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Some Properties of Rational Functions with Prescribed Poles

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Abstract. Let $P(z)$ be a polynomial of degree not exceeding n and let $W(z) = \prod_{j=1}^{n}(z - a_j)$ where $|a_j| > 1$, $j = 1, 2, \dots, n$. If the rational function $r(z) = P(z)/W(z)$ does not vanish in $|z| < k$, then for $k = 1$ it is known that

$$|r'(z)| \leq \frac{1}{2}|B'(z)| \operatorname*{Sup}_{|z|=1} |r(z)|$$

where $B(Z) = W^*(z)/W(z)$ and $W^*(z) = z^n\overline{W(1/\bar{z})}$. In the paper we consider the case when $k > 1$ and obtain a sharp result. We also show that

$$\operatorname*{Sup}_{|z|=1}\left\{\left|\frac{r'(z)}{B'(z)}\right| + \left|\frac{\big(r^*(z)\big)'}{B'(z)}\right|\right\} = \operatorname*{Sup}_{|z|=1} |r(z)|$$

where $r^*(z) = B(z)\overline{r(1/\bar{z})}$, and as a consquence of this result, we present a generalization of a theorem of O'Hara and Rodriguez for self-inversive polynomials. Finally, we establish a similar result when supremum is replaced by infimum for a rational function which has all its zeros in the unit circle.

1 Introduction and Statement of Results

Let $\mathcal{P}_n$ denote the class of all complex polynomials of degree at most n. Let D_{k-} denote the region inside the circle $T_k := \{z\,;|z| = k > 0\}$ and D_{k+} the region outside T_k. For $a_j \in C$ with $j = 1, 2, \dots, n$, write

$$W(z) = \prod_{j=1}^{n}(z - a_j), \quad B(z) = \prod_{j=1}^{n}\left(\frac{1 - \bar{a}_j z}{z - a_j}\right)$$

and

$$\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \dots, a_n) = \frac{P(z)}{W(z)}; \quad P \in \mathcal{P}_n.$$

Then $\mathcal{R}_n$ is the set of all rational functions with poles $a_1, a_2, \dots, a_n$ at most and with finite limit at infinity. We observe that $B(z) \in \mathcal{R}_n$. For f defined on T_k in the complex plane, we set $\|f\| = \operatorname{Sup}_{z\in T_k} |f(z)|$, the Chebyshev norm of f on T_1. Throughout this paper, we shall always assume that all poles $a_1, a_2, \dots, a_n$ are in D_{1+}.

The following famous result is due to Bernstein (for reference see [8]).

Theorem A *If $P \in \mathcal{P}_n$, then $\|P'\| \leq n\|P\|$.*

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As a refinement of Theorem A, we mention the following result due to A. Aziz [2] and M. A. Malik [5].

Theorem B *If $P \in \mathcal{P}_n$ and $P^*(z) = z^n\overline{P(1/\bar{z})}$, then*

$$\| \, |(P^*(z))'| + |P(z)| \, \| = n\|P\|.$$

The next result was conjectured by P. Erdős and later verified by P. D. Lax [4].

Theorem C *If $P \in \mathcal{P}_n$ and all the zeros of $P(z)$ lie in $T_1 \cup D_{1+}$, then for $z \in T_1$, we have*

$$\|P'\| \le \frac{n}{2}\|P\|. \tag{1}$$

Equality in (1) holds for $P(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta|$.

Recently Li, Mohapatra, and Rodriguez [6] (see also [3]) have proved Bernstein-type inequalities similar to Theorem A and Theorem C for rational functions with prescribed poles where they replaced z^n by Blaschke product $B(z)$. Among other things, they proved the following generalization of Theorem C.

Theorem D *Suppose $r \in \mathcal{R}_n$ and all the zeros of r lie in $T_1 \cup D_{1+}$, then for $z \in T_1$, we have*

$$|r'(z)| \le \frac{1}{2}|B'(z)| \, \|r\|. \tag{2}$$

Equality in (2) holds for $r(z) = \alpha B(z) + \beta$ with $|\alpha| = |\beta| = 1$.

