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PARTITION FORCING AND INDEPENDENT FAMILIES

JORGE A. CRUZ-CHAPITAL, VERA FISCHER, OSVALDO GUZMÁN, AND JAROSLAV ŠUPINA

Abstract. We show that Miller partition forcing preserves selective independent families and P-points, which implies the consistency of $\mathrm{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} < \mathfrak{a}_T = \omega_2$. In addition, we show that Shelah's poset for destroying the maximality of a given maximal ideal preserves tight mad families and so we establish the consistency of $\mathrm{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{i} = \omega_1 < \mathfrak{u} = \mathfrak{a}_T = \omega_2$.

§1. Introduction. One of the oldest questions regarding the theory of cardinal invariants of the continuum is the following question of Vaughan [50]: Is the inequality $\mathfrak{i} < \mathfrak{a}$ consistent?[1] The problem involves two fundamental objects in infinite combinatorics (maximal independent families and MAD families) and moreover, a positive answer will most likely require the development of new ideas and forcing techniques.[2] In order to gain more insight into the above question, we compare $\mathfrak{i}$ with the following cardinal invariant introduced by Miller in [40].

Definition. Define $\mathfrak{a}_T$ as the smallest size of a partition of ω^ω into compact sets.

It is well-known that the Baire space ω^ω is not σ-compact (see [32]), which implies that $\mathfrak{a}_T$ is uncountable. Furthermore, $\mathfrak{d}$ is the least size of a family of compact sets covering ω^ω (see [3]), so it follows that $\mathfrak{d} \leq \mathfrak{a}_T$. It is known that the compact subspaces of the Baire space are in correspondence with the finitely branching subtrees of $\omega^{<\omega}$. Using this correspondence and König's lemma, it is easy to prove that $\mathfrak{a}_T$ is equal to the least size of a maximal AD family of finitely branching subtrees of $\omega^{<\omega}$. Džamonja, Hrušák, and Moore proved that $\diamondsuit_{\mathfrak{d}}$ implies that $\mathfrak{a}_T = \omega_1$ (see Theorem 7.6 of [43]). Thus, since $\diamondsuit_{\mathfrak{d}}$ holds in most of the natural models of $\mathfrak{d} = \omega_1$ (see [28, 43] for a precise formulation of this statement), $\mathfrak{a}_T = \omega_1$ also holds in these models.

On the other hand, given a partition $\mathcal{C}$ of ω^ω into compact sets, Miller introduced a proper forcing notion $\mathbb{Q}(\mathcal{C})$ which has the Laver property and destroys $\mathcal{C}$, that is, $\mathcal{C}$ no longer covers ω^ω after forcing with $\mathbb{Q}(\mathcal{C})$. The forcing notion is known as Miller partition forcing and plays an important role in the current article (see Definition 2.1). Spinas showed that $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding, which together with Miller's result establishes the Sacks property of $\mathbb{Q}(\mathcal{C})$ (see [49]). Thus, every partition of ω^ω into compact sets can be destroyed with a proper forcing that has the Sacks

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[1] For definitions of cardinal characteristics, we refer the reader to [6].
[2] An interesting discussion of the problem can be found in Section 6.

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property, which implies the consistency of $cof(\mathcal{N}) < \mathfrak{a}_T$ and in particular the consistency of $\mathfrak{d} < \mathfrak{a}_T$. In [52, Proposition 4.1.31], Zapletal proved that $\mathbb{Q}(\mathcal{C})$ is forcing equivalent to the quotient of the Borel sets of ω^ω modulo a σ-ideal generated by closed sets. In this way, the forcing $\mathbb{Q}(\mathcal{C})$ falls into the scope of the theory developed in [51, 52].

In this article, we study the effect of Miller partition forcing on the independence number $\mathfrak{i}$ and obtain the consistency of $\mathfrak{i} < \mathfrak{a}_T$. The key argument is the fact that Miller partition forcing preserves selective independent families, a fact for which we provide two proofs: one building on the notion of fusion with witnesses (see Definition 2.5) and one building on Laflamme's filter games (see [34]). Both, the fusion with witnesses and the use of Laflamme's filter game in the context of Miller's partition forcing, are highly innovative and do not occur in earlier work on $\mathbb{Q}(\mathcal{C})$.

The notion of *selective independent family* was introduced by Shelah in his work on the consistency of $\mathfrak{i} < \mathfrak{u}$ (see [46]). Selective independent families are families with very strong combinatorial properties, which resemble the combinatorial features of Ramsey ultrafilters. Studying the similarities and differences between selective independent families and Ramsey ultrafilters remains a very interesting line of research. For more recent work on maximal independent families see [13, 14, 17, 18, 45].

Employing our notion of fusion with witnesses, we show also that $\mathbb{Q}(\mathcal{C})$ preserves *P*-points. Together with the fact that Miller partition forcing and its iterations preserve tight mad families (see [25]), we obtain the consistency of the following constellation:

THEOREM. *It is relatively consistent that* $\mathfrak{i} = \mathfrak{a} = \mathfrak{u} = \omega_1 < \mathfrak{a}_T$.

The question if one can increase simultaneously $\mathfrak{u}$ and $\mathfrak{a}_T$, while preserving small witnesses to $\mathfrak{a}$ and $\mathfrak{i}$ becomes of interest. Further, we show that Shelah's poset $\mathbb{Q}_\mathcal{I}$ for destroying the maximality of a given maximal ideal from [46] strongly preserves tight `MAD` families. The following result appears as Corollary 5.14 in the current article:

THEOREM. *It is relatively consistent that* $\mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{u}$.

Finally, combining Miller partition forcing, Shelah's $\mathbb{Q}_\mathcal{I}$, our preservation results, as well as the preservation results of [25, 46], in Corollary 5.15 we obtain:

THEOREM. *It is relatively consistent that* $\mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{u} = \mathfrak{a}_T = \omega_2$.

§2. Miller partition forcing.

2.1. Fusion with witnesses. Recall that Sacks forcing $\mathbb{S}$ consists of all perfect trees in $2^{<\omega}$ ordered by inclusion. That is, $p \in \mathbb{S}$ if and only if:

(1) $p \subseteq 2^{<\omega}$,
(2) $\forall \sigma \in p\ \forall \tau \in 2^{<\omega}\big(\tau \subseteq \sigma \to \tau \in p\big)$,
(3) $\forall \sigma \in p\ \exists \tau, \tau' \in p\big(\sigma \subseteq \tau \wedge \sigma \subseteq \tau' \wedge \tau \not\subseteq \tau' \wedge \tau' \not\subseteq \tau\big)$.

We will use standard notation: If $p \in \mathbb{S}$ and $\sigma \in p$ we let $p(\sigma) = \{\tau \in p \mid \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}$ and call σ a splitting node if $\sigma^\frown i \in p$ for each $i \in 2$. Let $split(p) = \{\sigma \in p \mid \sigma \text{ is a splitting node}\}$. For each $n \in \omega$ let $split_n(p) = \{\sigma \in split(p) \mid |\{\tau \in split(p) \mid \tau \subsetneq \sigma\}| = n\}$ and $stem(p)$ the unique element in $split_0(p)$. Finally, for

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$p \in \mathbb{S}$ let $[p] = \{f \in 2^\omega \mid \forall n \in \omega(f|_n \in p)\}$. More about Sacks forcing can be found in [4, 5, 15, 23, 29, 42, 53].

DEFINITION 2.1 (Miller partition forcing). Let $\mathcal{C} \subseteq \mathcal{P}(2^\omega)$ be an uncountable partition of 2^ω into closed sets and let

$$\mathbb{Q}(\mathcal{C}) = \{p \in \mathbb{S} \mid \text{for every } K \in \mathcal{C},\ K \cap [p] \text{ is nowhere dense in } [p]\}$$

ordered by reversed inclusion.

This forcing destroys the partition $\mathcal{C}$ in the following way. If G is a $\mathbb{Q}(\mathcal{C})$-generic filter, then

$$r_{gen} = \bigcup \bigcap G$$

is an element of 2^ω which does not belong to the interpretation in $V[G]$ of any element of $\mathcal{C}$. So in $V[G]$, $\mathcal{C}$ is no longer a partition of 2^ω. Thus, if we start with a model of CH and define $\mathbb{PM}$ as the resulting model after forcing with a countable support iteration of length ω_2 of all forcing notions of the form $\mathbb{Q}(\mathcal{C})$ with $\mathcal{C}$ ranging over all uncountable partitions in closed sets of 2^ω in all intermediate models, then $\mathbb{PM}$ will not have any uncountable partition in closed sets of 2^ω of size less than ω_2. Note that, Miller defined and used $\mathbb{PM}$ in [40] to show that $cov(\mathcal{M}) = \omega_1$ does not imply that $\mathfrak{a}_T = \omega_1$.

Notice that if $\mathcal{C}$ is the partition of 2^ω into singletons, then $\mathbb{Q}(\mathcal{C}) = \mathbb{S}$. Actually, it can be seen that if $\mathcal{C}$ is an analytic subset of $K(2^\omega)$, where $K(2^\omega)$ is the space of non-empty closed subsets of 2^ω equipped with the Vietoris topology, then $\mathbb{Q}(\mathcal{C})$ is forcing equivalent to Sacks forcing $\mathbb{S}$.

THEOREM 2.2. *Let $\mathcal{C} \subseteq K(2^\omega)$ be an uncountable analytic partition of 2^ω. Then $\mathbb{Q}(\mathcal{C})$ is forcing equivalent to Sacks forcing $\mathbb{S}$.*

PROOF. Let $p \in \mathbb{Q}(\mathcal{C})$. It is enough to find $q \in \mathbb{Q}(\mathcal{C})$ such that $q \leq p$ and $\{r \in \mathbb{Q}(\mathcal{C}) \mid r \leq q\} = \{r \in \mathbb{S} \mid r \subseteq q\}$. To do this consider continuous $f : K(2^\omega) \longrightarrow 2^\omega$ given by $f(A) = minA$ and let $X = \{K \cap [p] \mid K \in \mathcal{C}\} \setminus \{\emptyset\}$. Notice that X is an uncountable analytic subset of $K(2^\omega)$, $f|_X$ is injective, and $im(f|_X) \subseteq [p]$. This implies that there is $q \in \mathbb{S}$ such that $[q] \subseteq im(f|_X)$. It is easy to see that $q \subseteq p$ and $|[q] \cap K| \leq 1$ for every $K \in \mathcal{C}$. Checking that q is as desired is straightforward. ⊣

A main difficulty in adapting Sacks fusion sequences to $\mathbb{Q}(\mathcal{C})$ is guaranteeing that the fusion is indeed an element of $\mathbb{Q}(\mathcal{C})$. To have a better control over fusion sequences in $\mathbb{Q}(\mathcal{C})$ we introduce the notion of a fusion with witnesses (see Definition 2.5). We begin with some auxiliary notions.

DEFINITION 2.3. Let $\mathcal{C} = \{C_\alpha\}_{\alpha \in \omega_1}$ be an uncountable partition of 2^ω into closed sets.

(1) We say that $x, y \in {}^\omega 2$ are $\mathcal{C}$-different if x, y belong to different elements of $\mathcal{C}$.
(2) A tree $p \subseteq 2^{<\omega}$ is said to be $\mathcal{C}$-branching if for any $s \in p$ there are $\mathcal{C}$-different branches in $[p]$ extending s.

