\|y\| = 1$. This space does satisfy the conditions of our theorem, however, with $D = \{x \in P : |x_i| \le 1 \text{ for } i = 1, 2, 3, \ldots\}$ and

$$P = \{x \in \ell_1 : x_i \ge 0 \text{ for } i = 1, 2, 3, \ldots\}.$$

The resulting M-norm is, of course, the sup-norm for ℓ_1.

Now consider the space X of all real valued Lebesgue measurable functions on $[0, 1]$ ordered by taking the cone P to be the set of all functions which are almost everywhere non-negative. Then X is a boundedly complete (hence archimedian) linear lattice by Theorem 13 in Birkhoff [**1**, p. 361]. If P contained a set D which satisfied conditions (1) and (2) then an M-norm could be defined on X and Kakutani [**4**] has shown that non-zero lattice homomorphisms exist on such spaces. However, Kelley and Namioka [**5**, p. 55] show there is no non-negative linear functional on X which is not identically zero. Hence, there are no non-zero lattice homomorphisms on X, so X cannot contain a set D satisfying conditions (1) and (2).

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