We first prove the following generalization of Theorem D.

Theorem 1 *Suppose $r \in \mathcal{R}_n$ and all the zeros of r lie in $T_k \cup D_{k+}$, where $k \ge 1$, then for $z \in T_1$, we have*

$$|r'(z)| \le \frac{1}{2}\left\{|B'(z)| - \frac{n(k-1)}{(k+1)} \cdot \frac{|r(z)|^2}{\|r\|^2}\right\}\|r\|. \tag{3}$$

Equality in (3) holds for $r(z) = \big((z+k)/(z-a)\big)^n$ where $a > 1$, $k \ge 1$, and $B(z) = \big((1-az)/(z-a)\big)^n$ evaluated at $z = 1$.

For $k = 1$, this reduces to Theorem D.

The next result is a generalization of Theorem B for rational functions.

Theorem 2 *If $r \in \mathcal{R}_n$ and $r^*(z) = B(z)\overline{r(1/\bar{z})}$, then we have*

$$\operatorname*{Sup}_{z \in T_1}\left\{\left|\frac{r'(z)}{B'(z)}\right| + \left|\frac{(r^*(z))'}{B'(z)}\right|\right\} = \operatorname*{Sup}_{z \in T_1}|r(z)|. \tag{4}$$

Moreover, the suprema of both sides in (4) are attained at the same point $z_0 \in T_1$.

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The rational function $r \in \mathcal{R}_n$ is self-inversive if $r^*(z) = ur(z)$ for $u \in T_1$. The following result, which is a generalization of Theorem 1 of [7] for self-inversive polynomials, easily follows from Theorem 2.

Corollary 1 *If $r \in \mathcal{R}_n$ is a self-inversive rational function, then*

$$2 \operatorname*{Sup}_{z \in T_1} \left| \frac{r'(z)}{B'(z)} \right| = \operatorname*{Sup}_{z \in T_1} |r(z)|.$$

Finally we establish the following result which is a generalization of the corresponding result for polynomials [1, Remark 1].

Theorem 3 *Suppose $r \in \mathcal{R}_n$ has n zeros and all the zeros of r lie in $T_1 \cup D_{1-}$. If $r^*(z) = B(z)\overline{r(1/\bar{z})}$, then for $z \in T_1$,*

$$\operatorname*{Inf}_{z \in T_1} \left\{ \left| \frac{r'(z)}{B'(z)} \right| - \left| \frac{\big(r^*(z)\big)'}{B'(z)} \right| \right\} = \operatorname*{Inf}_{z \in T_1} |r(z)|. \tag{5}$$

Moreover, the infima of both sides in (5) are attained at the same point $z_0 \in T_1$.

From Theorem 3 we can easily deduce

Corollary 2 *Suppose $r \in \mathcal{R}_n$ and all the zeros of r lie in $T_1 \cup D_{1-}$, then for $z \in T_1$ we have*

$$\operatorname*{Inf}_{z \in T_1} \left\{ \left| \frac{r'(z)}{B'(z)} \right| \right\} \geq \operatorname*{Inf}_{z \in T_1} \{ |r(z)| \}.$$

Corollary 2 is a generalization of Theorem 1 of [1].

Lemmas

For the proof of these theorems we need the following lemmas. The first result is due to Li, Mohapatra and Rodriguez [6, Theorem 2].

Lemma 1 *If $r \in \mathcal{R}_n$ and $r^*(z) = B(z)\overline{r(1/\bar{z})}$, then for $z \in T_1$ we have*

$$\big|\big(r^*(z)\big)'\big| + |r'(z)| \leq |B'(z)|\, \|r\|. \tag{6}$$

Equality in (6) holds for $r(z) = uB(z)$ with $u \in T_1$.