Note that, a $\mathcal{C}$-branching tree is perfect. We will use the following notation: whenever $\mathcal{C}$ as above is given, for each $x \in 2^\omega$ we denote by α_x the unique ordinal such that $x \in C_{\alpha_x}$.

The equivalence of (a) and (c) in the lemma below can be found in [25].

Lemma 2.4. *Let $p \subseteq 2^{<\omega}$ be a tree. The following are equivalent*:

(a) $p \in \mathbb{Q}(\mathcal{C})$.
(b) *p is $\mathcal{C}$-branching.*
(c) *p is perfect and $[p]$ contains a countable dense subset with $\mathcal{C}$-different branches.*

Proof. ((a) $\Rightarrow$ (c)) Let $p \in \mathbb{Q}(\mathcal{C})$. p is a perfect tree by the definition. Thus arrange $\operatorname{split}(p)$ and assign by induction to each splitting node s a real x from $[p]$ extending s which was either already considered or belongs to a different set from $\mathcal{C}$ than all previously selected reals. This is possible since any $s \in \operatorname{split}(p)$ may be extended to $t \in \operatorname{split}(p)$ with $[p(t)]$ being disjoint with finitely many sets from $\mathcal{C}$ containing all previously selected reals. The set of all assigned branches is the required dense set.

((c) $\Rightarrow$ (b)) Trivial.

((b) $\Rightarrow$ (a)) Let $\beta < \omega_1$ and $s \in p$. There are $x, y \in [p]$ such that $s \subseteq x, y$ and $\alpha_x \neq \alpha_y$. We take $z \in \{x, y\}$ such that $\alpha_z \neq \beta$. Since $z \in [p] \setminus C_\beta$ and C_β is closed, there is $s \subseteq t \subseteq z$ such that $[p(t)] \cap C_\beta = \emptyset$. ⊣

The particular enumeration constructed in Lemma 2.4 will be applied several times. Therefore we state explicitly that we may assume the dense set in Lemma 2.4 is enumerated as $\{x_t : t \in p\}$ such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$.

Definition 2.5 (Fusion sequence with witnesses).

(1) Let p be a condition in $\mathbb{Q}(\mathcal{C})$. We say that a set $X \subseteq {}^\omega 2$ is a p-witness for the n-th level if $X \subseteq [p]$, for each $s \in \operatorname{split}_n(p)$ there is $x \in X$ extending s, and X has $\mathcal{C}$-different elements. Note that if X is a p-witness for the $(n+1)$-st level then each node from the n-th splitting level of p is contained in $\mathcal{C}$-different branches in X.
(2) Let $(p, X), (q, Y)$ be pairs with p, q conditions in $\mathbb{Q}(\mathcal{C})$. Let X be a p-witness for the $(n+1)$-st level and let Y be a q-witness for the n-th level. Then

$$(p, X) \leq^n (q, Y) \text{ if and only if } p \leq q \text{ and } X \supseteq Y.$$

Note that if $(p, X) \leq^n (q, Y)$ then $\operatorname{split}_{<n}(p) = \operatorname{split}_{<n}(q)$.
(3) A sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses if $(p_{n+1}, X_{n+1}) \leq^n (p_n, X_n)$ for each n.

Lemma 2.6. *If a sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses then the fusion $\bigcap\{p_n : n \in \omega\}$ is a condition in $\mathbb{Q}(\mathcal{C})$.*

Proof. We denote $p = \bigcap\{p_n : n \in \omega\}$, $X = \bigcup\{X_n : n \in \omega\}$, and we assume that we have $s \in p$. We take $n \in \omega$ and $t \in \operatorname{split}_n(p)$ such that t extends s. Since $\operatorname{split}_n(p) = \operatorname{split}_n(p_{n+1})$, the set X_{n+1} contains $\mathcal{C}$-different branches extending t. Hence, X is dense in $[p]$. One can easily see that X is contained in $[p]$. Finally, by Lemma 2.4 we conclude that $p \in \mathbb{Q}(\mathcal{C})$. ⊣

Miller [40] and Spinas [49] applied separate fusion arguments in their proofs, while Miller [40] introduced the notion of a fusion even formally. The partial order $\mathbb{Q}(\mathcal{C})$ was recently used in [25], where the notion of a nice sequence was isolated from Spinas's fusion arguments. Our definition of fusion sequence covers both approaches. The sequence $\{X_n\}_{n\in\omega}$ in our definition may be obtained as sets of

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leftmost branches in Miller's fusion argument, and as certain terms of nice sequence in Spinas's approach.

In addition to fusion sequences, we shall use two basic schemas to amalgamate conditions. Let us have a condition $p \in \mathbb{Q}(\mathcal{C})$, and for each $s \in \operatorname{split}_n(p)$, $i \in \{0,1\}$, a condition $q(s,i)$ extending $p(s^\frown i)$. Using Lemma 2.4, one can easily see that the tree

$$q = \bigcup \{q(s,i) \colon s \in \operatorname{split}_n(p), i \in \{0,1\}\}$$

is a condition in $\mathbb{Q}(\mathcal{C})$ as well. In the second amalgamation technique, we are given a decreasing sequence $\{q_i\}_{i\in\omega}$ of extensions of p with strictly increasing stems $s_n = \operatorname{stem} q_n$. We set $x = \bigcup_{i\in\omega} s_i$ and take $q = \bigcup_{i\in\omega} q_i(s_i^\frown\langle 1 - x(|s_i|)\rangle)$. Again, using Lemma 2.4, one can easily see that q is a condition in $\mathbb{Q}(\mathcal{C})$.

The proof of the fact that $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding is underlying many fusion arguments associated with $\mathbb{Q}(\mathcal{C})$. For convenience of the reader, we repeat it here. We will make use of the following two lemmas.

Lemma 2.7. *Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$ and let h be a function in ${}^\omega\omega \cap V$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$: There is a real $x \in [q]$ and a sequence $\{f_s\}_{s\in x\restriction \operatorname{split}(q)}$ of functions in ${}^{<\omega}\omega$ such that for any $s = x \restriction \operatorname{split}_n(q)$ we have $q(s) \Vdash \dot{f} \restriction h(n) = f_s$.*

Proof. Let $p \in \mathbb{Q}(\mathcal{C})$. One can construct a decreasing sequence $\{q_i\}_{i\in\omega}$ of extensions of p with strictly increasing stems such that $q_n \Vdash \dot{f} \restriction h(n) = f_n$ for some $f_n \in {}^{h(n)}\omega$. We denote $s_n = \operatorname{stem}(q_n)$ and we set $x = \bigcup_{i\in\omega} s_i$. Finally, we take the amalgamation $q = \bigcup_{i\in\omega} q_i(s_i^\frown\langle 1 - x(|s_i|)\rangle)$. ⊣

Lemma 2.8. *Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$: For all $m \in \omega$, for all $t \in \operatorname{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that*

$$q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t.$$

Proof. Let $p \in \mathbb{Q}(\mathcal{C})$. We build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \le p$ such that its fusion q has the required property. Let the condition q_0, branch x, and sequence $\{f_s\}_{s\in x\restriction \operatorname{split}(q_0)}$ be obtained from Lemma 2.7 for p and $h(n) = n+1$. We set $X_0 = \{x\}$.

Let $0 \le n < \omega$. Suppose we have defined $q_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [q_n]$. Let $s \in \operatorname{split}_n(q_n)$. Take the unique branch $x \in X_n$ extending s, node $r = x \restriction \operatorname{split}_{n+1}(q_n)$, and number $i = x(|s|)$ in $\{0,1\}$. We set $q(s,i) = q_n(r)$. Let $t \supseteq s^\frown\langle 1-i\rangle$ be such that $[q_n(t)] \cap C_{\alpha_x} = \emptyset$ for all already considered branches x (i.e., all branches in X_n and those assigned to previous nodes in some order of $\operatorname{split}_n(q_n)$). Use Lemma 2.7 for $q_n(t)$ and $h(j) = n + j + 2$ to obtain $q(s, 1-i) \le q_n(t)$, branch x, and sequence $\{f_s\}_{s\in x\restriction \operatorname{split}(q_n)}$.

Finally, let X_{n+1} be the set of all considered branches in this step, and

$$q_{n+1} = \bigcup \{q(s,i) \colon s \in \operatorname{split}_n(q_n), i \in \{0,1\}\}.$$

One can verify that the sequence $\{(q_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses. ⊣

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As an application, we obtain a straightforward proof of the fact that $\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding. Since the poset has the Laver property (see [40]), this also gives the Sacks property of $\mathbb{Q}(\mathcal{C})$.

LEMMA 2.9 (Spinas [49]). *The poset $\mathbb{Q}(\mathcal{C})$ has the Sacks property.*

PROOF. As explained above, by Miller's result it is sufficient to show that $\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding. Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$ and let $p \in \mathbb{Q}(\mathcal{C})$. We will show that there is $q \leq p$ and $g \in V \cap {}^{\omega}\omega$ such that $q \Vdash \dot{f} \leq^* \check{g}$.

By Lemma 2.8 we can assume that there is $q \leq p$ such that for all $m \in \omega$, for all $t \in \mathrm{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that $q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t$. Define $g \in {}^{\omega}\omega$ as follows:

$$g(n) = \max\{f_s(n) + 1 \colon s \in \mathrm{split}_n(q)\}.$$

Then $q \Vdash \forall n(\dot{f}(n) < g(n))$. ⊣

2.2. Preservation of *P*-points. Next, we show that Miller partition forcing preserves P-points. We will make use of the following notation: Given $\mathcal{G} \subseteq \mathcal{P}(\omega)$, let $\langle \mathcal{G} \rangle_{\mathrm{up}} = \{X \in \mathcal{P}(\omega) \colon \exists G \in \mathcal{G}(G \subseteq X)\}$ and $\langle \mathcal{G} \rangle_{\mathrm{dn}} = \{X \in \mathcal{P}(\omega) \colon \exists G \in \mathcal{G} (X \subseteq G)\}$.

THEOREM 2.10. *The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters.*

PROOF. We prove just the first part. The second claim follows from the first one and the fact that the forcing notion $\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding (see [26, Lemma 21.12]). Note that a filter base $\mathcal{G}$ generates an ultrafilter on ω if and only if $\mathcal{P}(\omega) = \langle \mathcal{G} \rangle_{\mathrm{up}} \cup \langle \mathcal{G}^* \rangle_{\mathrm{dn}}$.

Let $\mathcal{U}$ be a P-point in V. We shall prove that the family $\mathcal{U}$ generates an ultrafilter in $V^{\mathbb{Q}(\mathcal{C})}$, i.e., $V^{\mathbb{Q}(\mathcal{C})} \models \mathcal{P}(\omega) = \langle \mathcal{U} \rangle_{\mathrm{up}} \cup \langle \mathcal{U}^* \rangle_{\mathrm{dn}}$. Fix $p \in \mathbb{Q}(\mathcal{C})$ and a $\mathbb{Q}(\mathcal{C})$-name $\dot{Y}$ such that $p \Vdash \dot{Y} \subseteq \omega$. By Lemma 2.8 we can assume that for all $m \in \omega$, for all $t \in \mathrm{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that

$$p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t.$$

Note that the latter property remains true for any stronger condition q, since t in the m-th level of q is an extension of some s in the m-th level of p. Let $\{x_t \colon t \in p\} \subseteq [p]$ be a dense set in $[p]$ containing $\mathcal{C}$-different elements (enumerated such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$). Let $Y_t = \bigcup\{u_s \colon s \subseteq x_t\}$.

