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Equivariant images of projective space under the action of SL $(n, \mathbb{Z})$

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The point of this note is to answer in the affirmative a question of G. A. Margulis. In the course of his proof of the finiteness of either the cardinality or the index of a normal subgroup of an irreducible lattice in a higher rank semi-simple Lie group [**3**], [**4**], Margulis proves that if $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$, (X, μ) is a measurable Γ-space, μ quasi-invariant, and $\phi : \mathbb{P}^{n-1} \to X$ is a measure class preserving Γ-map, then either ϕ is a measure space isomorphism or μ is supported on a point. Margulis then asks whether the topological analogue of this result is true. This is answered in the following.

THEOREM. *Let* $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ *be a subgroup commensurable with* $\mathrm{SL}(n, \mathbb{Z})$, *where* $n \geq 3$. *Suppose* X *is a Hausdorff* Γ*-space and that there exists a continuous surjective* Γ*-map* $\phi : \mathbb{P}^{n-1} \to X$. *Then either* ϕ *is a homeomorphism or* X *is a point.*

For the proof, we collect some basic results on orbits of some natural actions of discrete linear groups. An action of a group on a metric space is called minimal if every orbit is dense. A p-frame, $1 \leq p \leq n$, in $\mathbb{R}^n$ is an ordered linearly independent set with p elements. The set of p-frames has a natural topology, and $\mathrm{GL}(n, \mathbb{R})$ acts transitively on the set of p-frames (p fixed). Following [**1**], we call a p-frame irrational if its $\mathbb{R}$-linear span contains no elements of $\mathbb{Q}^n$ other than 0. The results we need are the following.

(1) [**5**, lemma 8.5]. *If* $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ *is a lattice, then* Γ *acts minimally on* $\mathbb{P}^{n-1}$. (We recall that a lattice is a discrete subgroup with a fundamental domain of finite volume.)

(2) [**1**, theorem 3.5]. *For* $1 \leq p \leq n-1$, *the orbit of any irrational p-frame under* $\mathrm{SL}(n, \mathbb{Z})$ *is dense in the space of p-frames.*

(3) [**1**, proposition 4.2]. *Let* $V \subset \mathbb{R}^n$ *be a k-dimensional subspace with a basis in* $\mathbb{Q}^n$, *and* $\{v_1, \ldots, v_p\}$ *an irrational p-frame such that*

$$\mathrm{span}\{v_1, \ldots, v_p\} \cap V = \{0\}$$

and

$$p + k \leq n - 1.$$

Let

$$\Gamma_0 = \{\gamma \in \mathrm{SL}(n, \mathbb{Z}) \mid \gamma v = v \text{ for all } v \in V\}.$$

Then the Γ_0*-orbit of* $\{v_1, \ldots, v_p\}$ *is dense in the space of p-frames.*

Remarks. (*a*) We also have:

(4) [**6**, theorem 1.4]. *If* $\Gamma \subset \mathrm{SL}(n, \mathbb{Z})$ *is a uniform lattice* (*i.e.* $\mathrm{SL}(n, \mathbb{Z})/\Gamma$ *is also compact*), *then* Γ *acts minimally on the space of p-frames,* $1 \leq p \leq n-1$. (See also [**2**] for $p = 1$.)

This result easily implies the theorem if Γ is assumed to be a uniform lattice in $\mathrm{SL}(n, \mathbb{R})$, $n \geq 3$. Namely, suppose that $x, y \in \mathbb{P}^{n-1}$ and $\phi(x) = \phi(y)$ with $x \neq y$. Then, for any $z \neq x$, by Veech's theorem there exists a sequence $\gamma_n \in \Gamma$ such that

$$\gamma_n x \to x \quad \text{and} \quad \gamma_n y \to z.$$

Since ϕ is a Γ-map,

$$\phi(x) = \phi(z),$$

and so X is a point.

(*b*) Suppose the Baire category theorem holds in a Γ-space Y, where Γ is a discrete group, and Γ_y is dense for some $y \in Y$. Then, if $\Gamma_0 \subset \Gamma$ is of finite index, $\overline{\Gamma_0 y}$ contains an open set containing y. To see this, simply observe that, if

$$\Gamma = \bigcup \gamma_i \Gamma_0$$

for some finite set of γ_i, then

$$Y = \bigcup \gamma_i (\overline{\Gamma_0 y}).$$

One of these sets must contain an open set and, since γ_i acts by homeomorphisms, they all do. Choosing such an open set in $\overline{\Gamma_0 y}$ and saturating under the Γ_0 action verifies our assertion.

If $v \in \mathbb{R}^n$, $v \neq 0$, we let $[v]$ be its image in $\mathbb{P}^{n-1}$. We call $[v]$ rational if there is a point $w \in \mathbb{Q}^n$ with $[w] = [v]$. We assume the situation of the theorem. It clearly suffices to assume that $\Gamma \subset \mathrm{SL}(n, \mathbb{Z})$ is of finite index.