Lemma 2 *If $z \in T_1$, then*

$$\operatorname{Re}\left(\frac{zW'(z)}{W(z)} \right) = \frac{n - |B(z)|}{2}$$

and

$$\operatorname{Re}\left(\frac{z\big(W^*(z)\big)'}{W^*(z)} \right) = \frac{n + |B(z)|}{2}$$

where $W(z) = \prod_{j=1}^{n}(z - a_j)$, and $W^*(z) = z^n \overline{W(1/\bar{z})}$.

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Proof of Lemma 2 We have $W^*(z) = z^n \overline{W(1/\bar{z})}$. By direct calculation, we obtain

$$z\big(W^*(z)\big)' = nz^n\overline{W(1/\bar{z})} - z^{n-1}\overline{W'(1/\bar{z})}. \tag{7}$$

Now for $z \in T_1$, we have $\bar{z} = 1/z$ and from (7), we get

$$\frac{z\big(W^*(z)\big)'}{W^*(z)} = n - \overline{\left(\frac{zW'(z)}{W(z)}\right)}.$$

This gives for $z \in T_1$

$$\operatorname{Re}\frac{z\big(W^*(z)\big)'}{W^*(z)} + \operatorname{Re}\frac{zW'(z)}{W(z)} = n. \tag{8}$$

Also we have

$$B(z) = \prod_{j=1}^{n}\left(\frac{1-\bar{a}_j z}{z-a_j}\right),$$

which gives

$$\begin{aligned}\frac{zB'(z)}{B(z)} &= \sum_{j=1}^{n} z\left(\frac{-\bar{a}_j}{1-\bar{a}_j z} - \frac{1}{z-a_j}\right)\\ &= \sum_{j=1}^{n}\frac{z(|a_j|^2-1)}{(1-\bar{a}_j z)(z-a_j)}.\end{aligned}$$

This implies for $z \in T_1$,

$$\frac{zB'(z)}{B(z)} = \sum_{j=1}^{n}\frac{|a_j|^2-1}{|z-a_j|^2} > 0.$$

Hence

$$|B'(z)| = \frac{zB'(z)}{B(z)} \quad \text{for } z \in T_1. \tag{9}$$

Also, we have

$$\frac{zB'(z)}{B(z)} = \frac{z\big(W^*(z)\big)'}{W^*(z)} - \frac{zW'(z)}{W(z)}. \tag{10}$$

From (9) and (10), we get for $z \in T_1$

$$|B'(z)| = \frac{z\big(W^*(z)\big)'}{W^*(z)} - \frac{zW'(z)}{W(z)}.$$

This gives for $z \in T_1$,

$$\operatorname{Re}\left(\frac{z\big(W^*(z)\big)'}{W^*(z)}\right) - \operatorname{Re}\left(\frac{zW'(z)}{W(z)}\right) = |B'(z)|. \tag{11}$$

Finally, from (8) and (11), we obtain for $z \in T_1$,

$$\operatorname{Re}\left(\frac{z\big(W^*(z)\big)'}{W^*(z)}\right) = \frac{n + |B(z)|}{2}$$

and

$$\operatorname{Re}\left(\frac{zW'(z)}{W(z)}\right) = \frac{n - |B'(z)|}{2}.$$

This completes the proof of Lemma 2.

Proofs of the Theorems

Proof of Theorem 1 Let $r(z) = P(z)/W(z) \in \mathcal{R}_n$; if $b_1, b_2, \ldots, b_m$ are all the zeros of $P(z)$, then $m \le n$, $|b_j| \ge k > 1$, $j = 1, 2, \ldots, m$ and we have

$$\begin{aligned}\frac{zr'(z)}{r(z)} &= \frac{zP'(z)}{P(z)} - \frac{zW'(z)}{W(z)}\\ &= \sum_{j=1}^{m} \frac{z}{z - b_k} - \frac{zW'(z)}{W(z)}.\end{aligned}$$