CLAIM. We can assume that $\mathcal{Y}_0 = \{Y_s \colon s \in p\} \subseteq \mathcal{U}$ or $\mathcal{Y}_1 = \{\omega \setminus Y_s \colon s \in p\} \subseteq \mathcal{U}$.

PROOF. We set $U_0 = \{s \in p \colon Y_s \in \mathcal{U}\}$ and $U_1 = \{s \in p \colon (\omega \setminus Y_s) \in \mathcal{U}\}$. The sets U_0, U_1 are disjoint and their union is p. We may distinguish two cases:

(i) There is $s \in p$ such that $p(s) \subseteq U_0$. In this case, just take $p(s)$.

(ii) For each $s \in p$ there is $t \in p(s)$ such that $t \in U_1$. We build a fusion sequence $\{(p_n, X_n)\}_{n \in \omega}$ such that the fusion has the required properties. Taking $s \in \mathrm{split}_0(p)$ there is $t \in p(s)$ such that $t \in U_1$. We set $p_0 = p(t)$ and $X_0 = \{x_t\}$.

Let $0 \leq n < \omega$. Suppose we have defined $p_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [p_n]$. Let $s \in \mathrm{split}_n(p_n)$. Take node $r = x_s \restriction \mathrm{split}_{n+1}(p_n)$, and number $i = x_s(|s|)$ in $\{0, 1\}$. We set $p(s, i) = p_n(r)$. Let $t \supseteq s^\frown\langle 1 - i \rangle$ be splitting such that $t \in U_1$. We set $p(s, 1 - i) = p(t)$.

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Finally, let

$$p_{n+1} = \bigcup\{p(s,i)\colon s \in \mathrm{split}_n(p_n), i \in \{0,1\}\}.$$

and let X_{n+1} be the set of all x_t's for $t \in \mathrm{split}_{n+1}(p_{n+1})$. One can verify that the sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses. Moreover the fusion q of this sequence satisfies that $(\omega\setminus Y_s) \in \mathcal{U}$ for all $s \in q$. ⊣

We assume that $\mathcal{Y}_0 \subseteq \mathcal{U}$, the other case may be handled analogously. We take a pseudointersection Z of $\mathcal{Y}_0$ in $\mathcal{U}$, with $Z \subseteq Y_\emptyset$. We shall simultaneously build two fusion sequences with witnesses, namely $\{(p_n^0, X_n^0)\}_{n\in\omega}$, $\{(p_n^1, X_n^1)\}_{n\in\omega}$, and a partition of Z into two sets Z_0, Z_1 such that for their respective fusions $q_0, q_1 \leq p$ we obtain $q_0 \Vdash \check{Z}_0 \subseteq \dot{Y}$ and $q_1 \Vdash \check{Z}_1 \subseteq \dot{Y}$.

Let $p_0^0 = p_0^1 = p$, $X_0^0 = X_0^1 = \{Y_\emptyset\}$, and $k_0 = 0$, $k_1 = 2$. We assume that p_n^0, p_n^1, k_{2n}, and k_{2n+1} are constructed. Let $t \in \mathrm{split}_{k_{2n}}(p) \cap \mathrm{split}(p_n^0)$, and set $w^*(t) = x_t \restriction \mathrm{split}_{k_{2n+1}}(p)$. For each $i \in \{0,1\}$, we take $w^*(t,i) \in \mathrm{split}_{k_{2n+1}+1}(p)$ extending $w^*(t)^\frown i$. There is $k_{2n+2} > k_{2n+1}+1$ such that

$$Z \setminus k_{2n+2} \subseteq \bigcap\{Y_{w^*(t,i)}\colon t \in \mathrm{split}_{k_{2n}}(p) \cap \mathrm{split}(p_n^0), i \in \{0,1\}\}.$$

We set $w(t,i) = x_{w^*(t,i)} \restriction \mathrm{split}_{k_{2n+2}}(p)$. Take $p_{n+1}^0 = \bigcup\{p(w(t,i))\colon t \in \mathrm{split}_{k_{2n}}(p) \cap \mathrm{split}(p_n^0), i \in \{0,1\}\}$ and $X_{n+1}^0 = \{x_{w(t,i)}\colon t \in \mathrm{split}_{k_{2n}}(p) \cap \mathrm{split}(p_n^0), i \in \{0,1\}\}$. One can see that $p_n^0 \Vdash \check{Z} \cap [k_{2n}, k_{2n+1}) \subseteq \dot{Y}$. The construction of condition p_n^1 and the choice of number k_{2n+3} are done similarly, and leads to $p_n^1 \Vdash \check{Z} \cap [k_{2n+1}, k_{2n+2}) \subseteq \dot{Y}$. Finally, we define

$$Z_0 = Z \cap \bigcup\{[k_{2n}, k_{2n+1})\colon n \in \omega\} \text{ and } Z_1 = Z \cap \bigcup\{[k_{2n+1}, k_{2n+2})\colon n \in \omega\}.$$

Since $Z \in \mathcal{U}$, Z_0 or Z_1 is in $\mathcal{U}$, and so $q_0 \Vdash \dot{Y} \in \langle\mathcal{U}\rangle_{\mathrm{up}}$ or $q_1 \Vdash \dot{Y} \in \langle\mathcal{U}\rangle_{\mathrm{up}}$. ⊣

§3. Selective independence.

3.1. Dense maximality. Recall the definition:

Definition 3.1. A family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an *independent family* if for every distinct $A_0, \dots, A_n \in \mathcal{A}$ and $h : \{A_0, \dots, A_n\} \longrightarrow 2$, the set $\bigcap_{i\leq n} A_i^{h(A_i)}$ is infinite where $A_i^0 = \omega\setminus A_i$ and $A_i^1 = A_i$. It is *maximal independent*, if it is independent and maximal under inclusion.

We will be exclusively interested in infinite independent families. For an independent family $\mathcal{A}$ let $\mathrm{FF}(\mathcal{A})$ be the set of all finite partial functions from $\mathcal{A}$ to 2 and order it by inclusion. For $h \in \mathrm{FF}(\mathcal{A})$, we let $\mathcal{A}^h = \bigcap\{A^{h(A)} \mid A \in dom(h)\}$ where $A^0 = \omega\setminus A$ and $A^1 = A$ for $A \subseteq \omega$. The *density ideal* of $\mathcal{A}$, denoted $\mathrm{id}(\mathcal{A})$ is the set of all $X \subseteq \omega$ such that for all $h \in \mathrm{FF}(\mathcal{A})$ there is $h' \supseteq h$ in $\mathrm{FF}(\mathcal{A})$ such that $\mathcal{A}^{h'} \cap X$ is finite (or equivalently empty). Dual to the density ideal of $\mathcal{A}$ is the density filter of $\mathcal{A}$ denoted $\mathrm{fil}(\mathcal{A})$ and consisting of all $X \subseteq \omega$ such that for all $h \in \mathrm{FF}(\mathcal{A})$ there is $h' \supseteq h$ in $\mathrm{FF}(\mathcal{A})$ such that $\mathcal{A}^{h'}\setminus X$ is finite (or equivalently empty).[3]

[3]In the notation of [14], $\mathrm{fil}(\mathcal{A}) = \mathcal{F}_\mathcal{A}$ and $\mathbf{C}_\mathcal{A} = \mathrm{FF}(\mathcal{A})$. The density ideal and filter have been also studied in [17].

LEMMA 3.2. *Let $\mathcal{A}$ be an infinite independent family. The following are equivalent*:

(1) *For all $X \in \mathcal{P}(\omega)$ and all $h \in \mathrm{FF}(\mathcal{A})$ there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \cap X$ or $\mathcal{A}^{h'} \backslash X$ is finite.*
(2) *For all $h \in \mathrm{FF}(\mathcal{A})$ and all $X \subseteq \mathcal{A}^h$ either $\mathcal{A}^h \backslash X \in \mathrm{id}(\mathcal{A})$ or there is $h' \in \mathrm{FF}(\mathcal{A})$ such that $h' \supseteq h$ and $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \backslash X$.*
(3) *For each $X \in \mathcal{P}(\omega) \backslash \mathrm{fil}(\mathcal{A})$ there is $h \in \mathrm{FF}(\mathcal{A})$ such that $X \subseteq \omega \backslash \mathcal{A}^h$.*

PROOF. First we show that (1) implies (2). Let $h \in \mathrm{FF}(\mathcal{A})$, let $X \subseteq \mathcal{A}^h$, and suppose $\mathcal{A}^h \backslash X \notin \mathrm{id}(\mathcal{A})$. Thus, there is $h' \in \mathrm{FF}(\mathcal{A})$ such that for all $h'' \supseteq h'$ the set $\mathcal{A}^{h''} \cap (\mathcal{A}^h \backslash X)$ is non-empty. Note that if h and h' are incompatible, then $\mathcal{A}^{h'} \cap (\mathcal{A}^h \backslash X) = \emptyset$, which is a contradiction. Therefore h and h' are compatible and without loss of generality, we can assume that $h' \supseteq h$. Thus, we have that for all $h'' \supseteq h'$, the set $\mathcal{A}^{h''} \backslash X \neq \emptyset$. Now, since (1) holds, there is $h'' \supseteq h'$ such that $\mathcal{A}^{h''} \cap X = \emptyset$. That is, $\mathcal{A}^{h''} \subseteq \mathcal{A}^h \backslash X$.

Next, we show that (2) implies (3). Thus, consider any $X \in \mathcal{P}(\omega) \backslash \mathrm{fil}(\mathcal{A})$. Then, in particular $\omega \backslash X \notin \mathrm{id}(\mathcal{A})$ and so there is $h \in \mathrm{FF}(\mathcal{A})$ such that for all $h' \supseteq h$, $|\mathcal{A}^{h'} \cap (\omega \backslash X)| = |\mathcal{A}^{h'} \backslash X| = \omega$. Let $Y = \mathcal{A}^h \backslash X$. Thus, $Y \subseteq \mathcal{A}^h$. By part (2) either $\mathcal{A}^h \backslash Y \in \mathrm{id}(\mathcal{A})$ or there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \backslash Y$. Suppose $\mathcal{A}^h \backslash Y \in \mathrm{id}(\mathcal{A})$. Then, there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \cap (\mathcal{A}^h \backslash Y) = \mathcal{A}^{h'} \backslash Y = \emptyset$. However $\mathcal{A}^{h'} \backslash Y = \mathcal{A}^{h'} \cap X = \emptyset$ and so $X \subseteq \omega \backslash \mathcal{A}^{h'}$ and we are done. If there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \backslash Y = \mathcal{A}^h \cap X$, then $\mathcal{A}^{h'} \cap (\omega \backslash X) = \mathcal{A}^{h'} \backslash X = \emptyset$, contradicting the choice of h.