LEMMA. *Suppose* $[v] \in \mathbb{P}^{n-1}$ *is rational and X is not a point. Then*

$$\phi^{-1}(\phi([v])) = \{[v]\}.$$

Proof. Suppose first that

$$\phi([v]) = \phi(x),$$

where x is not rational. Let Γ_0 be as in (3) above, for $V = \mathbb{R}v$. By (3), $\Gamma_0 x$ is dense in $\mathbb{P}^{n-1}$. Since $\Gamma \cap \Gamma_0$ is of finite index in Γ_0, $\overline{(\Gamma \cap \Gamma_0)x}$ contains an open set U. But, for $\gamma \in \Gamma \cap \Gamma_0$,

$$\phi(\gamma x) = \phi([v]),$$

hence $\phi^{-1}(\phi([v]))$ contains U. Since $\Gamma \cdot U = \mathbb{P}^{n-1}$ by (1) above, there exist finitely many γ_i with

$$\bigcup_i \gamma_i U = \mathbb{P}^{n-1}.$$

This implies that

$$\mathbb{P}^{n-1} = \bigcup_i \phi^{-1}(\gamma_i \phi([v])),$$

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which implies by connectedness that

$$\phi(\mathbb{P}^{n-1}) = \phi([v]).$$

This contradicts the assumption that X is not a point.

On the other hand, suppose that

$$\phi([v]) = \phi([w]),$$

where $[w]$ is also rational. Then we can assume that $v, w \in \mathbb{Z}^n$, and we can choose a basis B of $\mathbb{R}^n$ containing v and w and consisting of elements of $\mathbb{Z}^n$. Let L be the lattice in $\mathbb{R}^n$ generated by B. Then the subgroup $\Gamma_1 \subset \Gamma$ which takes L onto itself is of finite index in Γ and consists of integral matrices when expressed as transformations of $\mathbb{R}^n$ in terms of the basis B. In fact, the image of Γ_1 in SL $(n, \mathbb{Z})$ with respect to this representation is of finite index in SL $(n, \mathbb{Z})$. Let $A \subset \mathbb{R}^n$ be the subspace spanned by $B - \{v\}$, and $[A]$ be the image in $\mathbb{P}^{n-1}$. Let

$$\Gamma_2 = \{\gamma \in \Gamma_1 \mid \gamma v = v, \gamma(A) = A\}.$$

We can view Γ_2 as a subgroup in SL (A) and, clearly, Γ_2 is a lattice in this group. Then, by (1) above, $\Gamma_2[w]$ is dense in $[A]$. But for $\gamma \in \Gamma_2$,

$$\phi(\gamma[w]) = \phi(\gamma[v]) = \phi([v]).$$

Thus,

$$\phi([A]) = \phi([v]).$$

But, since $\dim A \geq 2$, $[A]$ contains non-rational points. By the first part of the argument, this is impossible. This proves the lemma. □

We now complete the proof of the theorem. Suppose X is not a point and

$$\phi([v]) = \phi([w]), \quad [v] \neq [w].$$

If $\{v, w\}$ is an irrational 2-frame, then, by (3) and remark (*b*) above, there is an open set $U \subset \mathbb{P}^{n-1}$ containing $[w]$ such that, for any $x \in U$, there exists a sequence $\gamma_n \in \Gamma$ such that

$$\gamma_n[v] \to [v] \quad \text{and} \quad \gamma_n[w] \to x.$$

Thus

$$\phi(x) = \phi([v])$$

for all $x \in U$ and, arguing as in the proof of the lemma, we deduce that X is a point. Thus, in light of the lemma, it suffices to consider the case in which $[v]$, $[w]$ are both non-rational, but the linear span of $\{v, w\}$ contains a non-zero vector $y \in \mathbb{Q}^n$. Then we can write (changing v, w by scalars, if necessary),

$$w = v + y.$$

Let Γ_0 be as in (3) above with $V = \mathbb{R}y$, and let $\{\alpha_i\} \subset \Gamma_0$ be a finite set with

$$\Gamma_0 = \bigcup \alpha_i(\Gamma \cap \Gamma_0).$$

By (3), for each positive integer N we can find a sequence $\gamma_n \in \Gamma \cap \Gamma_0$ and $\alpha_{i(N)} \in \{\alpha_i\}$ such that

$$\alpha_{i(N)}\gamma_n v \to v/N.$$

For some sequence $N_j \to \infty$, we can assume that all

$$\alpha_{i(N_j)} = \alpha.$$

So, for each j, we have a sequence $\gamma_n \in \Gamma \cap \Gamma_0$ such that

$$\gamma_n v \to \alpha^{-1} v/N_j$$

and so

$$\gamma_n w \to (\alpha^{-1} v/N_j) + y.$$

Since ϕ is a Γ-map,

$$\phi([\alpha^{-1}v]) = \phi([(\alpha^{-1}v/N_j) + y]),$$

for all N_j. Letting $N_j \to \infty$, we obtain

$$\phi([\alpha^{-1}v]) = \phi([y]).$$

However, α^{-1} is an integral matrix, so $[\alpha^{-1}v]$ is not rational. Since $[y]$ is rational, this contradicts the lemma. This completes the proof of the theorem. □

Margulis' results in [4] for the measure theoretic case are significantly more general than the result we quoted above. It would be of interest to determine whether the topological analogues of these more general results are also true.

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