For $z \in T_1$, this gives, with the help of Lemma 2, that

$$\begin{aligned}\operatorname{Re}\frac{zr'(z)}{r(z)} &= \operatorname{Re}\sum_{j=1}^{m} \frac{z}{z - b_k} - \operatorname{Re}\frac{zW'(z)}{W(z)}\\ &= \operatorname{Re}\sum_{j=1}^{m} \frac{z}{z - b_k} - \left(\frac{n - |B'(z)|}{2}\right).\end{aligned} \tag{12}$$

Now it can be easily verified that for $z \in T_1$, $|b| \ge k > 1$,

$$\operatorname{Re}\left(\frac{z}{z - b}\right) \le \frac{1}{1 + k}.$$

Using this in (12), we get for $z \in T_1$,

$$\begin{aligned}\operatorname{Re}\frac{zr'(z)}{r(z)} &\le \frac{m}{1 + k} - \left(\frac{n - |B'(z)|}{2}\right)\\ &\le \frac{n}{1 + k} - \frac{n}{2} + \frac{|B'(z)|}{2}\\ &= \frac{|B'(z)|}{2} - \frac{n(k - 1)}{2(k + 1)}.\end{aligned}$$

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Hence for $z \in T_1$ we have [6, p. 529],

$$
\begin{aligned}
\left|\frac{z\big(r^*(z)\big)'}{r(z)}\right|^2 &= \left||B'(z)| - \frac{zr'(z)}{r(z)}\right|^2 \\
&= |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)| \operatorname{Re} \frac{zr'(z)}{r(z)} \\
&\geq |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - |B'(z)|\left\{|B'(z)| - \frac{n(k-1)}{k+1}\right\} \\
&= \left|\frac{zr'(z)}{r(z)}\right|^2 + \frac{n(k-1)}{(k+1)}|B'(z)|.
\end{aligned} \tag{13}
$$

This implies for $z \in T_1$,

$$
\left\{|r'(z)|^2 + \frac{n(k-1)}{(k+1)}|r(z)|^2\,|B'(z)|\right\}^{\frac{1}{2}} \leq \big|\big(r^*(z)\big)'\big|.
$$

Combining this with Lemma 1, we get

$$
|r'(z)| + \left\{|r'(z)|^2 + \frac{n(k-1)}{(k+1)}|r(z)|^2\,|B'(z)|\right\}^{\frac{1}{2}} \leq |B'(z)|\,\|r\|
$$

or equivalently,

$$
\begin{aligned}
|r'(z)|^2 + \frac{n(k-1)}{(k+1)}|r(z)|^2\,|B'(z)| &\leq \{|B'(z)|\,\|r\| - |r'(z)|\}^2 \\
&= |B'(z)|^2\,\|r\|^2 - 2|B'(z)|\,|r'(z)|\,\|r\| + |r'(z)|^2
\end{aligned}
$$

which after a short simplification yields for $z \in T_1$ that

$$
|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{n(k-1)}{(k+1)}\frac{|r(z)|^2}{\|r\|^2}\right\}\|r\|.
$$

This proves (3).

To show equality in (3) holds for $r(z) = \big((z+k)/(z-a)\big)^n$ and $B(z) = \big((1-az)/(z-a)\big)^n$, $a > 1$, $k \geq 1$ at $z = 1$, we observe that

$$
\sup_{z\in T_1} |r(z)| = \left\{\frac{(1+k)}{(a-1)}\right\}^n = |r(1)| = \|r\|, \quad B(1) = 1,
$$

$$
|r'(1)| = n\left(\frac{1+k}{a-1}\right)^{n-1}\left(\frac{(k+a)}{(a-1)^2}\right) \quad \text{and} \quad |B'(1)| = \frac{n(a+1)}{(a-1)}.
$$

It can now be seen easily that two sides of (3) are equal. This completes the proof of Theorem 1.