To see that (3) implies (2), consider any $h \in \mathrm{FF}(\mathcal{A})$ and $X \subseteq \mathcal{A}^h$. Let $Y = \mathcal{A}^h \backslash X$. If $\omega \backslash Y \in \mathrm{fil}(\mathcal{A})$, then $Y = \mathcal{A}^h \backslash X \in \mathrm{id}(\mathcal{A})$. Otherwise, there is h^* such that $\omega \backslash Y \subseteq \omega \backslash \mathcal{A}^{h^*}$, which implies that $\mathcal{A}^{h^*} \subseteq Y = \mathcal{A}^h \backslash X$ and so $\mathcal{A}^{h^* \cup h} \subseteq \mathcal{A}^{h^*} \subseteq \mathcal{A}^h \backslash X$.

To see that (2) implies (1) consider any $X \in [\omega]^\omega$ and let $h \in \mathrm{FF}(\mathcal{A})$. We want to show that there is $h' \supseteq h$ such that either $\mathcal{A}^{h'} \cap X = \emptyset$, or $\mathcal{A}^{h'} \backslash X = \emptyset$. Let $Y = X \cap \mathcal{A}^h$. Thus, $Y \subseteq \mathcal{A}^h$. If $\mathcal{A}^h \backslash Y \in \mathrm{id}(\mathcal{A})$, then $\mathcal{A}^h \backslash X \in \mathrm{id}(\mathcal{A})$ and so there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \cap (\mathcal{A}^h \backslash X) = \mathcal{A}^{h'} \backslash X = \emptyset$. Otherwise, there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \backslash Y$ and so $\mathcal{A}^{h'} \cap Y = \emptyset$. However, $\mathcal{A}^{h'} \cap Y = \mathcal{A}^{h'} \cap (X \cap \mathcal{A}^h) = \mathcal{A}^{h'} \cap X = \emptyset$. ⊣

An independent family $\mathcal{A}$ is said to be *densely maximal* if any one of the above three properties holds. Note that if an independent family is densely maximal, then it is also maximal. The notion of dense maximality of independent families appears (to the best knowledge of the authors) for the first time in [24]. In particular, we obtain:

COROLLARY 3.3. *Let $\mathcal{A}$ be an infinite independent family. Then, $\mathcal{A}$ is densely maximal iff*

$$\mathcal{P}(\omega) = \mathrm{fil}(\mathcal{A}) \cup \langle \omega \backslash \mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A}) \rangle_{dn}.$$[4]

The fact that partial orders has the Sacks property implies in particular:

LEMMA 3.4 [14]. *Let W be a $\mathbb{P}$-generic extension of V, where $\mathbb{P}$ has the Sacks property. If $\mathcal{A} \in V$ is an independent family, then in W, $\mathrm{fil}(\mathcal{A})$ is generated by $\mathrm{fil}(\mathcal{A})^V$.*

[4]Thus, in the notation of [14], $\mathcal{A}$ is densely maximal iff $\mathcal{P}(\omega) = \mathcal{F}_{\mathcal{A}} \cup \langle \mathcal{C}_{\mathcal{A}} \rangle_{dn}$.

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3.2. Selectivity. Recall the following definitions. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$. Then $\mathcal{F}$ is centered if for every finite subfamily $\mathcal{H}$, $\bigcap \mathcal{H} \in \mathcal{F}$; $\mathcal{F}$ is a P-set, if every countable subfamily has a pseudo-intersection in $\mathcal{F}$; $\mathcal{F}$ is a Q-set, if for every partition $\mathcal{E}$ of ω into finite sets, there is an $X \in \mathcal{F}$ meeting each element of the partition in at most one point, i.e., $|X \cap E| \leq 1$ for each $E \in \mathcal{E}$.

Definition 3.5. Let $\mathcal{F}$ be a filter over ω containing the Fréchet filter. We say that $\mathcal{F}$ is a *selective filter* if and only if for every partition $\{X_i\}_{i\in\omega}$ of ω into elements of $\mathcal{F}^*$, where $\mathcal{F}^*$ is the dual ideal of $\mathcal{F}$, there exists $Y \in \mathcal{F}$ such that $|Y \cap X_i| \leq 1$ for each $i \in \omega$.

Note that a filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is selective (also called Ramsey) if and only if $\mathcal{F}$ extends the Fréchet filter and is both a P-set and a Q-set.

Definition 3.6. An independent family is said to be *selective* if it is densely maximal and $\text{fil}(\mathcal{A})$ is a selective filter.

Selective independent families exist under CH, a result which is due to Shelah (see [46]). Further studies of selective independent families can be found in [14, 17].

Of particular interest for us will be the following combinatorial characterization of Q-filters which is similar to a characterization of happy families (see Proposition 0.7 in the work of Mathias [37] or Proposition 11.6 in [26]), as well as the fact that Q-filters are preserved by ${}^{\omega}\omega$-bounding forcing notions (see Lemma 3.8).

Lemma 3.7. *Let $\mathcal{F}$ be a filter. The following are equivalent*:

(a) *$\mathcal{F}$ is a Q-filter.*

(b) *For any increasing function $f \in {}^{\omega}\omega$ there is $\{k(n) \colon n \in \omega\} \in \mathcal{F}$ such that $f(k(n)) < k(n+1)$.*

Proof. ((a) $\Rightarrow$ (b)) Note that the partition relations occurring in this proof, and so implicitly the proof itself, can be found in [46]. Inductively, choose a sequence $\{n(l)\}_{l\in\omega}$ such that $n(0) = 0$ and

$$n(l+1) = \min\{n \colon n(l) < n \text{ and } \forall m \leq n(l)(f(m) \leq n)\}.$$

We consider the partition $\mathcal{E}_0 = \{[n(3l), n(3l+3))\}_{l\in\omega}$. There is $C_1 \in \mathcal{F}$ such that C_1 is a selector for $\mathcal{E}_0$. Now, consider the relation $\mathcal{E}_1$ on C_1 defined as follows:

$$m \sim_{\mathcal{E}_1} k \text{ iff } m = k \vee m < k \leq f(m) \vee k < m \leq f(k).$$

Note that $\mathcal{E}_1$ is clearly reflexive and symmetric. Transitivity holds, since no three pairwise distinct elements of C_1 are $\mathcal{E}_1$-equivalent: Indeed, suppose $m_1 < m_2 < m_3$ are such that $m_1 \mathcal{E}_1 m_2$ and $m_2 \mathcal{E}_1 m_3$. Then $m_1 < m_2 < m_3 \leq f(m_1)$. There are $l_1 < l_2 < l_3$ such that $m_i \in [n(3l_i), n(3l_i + 3))$. Then $m_1 < n(3l_2) \leq m_2 < n(3l_3) \leq m_3 \leq f(m_1)$. However, on the other hand by the definition of sequence $\{n(l)\}_{l\in\omega}$ we have $f(m_1) \leq n(3l_2 + 1) < n(3l_3)$, a contradiction. Thus, $\mathcal{E}_1$ is an equivalence relation on C_1 and each $\mathcal{E}_1$-equivalence class has at most two elements.

Extend $\mathcal{E}_1$ to an equivalence relation $\mathcal{E}_2$ on ω by defining

$$m \sim_{\mathcal{E}_2} k \text{ iff } m = k \vee m \sim_{\mathcal{E}_1} k.$$

There is C_2 in $\mathcal{F}$ such that C_2 is a selector for $\mathcal{E}_2$. Without loss of generality $C_2 \subseteq C_1$ and $0 \in C_2$. Let $\{k(n)\}_{n\in\omega}$ enumerate in increasing order C_2. Thus for all n, n' we

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have that $k(n) \not\sim_{\mathcal{E}_2} k(n')$. Thus, if $n < n'$ then $k(n') \not\leq f(k(n))$ and so for all $n \in \omega$, $f(k(n)) < k(n+1)$.

((b) $\Rightarrow$ (a)) Let $\mathcal{E}$ be a partition of ω into finite sets. We set

$$f(n) = \max \bigcup \{E \in \mathcal{E} \colon (\exists i \leq n)\ i \in E\}.$$

There is $\{k(n) \colon n \in \omega\} \in \mathcal{F}$ such that $f(k(n)) < k(n+1)$ for each $n \in \omega$. The set $\{k(n) \colon n \in \omega\}$ is a selector for $\mathcal{E}$. Indeed, $k(n) \leq f(k(n)) < k(n+1)$ and therefore $k(n+1)$ is from a different set of partition $\mathcal{E}$ than all $k(i)$ for $i \leq n$. $\dashv$

In particular, we get the following, which we state for completeness.

Lemma 3.8. *An ${}^{\omega}\omega$-bounding forcing notion preserves Q-filters.*

Proof. If $\mathbb{P}$ is an ${}^{\omega}\omega$-bounding forcing notion, $\mathcal{F}$ a Q-filter in V, then we use part (2) of Lemma 3.7 for $f \in V \cap {}^{\omega}\omega$ dominating function $g \in V^{\mathbb{P}} \cap {}^{\omega}\omega$. $\dashv$

Selective filter has a property similar to Mathias' notion of a happy family [37] (see [26] as well). Note that Mathias [37, Proposition 0.10] has shown that an ultrafilter $\mathcal{G}$ is Ramsey if and only if $\mathcal{G}$ is happy (see Proposition 11.7 in [26] as well).

Lemma 3.9. *Let $\mathcal{F}$ be a filter over ω containing the Fréchet filter. The following are equivalent*:

(a) *$\mathcal{F}$ is selective.*
(b) *For any sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of finite subsets of $\mathcal{F}$ there is $a \in \mathcal{F}$ such that*

$$a(n+1) \in \bigcap \mathcal{G}_{a(n)}.$$

Proof. ((a) $\Rightarrow$ (b))$\mathcal{F}$ is a P-set and therefore there is $C_0 \in \mathcal{F}$ such that $C_0 \subseteq^* G$ for each $G \in \bigcup \{\mathcal{G}_n \colon n \in \omega\}$. Thus, for some function $f \in {}^{\omega}\omega$

$$(\forall n \in \omega)\ C_0 \setminus f(n) \subseteq \bigcap \mathcal{G}_n.$$

Let us take $\{k(n) \colon n \in \omega\} \in \mathcal{F}$ from Lemma 3.7 such that $C = \{k(n+1) \colon n \in \omega\} \subseteq C_0$. Hence, we have $k(n+1) \in C_0 \setminus f(k(n))$, and so $k(n+1) \in \bigcap \mathcal{G}_{k(n)}$.

((b) $\Rightarrow$ (a)) First we shall show that $\mathcal{F}$ is a P-set. Let $\{G_i\}_{i\in\omega}$ be a sequence in $\mathcal{F}$. We set $\mathcal{G}_i = \{G_0 \cap G_1 \cap \cdots \cap G_i\}$, and we take $a \in \mathcal{F}$ such that $a(n+1) \in \bigcap \mathcal{G}_{a(n)}$. The set a is a pseudointersection of $\{G_i\}_{i\in\omega}$. Indeed, if G_j is such that $j \leq a(n)$ then $\{a(k) \colon k \geq n+1\} \subseteq G_j$.

We shall show that $\mathcal{F}$ is a Q-set using Lemma 3.7. Indeed, let the function $f \in {}^{\omega}\omega$ be increasing. We consider sets $\mathcal{G}_i = \{(f(i), +\infty)\}$, and we take $a \in \mathcal{F}$ such that $a(n+1) \in \bigcap \mathcal{G}_{a(n)}$. Hence, $a(n+1) > f(a(n))$. $\dashv$

The following preservation theorem will be central to the proof that iterations of Miller partition forcing, as well as other partial orders which are of interest for this article, preserve selective independent families.

Lemma 3.10 [46, Lemma 3.2]. *Let $\mathcal{F}$ be a selective filter and let $\mathcal{H} \subseteq P(\omega)\setminus\mathcal{F}$ be cofinal in $P(\omega)\setminus\mathcal{F}$ with respect to $\subseteq^*$. If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta\rangle$ is a countable support iteration of ω^ω-bounding proper forcing notions such that for all $\alpha < \delta$, we have $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash$ "$\mathcal{H}$ is cofinal in $P(\omega)\setminus\langle\mathcal{F}\rangle$," then the same holds for δ.*

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The forcing iterations, that we will be interested in, all have the Sacks property. Thus, in the corresponding generic extensions $\text{fil}(\mathcal{A})$ is generated by $\text{fil}(\mathcal{A}) \cap V$, where V denotes the ground model (see Lemma 3.4). Thus, if $\text{fil}(A)$ is selective in the ground model, then it will remain selective in the desired generic extensions. Thus, the above preservation theorem implies that in order to guarantee that a given selective independent family remains selective (in our desired generic extensions), it is sufficient to guarantee that each iterand preserves the dense maximality of the family. Note that the fact that the density filter is selective will play a crucial role in this preservation arguments. That is, our techniques do not imply that densely maximal independent families are preserved, but only selective ones. Before giving detailed proofs of these crucial preservation properties of Miller partition forcing (see Theorem 4.1 and Corollary 4.10) which are also the most technical arguments in the paper, we state our main result:

THEOREM 3.11. *Assume CH. There is a cardinals preserving generic extension in which*

$$\text{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} = \omega_1 < \mathfrak{a}_T = \omega_2.$$

PROOF. Let V denote the ground model. We assume that $\mathcal{A}$ is a selective independent family in V, $\mathcal{U}$ is a P-point in V, and $\mathcal{E}$ is a tight MAD family in V (according to [25]). Using an appropriate bookkeeping device define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets such that for each α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}(\mathcal{C})$ for some uncountable partition $\mathcal{C}$ of 2^ω into compact sets and such that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{a}_T = \omega_2$. $\mathbb{P}_{\omega_2}$ has the Sacks property and therefore $\text{cof}(\mathcal{N}) = \omega_1$. By Shelah's preservation Theorem 3.10 and Corollary 4.10, or alternatively by Theorem 4.1, the family $\mathcal{A}$ remains selective independent in $V^{\mathbb{P}_{\omega_2}}$ and so a witness to $\mathfrak{i} = \omega_1$. Similarly, $\mathcal{U}$ generates a P-point in $V^{\mathbb{P}_{\omega_2}}$, so $\mathfrak{u} = \omega_1$ as well. And finally, $\mathfrak{a} = \omega_1$ since $\mathcal{E}$ is a tight MAD family (see [25]). ⊣

§4. Miller partition forcing preserves selective independent families. We proceed with taking care of the successor stages of our forcing construction, i.e., the fact that Miller partition forcing preserves selective independent families. In Section 4.1 using the technique of fusion with witnesses, we give a proof of this fact, while in Section 4.2 we give yet one more proof using Laflamme's filter games.

4.1. Fusion sequences and selectivity.

THEOREM 4.1. *Let $\mathcal{A}$ be a selective independent family and let G be a $\mathbb{Q}(\mathcal{C})$-generic filter. Then, in $V[G]$, $\mathcal{A}$ is still selective independent.*

PROOF. Recall that by Lemmas 3.4 and 3.8, $\text{fil}(\mathcal{A})$ remains a selective filter. Thus, it is sufficient to show that $\mathcal{A}$ remains densely maximal in the generic extension. In $V^{\mathbb{Q}(\mathcal{C})}$, take any $Y \in \mathcal{P}(\omega) \backslash \langle \text{fil}(\mathcal{A}) \cap V \rangle_{\text{up}}$. Suppose $Y \notin \langle \{\omega \backslash \mathcal{A}^h : h \in \text{FF}(\mathcal{A})\} \rangle_{\text{dn}}$. Thus, for all $h \in \text{FF}(\mathcal{A})$, $Y \not\subseteq \omega \backslash \mathcal{A}^h$ and so for all $h \in \text{FF}(\mathcal{A})$, $|Y \cap \mathcal{A}^h| = \omega$. Therefore in V we can fix $p \in \mathbb{Q}(\mathcal{C})$ and a $\mathbb{Q}(\mathcal{C})$-name $\dot{Y}$ for Y such that for all $h \in \text{FF}(\mathcal{A})$, $p \Vdash |\dot{Y} \cap \mathcal{A}^h| = \infty$.

By Lemma 2.8 we can assume that for all $m \in \omega$, for all $t \in \text{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that $p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$. Now, in V for each $t \in p$, let

$$Y_t = \{m \in \omega : p(t) \nVdash \check{m} \notin \dot{Y}\}.$$

CLAIM 4.2.
(i) $p(t) \Vdash \dot{Y} \subseteq \check{Y}_t$.
(ii) If $s \subseteq t$ then $Y_t \subseteq Y_s$.
(iii) $Y_t \in \text{fil}(\mathcal{A}) \cap V$.
(iv) If $m \in Y_s$ for $s \in \text{split}_n(p)$, and $n < m$ then there is $t \in \text{split}_m(p)$ extending s such that $p(t) \Vdash \check{m} \in \dot{Y}$.

PROOF. (i) Let $m \in \dot{Y}[G]$ for a generic G containing $p(t)$. If $p(t) \Vdash \check{m} \notin \dot{Y}$ then $m \notin \dot{Y}[G]$, a contradiction.
(ii) Since $p(t) \subseteq p(s)$, from $p(t) \nVdash \check{m} \notin \dot{Y}$ we obtain $p(s) \nVdash \check{m} \notin \dot{Y}$.
(iii) If $Y_t \notin \text{fil}(\mathcal{A}) \cap V$ then by Lemma 3.2(3), there is $h \in \text{FF}(\mathcal{A})$ such that $Y_t \subseteq \omega \backslash \mathcal{A}^h$, i.e., $Y_t \cap \mathcal{A}^h = \emptyset$. Since $p(t) \Vdash \dot{Y} \subseteq \check{Y}_t$, $p(t) \Vdash \mathcal{A}^h \cap \dot{Y} = \emptyset$. However, $p(t) \Vdash |\dot{Y} \cap \mathcal{A}^h| = \infty$, which is a contradiction.
(iv) Since $p(s) \nVdash \check{m} \notin \dot{Y}$ there is a condition $q \leq p(s)$ such that $q \Vdash \check{m} \in \dot{Y}$. However, by our assumption on p due to Lemma 2.8, for any $t \in \text{split}_m(p)$ we have either $p(t) \Vdash \check{m} \in \dot{Y}$ or $p(t) \Vdash \check{m} \notin \dot{Y}$. Since $\{p(t) : t \in \text{split}_m(p), t \supseteq s\}$ is pre-dense in $p(s)$, there is $t \in \text{split}_m(p)$ extending s such that $p(t) \Vdash \check{m} \in \dot{Y}$. ⊣

CLAIM 4.3. We can assume that a dense set $X \subseteq [p]$ with $\mathcal{C}$-different elements has the associated family $\{y_x : x \in X\}$ of sets in $\text{fil}(\mathcal{A})$ such that if $t = x \restriction \text{split}_n(p)$ then $p(t) \Vdash y_x(n) \in \dot{Y}$.

PROOF. By Claim 4.2(iii), $Y_t \in \text{fil}(\mathcal{A}) \cap V$ for each $t \in \text{split}(p)$. By Lemma 3.9 for $\mathcal{G}_n$ being the family of all Y_t's with $t \in \text{split}_{\leq n}(p)$, we obtain $\{k(n) : n \in \omega\} \in \text{fil}(\mathcal{A})$ such that

$$k(n+1) \in \bigcap \{Y_t : t \in \text{split}_{\leq k(n)}(p)\}.$$

Moreover, by part (iv) of Claim 4.2 for any $s \in \text{split}_{k(n)}(p)$ there is $t \in \text{split}_{k(n+1)}(p)$ extending s such that $p(t) \Vdash \check{k}(n+1) \in \dot{Y}$. For each branch $x \in [p]$ we consider the set

$$i(x) = \{i : p(t) \Vdash \check{k}(i+1) \in \dot{Y} \text{ for } t = x \restriction \text{split}_{k(i+1)}(p)\}.$$

We say that $x \in [p]$ is an acceptable branch if $i(x)$ is cofinite. The smallest n with $i(x) \supseteq [n, \infty)$ is called the degree of acceptability of x. Note that for each acceptable branch x, $y_x = \{k(i+1) : i \in i(x)\} \in \text{fil}(\mathcal{A})$. Moreover, each $s \in p$ can be extended to an acceptable branch. Indeed, if $s \in \text{split}_l(p)$, take j such that $l \leq k(j)$. Thus, we assume that $k(j+1) \in Y_s$. By part (iv) of Claim 4.2 there is $t \in \text{split}_{k(j+1)}(p)$ extending s such that $p(t) \Vdash \check{k}(j+1) \in \dot{Y}$. Repeat the procedure recursively to build an acceptable branch extending s. In the following, we continue using a fusion argument. We build a fusion sequence $\{(p_n, X_n)\}_{n \in \omega}$.

To define p_0, take some acceptable branch x extending some node in $\text{split}_{k(0)+1}(p)$ with degree of acceptability at most 0, and a node $s = x \restriction \text{split}_{k(1)}(p)$. We set $p_0 = p(s)$ and $X_0 = \{x\}$.

Let us assume that p_n and X_n are defined, and consider $s \in \operatorname{split}_{k(n)}(p) \cap \operatorname{split}(p_n)$. Take the unique acceptable branch $x \in X_n$ extending s. Define $i = x(|s|) \in \{0,1\}$ and $s_i = x \upharpoonright \operatorname{split}_{k(n+1)}(p)$. Then we set s_{1-i} to be an extension of $s^\frown\langle 1-i\rangle$ such that:

(i) $[p(s_{1-i})] \cap C_{\alpha_x} = \emptyset$ for all already considered acceptable branches x (i.e., all branches in X_n and those assigned to previous nodes in some order of $\operatorname{split}_{k(n)}(p) \cap \operatorname{split}(p_n)$). This can be easily achieved since each C_{α_x} is nowhere dense in $[p]$.

(ii) $s_{1-i} = x \upharpoonright \operatorname{split}_{k(j+1)}(p)$ with $j \in i(x)$ for some acceptable branch $x \in [p]$ with degree of acceptability at most j. Let us recall that each node can be extended to an acceptable branch.

Finally, let X_{n+1} be the set of all considered acceptable branches in this step, and

$$p_{n+1} = \bigcup\{p(s_i) \colon s \in \operatorname{split}_{k(n)}(p) \cap \operatorname{split}(p_n), i \in \{0,1\}\}.$$

One can see that the sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses. Let $q = \bigcap\{p_n \colon n \in \omega\}$ and let $X = \bigcup\{X_n \colon n \in \omega\}$.