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Proof of Theorem 2 Let $r(z) \in \mathcal{R}_n$, then we have

$$r^*(z) = B(z)\overline{r(1/\bar{z})}.$$

A straightforward computation shows that

$$\frac{z\big(r^*(z)\big)'}{r^*(z)} = \frac{zB'(z)}{B(z)} - \frac{\overline{r'(1/\bar{z})}}{\overline{zr(1/\bar{z})}}.$$

This implies with the help of (9) for $z \in T_1$ that

$$\frac{z\big(r^*(z)\big)'}{r^*(z)} = |B'(z)| - \overline{\left(\frac{zr'(z)}{r(z)}\right)},$$

which gives

$$\left|\frac{z\big(r^*(z)\big)'}{r^*(z)}\right| + \left|\overline{\left(\frac{zr'(z)}{r(z)}\right)}\right| \geq \left|\overline{\left(\frac{zr'(z)}{r(z)}\right)} + \frac{z\big(r^*(z)\big)'}{r^*(z)}\right| = |B'(z)|. \tag{14}$$

Using the fact that for $z \in T_1$, $|r(z)| = |r^*(z)|$, it follows from (14) that

$$|r'(z)| + |\big(r^*(z)\big)'| \geq |B'(z)|\,|r(z)|. \tag{15}$$

This gives

$$\operatorname*{Sup}_{z\in T_1}\left\{\left|\frac{r'(z)}{B'(z)}\right| + \left|\frac{\big(r^*(z)\big)'}{B'(z)}\right|\right\} \geq \operatorname*{Sup}_{z\in T_1}|r(z)|. \tag{16}$$

Also from Lemma 1, we easily get

$$\operatorname*{Sup}_{z\in T_1}\left\{\left|\frac{r'(z)}{B'(z)}\right| + \left|\frac{\big(r^*(z)\big)'}{B'(z)}\right|\right\} \leq \operatorname*{Sup}_{z\in T_1}|r(z)|. \tag{17}$$

From (16) and (17), we obtain

$$\operatorname*{Sup}_{z\in T_1}\left\{\left|\frac{r'(z)}{B'(z)}\right| + \left|\frac{\big(r^*(z)\big)'}{B'(z)}\right|\right\} = \operatorname*{Sup}_{z\in T_1}|r(z)|.$$

This proves (4).

We now show the suprema of both sides in (4) are attained at the same point $z_0 \in T_1$. Let

$$\operatorname*{Sup}_{z\in T_1}|r(z)| = |r(z_0)|;$$

then from Lemma 1, we get

$$\left|\frac{r'(z_0)}{B'(z_0)}\right| + \left|\frac{\big(r^*(z_0)\big)'}{B'(z_0)}\right| \leq |r(z_0)| \quad z_0 \in T_1. \tag{18}$$

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Also from (15), we have

$$\left|\frac{r'(z_0)}{B'(z_0)}\right| + \left|\frac{\big(r^*(z_0)\big)'}{B'(z_0)}\right| \geq |r(z_0)| \quad z_0 \in T_1. \tag{19}$$

From (18) and (19), we obtain

$$\left|\frac{r'(z_0)}{B'(z_0)}\right| + \left|\frac{\big(r^*(z_0)\big)'}{B'(z_0)}\right| = |r(z_0)|.$$

This completes the proof of Theorem 2.

Proof of Theorem 3 Suppose all the n zeros of $r \in \mathcal{R}_n$ lie in $T_1 \cup D_{1-}$ and let $m = \operatorname{Inf}_{z\in T_1} |r(z)|$, then we have $m \leq |r(z)|$ for $z \in T_1$. We show for every complex number α with $|\alpha| < 1$, the rational function $F(z) = r(z) + \alpha m$ has all its zeros in $T_1 \cup D_{1-}$. This is obvious if $m = 0$, that is, if $r(z)$ has a zero on T_1. So we suppose all the zeros of $r(z)$ lie in D_{1-} so that $m \neq 0$ and we have