We shall show that the family $\{y_x \colon x \in X\}$ possesses the desired properties. Indeed, let $x \in X$. For each $n \in \omega$ we have $y_x(n) = k(i(x)(n)+1)$. Due to construction of q we have $x \upharpoonright \operatorname{split}_n(q) = x \upharpoonright \operatorname{split}_{k(j_n+1)}(p)$ for some increasing sequence $\{j_i\}_{i\in\omega}$, and if $t = x \upharpoonright \operatorname{split}_n(q)$ then $p(t) \Vdash \check{k}(j_n+1) \in \dot{Y}$. Thus $k(j_n+1) \in y_x$ and consequently $j_n \geq i(x)(n)$ for each n. Let us now fix n and consider $t = x \upharpoonright \operatorname{split}_n(q)$. The definition of y_x guarantees that $p(s) \Vdash \check{k}(i(x)(n)+1) \in \dot{Y}$ for $s = x \upharpoonright \operatorname{split}_{k(i(x)(n)+1)}(p)$. Thus we have $q(s) \Vdash \check{y}_x(n) \in \dot{Y}$. On the other hand, $s = x \upharpoonright \operatorname{split}_{k(i(x)(n)+1)}(p) \subseteq x \upharpoonright \operatorname{split}_{k(j_n+1)}(p) = x \upharpoonright \operatorname{split}_n(q) = t$. ⊣

The last part of our proof resembles the proof of the previous claim. Let x_s for $s \in \operatorname{split}(p)$ be the branch in X extending s such that if $s \subseteq t \subseteq x_s$ then $x_t = x_s$. The corresponding y_{x_s} is denoted by y_s. The set y_s belongs to $\operatorname{fil}(\mathcal{A}) \cap V$. By Lemma 3.9 for $\mathcal{G}_n$ being the family of all y_t's with $t \in \operatorname{split}_{\leq n}(p)$, we obtain $\{l(n) \colon n \in \omega\} \in \operatorname{fil}(\mathcal{A})$ such that

$$l(n+1) \in \bigcap\{y_t \colon t \in \operatorname{split}_{\leq l(n)}(p)\}.$$

Let us denote $C = \{l(n+1) \colon n \in \omega\}$. We shall construct a condition $q^* \leq p$ such that $q^* \Vdash \check{C} \subseteq \dot{Y}$. Then $q^* \Vdash \dot{Y} \in \operatorname{fil}(\mathcal{A})$ which is a contradiction.

We build a fusion sequence $\{(p_n, X_n)\}_{n\in\omega}$. Let $p_0 = p$, $X_0 = \{x_t\}$ for $t \in \operatorname{split}_0(p)$, and suppose we have defined p_n. For each $t \in \operatorname{split}_n(p_n) \subseteq \operatorname{split}_{l(n)}(p)$ and each $i \in \{0,1\}$ take $w^*(t,i) \in \operatorname{split}_{l(n)+1}(p)$ such that $w^*(t,i)$ end-extends $t^\frown i$. Then

$$l(n+1) \in \bigcap\{y_{w^*(t,i)} \colon t \in \operatorname{split}_n(p_n), i \in \{0,1\}\}$$

and so for each t, i we take $w(t,i) = x_{w^*(t,i)} \upharpoonright \operatorname{split}_{l(n+1)}(p)$. Note that by Claim 4.3 and the fact that $l(n+1) \geq j$ for $l(n+1) = y_{w^*(t,i)}(j)$ we obtain

$$p(w(t,i)) \Vdash \check{l}(n+1) \in \dot{Y}.$$

Take $p_{n+1} = \bigcup\{p(w(t,i)) \colon t \in \operatorname{split}_n(p_n), i \in \{0,1\}\}$, $X_{n+1} = \{x_{w(t,i)} \colon t \in \operatorname{split}_n(p_n), i \in \{0,1\}\}$. Finally, let q^* be the fusion of $\{(p_n, X_n)\}_{n\in\omega}$. Then, since $q^* \Vdash l(n+1) \in \dot{Y}$ for all $n \in \omega$, q^* is as required. ⊣

4.2. Laflamme's filter game. We give yet one more proof of the fact that Miller partition forcing preserves selective independent families, which is based on Laflamme's filter game. We will make use of the following property, which can be deduced from [52, Propositions 4.1.2 and 4.1.31].

LEMMA 4.4. *Let $p \in \mathbb{Q}(\mathcal{C})$ and let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name such that $p \Vdash \dot{f} \in 2^\omega$. Then there exists $q \leq p$ and a continuous $H : [q] \longrightarrow 2^\omega$ such that $q \Vdash H(\dot{r}_{gen}) = \dot{f}$.*

DEFINITION 4.5 (Laflamme [34]). Let $\mathcal{F}$ be a filter over ω. The game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$ is defined as follows. On the n turn, Player I plays some $U_n \in \mathcal{F}$ and Player II responds with some $a_n \in U_n$. After ω turns, Player II wins if the sequence $\{a_n\}_{n\in\omega}$ belongs to $\mathcal{F}$. Otherwise, Player I wins.

It is not hard to prove that Player II never has a winning strategy in this game. On the other hand, we have the following theorem of Laflamme, see [34] (as well as [35]).

THEOREM 4.6 (Laflamme). *A filter $\mathcal{F}$ is not selective if and only if Player I does have a winning strategy for the game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$.*

LEMMA 4.7. *Let $\mathcal{F}$ be a selective filter, $p \in \mathbb{S}$, and $H : [p] \longrightarrow P(\omega)$ be a continuous function such that for every $s \in p$,*

$$\bigcup H\big[[p(s)]\big] \in \mathcal{F}.$$

Then there are $q \in \mathbb{S}$, $Y \in \mathcal{F}$ such that $q \subseteq p$ and for every $f \in [q]$, $Y \subseteq H(f)$.

PROOF. For every $q \in \mathbb{S}$ such that $q \subseteq p$, let $L(q) = \bigcup H\big[[q]\big]$. Now consider the game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$. Player I will play the following strategy, while constructing a sequence $\{t_\sigma\}_{\sigma\in 2^{<\omega}} \subseteq p$ such that:

(a) $\forall \sigma \in 2^{<\omega} \forall i \in 2\big(t_\sigma \subsetneq t_{\sigma^\frown i}\big)$.
(b) $\forall \sigma, \tau \in 2^n\big(\sigma \neq \tau \to (t_\sigma \text{ and } t_\tau \text{ are incomparable })\big)$.

On the first turn Player I defines $t_\emptyset = \emptyset$ and plays $U_0 = L(p(t_\emptyset))$. As the rules dictate, Player II responds with some $a_0 \in U_0$. Since $a_0 \in \bigcup H\big[[p(t_\emptyset)]\big]$, there is $f \in [p(t_\emptyset)]$ such that $a_0 \in H(f)$. As H is continuous there is $k \in \omega$ such that for every $g \in [p(f|_k)]$ we have $a_0 \in H(g)$. Now Player I extends $f|_k$ to incomparable $t_0, t_1 \in p$ such that $t_\emptyset \subsetneq t_0, t_1$ and plays $U_1 = L(p(t_0)) \cap L(p(t_1))$. As the rules dictate Player II responds with some $a_1 \in U_1$.

In general, suppose that it is the $n+1$ turn and that Player I has constructed $\{t_\sigma\}_{\sigma\in 2^{\leq n}}$ and for every $m \leq n$ played $U_m = \bigcap_{\sigma\in 2^{\leq m}} L(p(t_\sigma))$. As $a_n \in \bigcap_{\sigma\in 2^n} \big(\bigcup H\big[[p(t_\sigma)]\big]\big)$, for every $\sigma \in 2^n$ there is $f_\sigma \in [p(t_0)]$ such that $a_n \in H(f_\sigma)$. Since H is continuous, there is $k \in \omega$ such that for every $\sigma \in 2^n$ and every $g \in [p(f_\sigma|_k)]$, $a_n \in H(g)$. Now Player I extends each $f_\sigma|_k$ to incomparable $t_{\sigma^\frown 0}, t_{\sigma^\frown 1} \in p$ such that $t_\sigma \subsetneq t_{\sigma^\frown 0}, t_{\sigma^\frown 1}$ and plays $U_{n+1} = \bigcap_{\sigma\in 2^{\leq n+1}} L(p(t_\sigma))$. As the rules dictate, Player II responds with some $a_{n+1} \in U_{n+1}$.

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Since $\mathcal{F}$ is a selective filter, this is not a winning strategy for Player I. Therefore there is a match where Player I plays by the above strategy, but Player II wins. Let $\{a_n\}_{n\in\omega}$ and $\{t_\sigma\}_{\sigma\in 2^{<\omega}}$ be the sequences associated with one of these matches and let $q = \{\tau \in p \mid \exists\sigma \in 2^{<\omega}(\tau \subseteq t_\sigma)\}$. It is straightforward that q and $Y = \{a_n\}_{n\in\omega}$ are the objects we are looking for. ⊣

THEOREM 4.8. *Let $\mathcal{F}$ be a selective filter and let G be a $\mathbb{Q}(\mathcal{C})$-generic filter. In $V[G]$, for every $X \in P(\omega)$ one of the following statements occurs*:

(a) *There is $Y \in \mathcal{F} \cap V$ such that $Y \subseteq X$.*
(b) *There is $Z \in V$, such that $Z \notin \mathcal{F}$ and $X \subseteq Z$.*

PROOF. Let $\dot{X}$ be a name for a subset of ω, $p \in \mathbb{Q}(\mathcal{C})$, and suppose no condition below p forces (b). By Lemma 4.4 there is a continuous $H : [p] \longrightarrow P(\omega)$ such that $p \Vdash H(\dot{r}_{gen}) = \dot{X}$. For every $q \in \mathbb{S}$ such that $q \subseteq p$, let $L(q) = \bigcup H\big[[q]\big]$ and note that if $q \in \mathbb{Q}(\mathcal{C})$ then $L(q) \in \mathcal{F}$. This holds, because $q \Vdash \dot{X} \subseteq L(q)$ and q does not force (b). We say that a condition $q \in \mathbb{S}$, which is not necessarily in $\mathbb{Q}(\mathcal{C})$, is *special* if $q \subseteq p$ and for every $s \in q$ we have that $L(q(s)) \in \mathcal{F}$.

We will make use of the following notion: Given $s \in p$ we say that the pair (q, T) is *s-special* if $q \in \mathbb{S}$ is special, $T \in \mathcal{C}$, $[q] \subseteq [p(s)] \cap T$. We divide the proof in cases.

CASE 1. For every $s \in p$, there is an s-special pair (q, T).

In this case, consider the game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$. Player I will play by the following strategy, while recursively constructing sequences $\{q_\sigma\}_{\sigma\in 2^{<\omega}} \subseteq \mathbb{S}$, $\{s_\sigma\}_{\sigma\in 2^{<\omega}} \subseteq p$, and $\{T_\sigma\}_{\sigma\in 2^{<\omega}} \subseteq \mathcal{C}$ such that:

(a) $\forall\sigma \in 2^{<\omega}$ the pair (q_σ, T_σ) is s_σ-special;
(b) $\forall\sigma \in 2^{<\omega}\big(q_{\sigma^\frown 0} \subseteq q_\sigma\big)$;
(c) $\forall\sigma \in 2^{<\omega}\big(T_{\sigma^\frown 0} = T_\sigma\big)$;
(d) $\forall\sigma, \tau \in 2^n\big(\sigma \neq \tau \to T_\sigma \neq T_\tau\big)$;
(e) $\forall\sigma \in 2^{<\omega}\forall i \in 2\big(s_\sigma \subsetneq s_{\sigma^\frown i}\big)$;
(f) $\forall\sigma \in 2^{<\omega}\big(s_{\sigma^\frown 0} \in q_\sigma \wedge [p(s_{\sigma^\frown 1})] \cap T_\sigma = \emptyset\big)$.