$$|\alpha m| < m \leq |r(z)| \quad \text{for } z \in T_1.$$

Applying Rouché's theorem, it follows that $F(z) = r(z) + \alpha m$ has all its zeros in D_{1-}. Hence in any case $F(z)$ has all its zeros in $T_1 \cup D_{1-}$ for every α with $|\alpha| < 1$. Let

$$\begin{aligned} F^*(z) &= B(z)\overline{F(1/\bar{z})} \\ &= B(z)\overline{r(1/\bar{z})} + \bar{\alpha} m B(z) \\ &= r^*(z) + \bar{\alpha} m B(z). \end{aligned}$$

Then all the zeros of $F^*(z)$ lie in $T_1 U D_{1+}$. Now it follows from (13) with $k = 1$ and r replaced by F^*,

$$|\big(F^*(z)\big)'| \leq |F'(z)| \quad \text{for } z \in T_1,$$

or equivalently,

$$|\big(r^*(z)\big)' + \bar{\alpha} m B'(z)| \leq |r'(z)| \quad \text{for } z \in T_1.$$

Choosing argument of α suitably, we get

$$|\big(r^*(z)\big)'| + |\alpha| m |B'(z)| \leq |r'(z)| \quad \text{for } z \in T_1.$$

Letting $|\alpha| \to 1$, we obtain

$$|r'(z)| - |\big(r^*(z)\big)'| \geq m|B'(z)| \quad \text{for } z \in T_1. \tag{20}$$

which implies

$$\operatorname*{Inf}_{z\in T_1}\left\{\left|\frac{r'(z)}{B'(z)}\right| - \left|\frac{\big(r^*(z)\big)'}{B'(z)}\right|\right\} \geq \operatorname*{Inf}_{z\in T_1} |r(z)|. \tag{21}$$

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Again $r^*(z) = B(z)\overline{r(1/\bar{z})}$ and it can be easily verified that for $z \in T_1$,

$$\begin{aligned} |\big(r^*(z)\big)'| &= |B'(z)r(z) - r'(z)B(z)| \\ &\geq |r'(z)| - |B'(z)|\,|r(z)|. \end{aligned}$$

This gives

$$|B'(z)|\,|r(z)| \geq |r'(z)| - |\big(r^*(z)\big)'| \quad \text{for } z \in T_1 \tag{22}$$

$$\operatorname*{Inf}_{z\in T_1}\left\{\left|\frac{r'(z)}{B'(z)}\right| - \left|\frac{\big(r^*(z)\big)'}{B'(z)}\right|\right\} \leq \operatorname*{Inf}_{z\in T_1}|r(z)| = m. \tag{23}$$

Combining (21) and (23), we get

$$\operatorname*{Inf}_{z\in T_1}\left\{\left|\frac{r'(z)}{B'(z)}\right| - \left|\frac{\big(r^*(z)\big)'}{B'(z)}\right|\right\} = \operatorname*{Inf}_{z\in T_1}|r(z)|.$$

This proves (5).

We now show the infima of both sides in (5) are attained at the same point $z_0 \in T_1$. Let

$$\operatorname*{Inf}_{z\in T_1}|r(z)| = |r(z_0)|.$$

Then from (20), we get

$$\left|\frac{r'(z_0)}{B'(z_0)}\right| - \left|\frac{\big(r^*(z_0)\big)'}{B'(z_0)}\right| \geq |r(z_0)| \quad \text{for } z_0 \in T_1. \tag{24}$$

Also from (22), we obtain

$$\left|\frac{r'(z)}{B'(z_0)}\right| - \left|\frac{\big(r^*(z_0)\big)'}{B'(z_0)}\right| \leq |r(z_0)|.$$

From (24) and (25), it follows that

$$\left|\frac{r'(z_0)}{B'(z_0)}\right| - \left|\frac{\big(r^*(z_0)\big)'}{B'(z_0)}\right| = |r(z_0)|.$$

This completes the proof of Theorem 3.

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