On the first turn, Player I defines $s_\emptyset = \emptyset$, an $s_\emptyset$-special pair $(q_\emptyset, T_\emptyset)$ and plays $U_0 = L(q_\emptyset)$. As the rules dictate, Player II responds with some $a_0 \in U_0$. Since $a_0 \in \bigcup H\big[[q_\emptyset]\big]$, there is some $f \in [q_\emptyset]$ such that $a_0 \in H(f)$ and since H is continuous there is $k \in \omega$ such that for every $g \in [p(f|_k)]$, $a_0 \in H(g)$. Notice that since $(q_\emptyset, T_\emptyset)$ is $s_\emptyset$-special, we have that $f|_k$ is compatible with $s_\emptyset$ and moreover we can extend $f|_k$ to incomparable $s_0, s_1 \in p$ such that $s_\emptyset \subsetneq s_0, s_1$, $s_0 \in q_\emptyset$ and $[p(s_1)] \cap T_\emptyset = \emptyset$. Now Player I defines $q_0 = q_\emptyset(s_0)$, $T_0 = T_\emptyset$, an s_1-special pair (q_1, T_1) and plays $U_1 = L(q_0) \cap L(q_1)$. As the rules dictate, Player II responds with some $a_1 \in U_1$.

In general, suppose that it is the $n + 1$ turn and Player I has constructed q_σ, s_σ, and T_σ for every $\sigma \in 2^{\leq n}$. Moreover, suppose that for every $m \leq n$ Player I has played $U_m = \bigcap_{\sigma\in 2^m} L(q_\sigma)$. Since $a_n \in \bigcap_{\sigma\in 2^n}\big(\bigcup H\big[[q_\sigma]\big]\big)$, for every $\sigma \in 2^n$ there is $f_\sigma \in [q_\sigma]$ such that $a_n \in H(f_\sigma)$. As H is continuous, there is $k \in \omega$ such that for every $\sigma \in 2^n$ and every $g \in [p(f_\sigma|_k)]$, $a_n \in H(g)$. As each (q_σ, T_σ) is s_σ-special, we have that $f_\sigma|_k$ is compatible with s_σ and that $\bigcup_{\sigma\in 2^n} T_\sigma \cap [p]$ is nowhere dense in $[p]$. Then we can extend each $f_\sigma|_k$ to incomparable $s_{\sigma^\frown 0}, s_{\sigma^\frown 1} \in p$ such that $s_\sigma \subsetneq s_{\sigma^\frown 0}, s_{\sigma^\frown 1}$, $s_{\sigma^\frown 0} \in q_\sigma$

and $[p(s_{\sigma^\frown 1})] \cap T_\sigma = \emptyset$. Now Player I defines $q_{\sigma^\frown 0} = q_\sigma(s_{\sigma^\frown 0})$, $T_{\sigma^\frown 0} = T_\sigma$, an $s_{\sigma^\frown 1}$-special pair $(q_{\sigma^\frown 1}, T_{\sigma^\frown 1})$ and plays $U_{n+1} = \bigcap_{\sigma \in 2^{n+1}} L(q_\sigma)$. As the rules dictate, Player II responds with some $a_{n+1} \in U_{n+1}$.

Since $\mathcal{F}$ is a selective filter, the above is not a winning strategy for Player I and so, there is a match where Player I follows the strategy, but Player II wins. Let $\{a_n\}_{n\in\omega}$, $\{q_\sigma\}_{\sigma\in 2^\omega}$, $\{s_\sigma\}_{\sigma\in 2^\omega}$, and $\{T_\sigma\}_{\sigma\in 2^\omega}$ be the sequences associated with one of these matches. To finish Case 1, define $q = \{\tau \in p \mid \exists \sigma \in 2^\omega(\tau \subseteq s_\sigma)\}$. Moreover, if c_0 is the constant 0 function in 2^ω and $\sigma \in 2^{<\omega}$ then $g_\sigma = \bigcup\{s_\tau \mid \tau \subseteq \sigma^\frown c_0\} \in T_\sigma$ and the set $Q = \{g_\sigma \mid \sigma \in 2^{<\omega}\}$ is dense in $[q]$. But then, by Lemma 2.4, $q \in \mathbb{Q}(\mathcal{C})$. Since for every $n \in \omega$, every $\sigma \in 2^{n+1}$, and every $g \in [p(s_\sigma)]$, we have that $a_n \in H(g)$ and every $g \in [q]$ satisfies this condition for some $\sigma \in 2^{n+1}$, we obtain that for every $g \in [q]$, $\{a_n\}_{n\in\omega} \subseteq H(g)$. In particular we have that $q \Vdash \{a_n\}_{n\in\omega} \subseteq \dot{X}$. Since $\{a_n\}_{n\in\omega} \in \mathcal{F}$, we are done.

Case 2. There is $s_0 \in p$ for which there is no s_0-special pair (q, T). That is, every ordered pair (q, T) does not satisfy one of the following conditions: $q \in \mathbb{S}$ is special, $T \in \mathcal{C}$, or $[q] \subseteq [p(s_0)] \cap T$.

In this case, we use Lemma 4.7 to find $q \in \mathbb{S}$ and $Y \in \mathcal{F}$ such that $q \subseteq p(s_0)$ and for every $f \in [q]$, $Y \subseteq H(f)$. Notice that q is special. Suppose towards a contradiction that $q \notin \mathbb{Q}(\mathcal{C})$. Since every element of $\mathcal{C}$ is closed, this means that there is some $T \in \mathcal{C}$ such that $T \cap [q]$ has non-empty interior in $[q]$ and so we can find $\tau \in q$ such that $[q(\tau)] \subseteq T$. Then $(q(\tau), T)$ is s_0-special, which is a contradiction. Therefore $q \in \mathbb{Q}(\mathcal{C})$. To finish this case, just note that as before $q \Vdash Y \subseteq \dot{X}$. ⊣

Suppose that $\mathcal{C}$ is the partition of 2^ω in singletons. Then $\mathbb{Q}(\mathcal{C}) = \mathbb{S}$ and so *Case 1* of Theorem 4.8 never occurs. Therefore Lemma 4.7 actually yields a complete proof of Theorem 4.8 for Sacks forcing. Additionally, we obtain once again:

Corollary 4.9. *The poset $\mathbb{Q}(\mathcal{C})$ preserves selective ultrafilters.*

Corollary 4.10. *Let $\mathcal{A}$ be a selective independent family and let G be a $\mathbb{Q}(\mathcal{C})$-generic filter. Then, in $V[G]$, $\mathcal{A}$ is still selective independent.*

Proof. Since $\mathbb{Q}(\mathcal{C})$ is proper and has the Sacks property, $\langle \mathrm{fil}(\mathcal{A})^V \rangle$ is a selective filter in $V[G]$, but by Lemma 3.4 we know that $\mathrm{fil}(\mathcal{A})^{V[G]} = \langle \mathrm{fil}(\mathcal{A})^V \rangle$. To show that $\mathcal{A}$ remains densely maximal in $V[G]$, note that by Theorem 4.8, the family $P(\omega)^V \setminus \mathrm{fil}(\mathcal{A})^V$ is cofinal in $P(\omega)^{V[G]} \setminus \langle \mathrm{fil}(\mathcal{A})^V \rangle$. However, by hypothesis $\{\omega \setminus \mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\}$ is cofinal in $P(\omega)^V \setminus \mathrm{fil}(\mathcal{A})^V$ and so we are done. ⊣

§5. No small ultrafilter bases and tightness.

5.1. The poset $\mathbb{Q}_\mathcal{I}$. For a maximal ideal $\mathcal{I}$ on ω, $\mathbb{Q}_\mathcal{I}$ denotes the forcing notion introduced by Shelah in [46] for obtaining the consistency of $\mathfrak{i} < \mathfrak{u}$. In [46] it is shown that $\mathbb{Q}_\mathcal{I}$ is proper [46, Claim 1.13], ${}^\omega\omega$-bounding [46, Claim 1.12], and even has the Sacks property [46, Claim 1.12]. In the $\mathbb{Q}_\mathcal{I}$-generic extension, $\mathcal{I}$ is no longer a maximal ideal [46, Claim 1.5]. For completeness of the presentation we repeat below the definition and some of the key properties of $\mathbb{Q}_\mathcal{I}$.

DEFINITION 5.1. Let $\mathcal{I}$ be an ideal on ω.

(1) An equivalence relation E on a subset of ω is an $\mathcal{I}$-equivalence relation if $\operatorname{dom} E \in \mathcal{I}^*$ and each E-equivalence class is in $\mathcal{I}$.
(2) For $\mathcal{I}$-equivalence relations E_1, E_2, we denote $E_1 \leq_{\mathcal{I}} E_2$ if $\operatorname{dom} E_1 \subseteq \operatorname{dom} E_2$, and E_1-equivalence classes are unions of E_2-equivalence classes.
(3) Let $A \subseteq \omega$. A function g is A-n-determined if $g\colon {}^{A}\{0,1\} \to \{0,1\}$ and there is $w \subseteq A \cap (n+1)$ such that for any $\eta, \nu \in {}^{A}\{0,1\}$ with $\eta \restriction w = \nu \restriction w$ we have $g(\eta) = g(\nu)$.

For $i \in A$, by g_i we denote the function from ${}^{A}\{0,1\}$ to $\{0,1\}$ which maps $\eta \in {}^{A}\{0,1\}$ to $\eta(i)$.

CLAIM 5.2. Each A-n-determined function is equal to a function $\varphi(g_0, \dots, g_n)$ which is obtained as an interpretation of a formula $\varphi(a_0, \dots, a_n)$ of propositional calculus. The symbols $\wedge$, $\vee$, $\neg$ are interpreted as a maximum, minimum, and complement (i.e., $1 - g_i$), respectively. The formula $\varphi(a_0, \dots, a_n)$ may contain constant symbols $0, 1$ which are interpreted as constant functions $0, 1$.

For an $\mathcal{I}$-equivalence relation E we denote $A = A(E) = \{x\colon x \in \operatorname{dom} E, x = \min[x]_E\}$.

DEFINITION 5.3 (Set of conditions in $\mathbb{Q}_{\mathcal{I}}$). Let $\mathcal{I}$ be an ideal on ω. We define a forcing notion $\mathbb{Q}_{\mathcal{I}}$:

$$p \in \mathbb{Q}_{\mathcal{I}} \text{ iff } p = (H, E) = (H^p, E^p) \text{ where:}$$

(1) E is an $\mathcal{I}$-equivalence relation,
(2) H is a function with $\operatorname{dom} H = \omega$,
(3) a value $H(n)$ is an $A(E)$-n-determined function,
(4) if $n \in A(E)$ then $H(n) = g_n$,
(5) if $n \in \operatorname{dom} E \setminus A(E)$ and nEi for $i \in A(E)$ then $H(n)$ is g_i or $1 - g_i$.

For a condition $q \in \mathbb{Q}_{\mathcal{I}}$, let A^q be $A(E^q)$ in the following.

DEFINITION 5.4. If $p, q \in \mathbb{Q}_{\mathcal{I}}$ with $A^p \subseteq A^q$ then for each $n \in \omega \setminus \operatorname{dom}(E^q)$ we write $H^p(n) =^{**} H^q(n)$ if for each $\eta \in {}^{A^p}\{0,1\}$ we have $H^p(n)(\eta) = H^q(n)(\eta')$ where

$$\eta'(j) = \begin{cases} \eta(j), & j \in A^p, \\ H^p(j)(\eta), & j \in A^q \setminus A^p. \end{cases}$$

DEFINITION 5.5 (The order of $\mathbb{Q}_{\mathcal{I}}$). If $p, q \in \mathbb{Q}_{\mathcal{I}}$ then $p \leq q$ if:

(1) $E^p \leq_{\mathcal{I}} E^q$.
(2) If $H^q(n) = g_i$ for $n \in \operatorname{dom} E^q$ then $H^p(n) = H^p(i)$.
(3) If $H^q(n) = 1 - g_i$ for $n \in \operatorname{dom} E^q$ then $H^p(n) = 1 - H^p(i)$.
(4) If $n \in \omega \setminus \operatorname{dom} E^q$ then $H^p(n) =^{**} H^q(n)$.

Finally, $p \leq_n q$ if $p \leq q$ and A^p contains the first n elements of A^q.

The following has been proven in [46]. Items (1) and (2) correspond to [46, Claim 1.7(2)], and item (3) is a straightforward modification of [46, Claim 1.8].

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Claim 5.6. Let $p \in \mathbb{Q}_{\mathcal{I}}$. For an initial segment u of A^p, and $h\colon u \to \{0,1\}$, let $p^{[h]}$ be the pair $q = (H^q, E^q)$ defined by (i) and (ii) below:

(i) $E^q = E^p \restriction \bigcup\{[i]_{E^p} : i \in A^p \setminus u\}$.
(ii) If $H^p(n)$ is $\varphi(g_0, \dots, g_n)$ then $H^q(n)$ is $\varphi(g_0, \dots, g_i/h(i), \dots, g_n)$, where the substitution is done just for $i \in u$.

Then we have:

(1) $p^{[h]}$ is a condition in $\mathbb{Q}_{\mathcal{I}}$ stronger than p.
(2) The set $\{p^{[h]} : h \in {}^u\{0,1\}\}$ is predense below p.
(3) If u is the set of first n elements of A^p, D a dense subset of $\mathbb{Q}_{\mathcal{I}}$ then there is $q \in \mathbb{Q}_{\mathcal{I}}$ such that $q \leq_n p$ and $q^{[h]} \in D$ for any $h \in {}^u\{0,1\}$.

Definition 5.7 (The game $\mathrm{GM}_{\mathcal{I}}(E)$). $\mathrm{GM}_{\mathcal{I}}(E)$ is the following game. In the n-th move, the first player chooses an $\mathcal{I}$-equivalence relation $E^1_n \leq_{\mathcal{I}} E^2_{n-1}$ ($E^1_0 = E$), and the second player chooses an $\mathcal{I}$-equivalence relation $E^2_n \leq_{\mathcal{I}} E^1_n$. In the end, the second player wins if

$$\bigcup_{n>0} (\operatorname{dom} E^1_n \setminus \operatorname{dom} E^2_n) \in \mathcal{I}^*.$$

Otherwise, the first player wins.

Remark 5.8. If the second player wins in the game $\mathrm{GM}_{\mathcal{I}}(E)$, then the game is invariant to taking subsets. That is, the game is invariant to taking $\leq_{\mathcal{I}}$-extensions $\{E^{2,*}_n\}_{n\in\omega}$ with $\operatorname{dom}(E^{2,*}_n) \subseteq \operatorname{dom} E^2_n$.

The next lemma corresponds to [46, Claim 1.10(1)]

Lemma 5.9. *The game $GM_{\mathcal{I}}(E)$ is not determined for a maximal ideal $\mathcal{I}$.*

5.2. Tight MAD families. Tight MAD families were investigated in [25, 33, 36]. An AD family $\mathcal{A}$ is called tight if for every $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$.

A preservation theorem for tight MAD family under countable support iteration of proper forcing notions was developed by Guzmán, Hrušák, and Téllez [25].

Definition 5.10. Let $\mathcal{A}$ be a tight MAD family. A proper forcing $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$ if for every $p \in \mathbb{P}$, M a countable elementary submodel of $H(\kappa)$ (where κ is a large enough regular cardinal) such that $\mathbb{P}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, there is $q \leq p$ an $(M, \mathbb{P})$-generic condition such that

$$q \Vdash \text{“}(\forall \dot{Z} \in \mathcal{I}(\mathcal{A}) \cap M[\dot{G}])\ |\dot{Z} \cap B| = \omega,\text{”}$$

where $\dot{G}$ denotes the name of the generic filter.

We restate Corollary 32 of [25] which is crucial for preserving MAD families in the forthcoming model.

Theorem 5.11 (Guzmán, Hrušák, and Téllez). *Let $\mathcal{A}$ be a tight MAD family. If the sequence $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ is a countable support iteration of proper posets*

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such that

$$\mathbb{P}_\alpha \Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \textit{ strongly preserves the tightness of } \mathcal{A}\text{,”}$$

then $\mathbb{P}_{\omega_2} \Vdash_\alpha$ *“$\mathcal{A}$ is a tight* MAD *family.”*

We need the following fact about the outer hulls observed in [25].

LEMMA 5.12. *Let $\mathcal{A}$ be an* AD *family, $\mathbb{P}$ a partial order, $\dot{B}$ a $\mathbb{P}$-name for a subset of ω, and $p \in \mathbb{P}$ such that $p \Vdash$ “$\dot{B} \in \mathcal{I}(\mathcal{A})^+$.” Then the set $\{n : (\exists q \le p)\, q \Vdash \text{“}n \in \dot{B}\text{”}\}$ is in $\mathcal{I}(\mathcal{A})^+$.*

And now we are ready to show the main result of this section.

THEOREM 5.13. *Let $\mathcal{A}$ be a tight* MAD *family, $\mathcal{I}$ being a maximal proper ideal on ω. The poset $\mathbb{Q}_\mathcal{I}$ strongly preserves the tightness of $\mathcal{A}$.*

PROOF. Let $p \in \mathbb{Q}_\mathcal{I}$, M a countable elementary submodel of $H(\kappa)$ such that $\mathcal{I}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$. We fix an enumeration $\{D_n : n \in \omega\}$ of all open dense subsets of $\mathbb{Q}_\mathcal{I}$ that are in M, and an enumeration $\{\dot{Z}_n : n \in \omega\}$ with infinite repetitions of all $\mathbb{Q}_\mathcal{I}$-names for elements of $\mathcal{I}(\mathcal{A})^+$ that are in M.

We define a strategy for the first player in the game $\mathrm{GM}_\mathcal{I}(E)$, which cannot be winning in all rounds. Work in M. We set $p_0 = q_0 = p$ and $u_0 = \emptyset$. We assume that the first player has chosen E^1_n, q_n, p_n, u_n, and the second one has chosen E^2_n. We give instructions to choose E^1_{n+1}, q_{n+1}, p_{n+1}, u_{n+1}. We begin with q_{n+1}:

(1) $\operatorname{dom} E^{q_{n+1}} = \operatorname{dom} E^{p_n}$.
(2) $xE^{q_{n+1}}y$ iff one of the following holds:
  (i) $xE^2_n y$.
  (ii) There is $k \in u_n$ with $x, y \in [k]_{E^{p_n}}$ and $x, y \notin \operatorname{dom} E^2_n$.
  (iii) $x, y \notin \bigcup\{[i]_{E^{p_n}} : i \in u_n\} \cup \operatorname{dom} E^2_n$.
(3) $H^{q_{n+1}}$ is chosen such that:
  (i) If $l \in A^{q_{n+1}}$, then $H^{q_{n+1}}(l) = g_l$.
  (ii) If $l \in A^{p_n} \setminus A^{q_{n+1}}$, then $H^{q_{n+1}}(l) = g_{\min[l]_{E^{q_{n+1}}}}$.
  (iii) If $l \in \operatorname{dom} E^{p_n} \setminus A^{p_n}$, $H^{p_n}(l) = g_i$ then $H^{q_{n+1}}(l) = H^{q_{n+1}}(i)$.
  (iv) If $l \in \operatorname{dom} E^{p_n} \setminus A^{p_n}$, $H^{p_n}(l) = 1 - g_i$ then $H^{q_{n+1}}(l) = 1 - H^{q_{n+1}}(i)$.
  (v) If $l \in \omega \setminus \operatorname{dom} E^{p_n}$ then $H^{q_{n+1}}(l) =^{**} H^{p_n}(l)$.

Note that for the already defined condition q_{n+1} we have $q_{n+1} \le_n p_n$. Let z^n be the least element of $\operatorname{dom} E^{q_{n+1}}$ generating the $E^{q_{n+1}}$-equivalence class of (2)(iii) above and let $u_{n+1} = u_n \cup \{z^n\}$. By Lemma 5.12, the set $D'_n = \{r \in \mathbb{Q}_\mathcal{I} : r \Vdash \text{“}(\dot{Z}_n \cap B) \setminus n \ne \emptyset\text{”}\}$ is open dense below p (and also below q_{n+1}). Then $D'_n \cap D_n$ is dense below q_{n+1}. Therefore we can apply Claim 5.6(3) to obtain $p_{n+1} \le_{n+1} q_{n+1}$ such that for each $h \in {}^{u_{n+1}}\{0,1\}$, the condition $p^{[h]}_{n+1} \in D'_n \cap D_n \cap M$. In particular, if $h \in {}^{u_{n+1}}\{0,1\}$ then $p^{[h]}_{n+1} \Vdash$ “$(\dot{Z}_n \cap B) \setminus n \ne \emptyset$” and $p^{[h]}_{n+1} \in D_n \cap M$. By Claim 5.6(2) we have $p_{n+1} \Vdash$ “$(\dot{Z}_n \cap B) \setminus n \ne \emptyset$.” Finally, we set

$$E^1_{n+1} = E^{p_{n+1}} \restriction (\operatorname{dom} E^{p_{n+1}} \setminus \bigcup\{[i]_{E^{p_{n+1}}} : i \in u_{n+1}\}).$$

We define a fusion q of the sequence $\langle p_n : n \in \omega\rangle$. The relation E^q has $\operatorname{dom} E^q = \bigcap_{n\in\omega} \operatorname{dom} E^{p_n}$ and $xE^q y$ if for every n large enough, $xE^{p_n}y$. The function H^q

is equal to H^{p_n} for large enough n. In order to guarantee that $\operatorname{dom} E^q \in \mathcal{I}^*$, it is sufficient to choose a play with the first player using the described strategy, but loosing, which by Lemma 5.9 exists. The other properties for $q \in \mathbb{Q}_{\mathcal{I}}$ are satisfied by the definition of q.

Finally q is $(M, \mathbb{Q}_{\mathcal{I}})$-generic, $q \leq_n p_n$ for all n and so $q \Vdash$ "$(\forall \dot{Z} \in \mathcal{I}(\mathcal{A}) \cap M[\dot{G}])\ |\dot{Z} \cap B| = \omega$." ⊣

As a corollary we obtain that in Shelah's model of $\mathfrak{i} < \mathfrak{u}$ (see [46]), also the almost disjointness number is small.

Corollary 5.14. *It is relatively consistent that* $\mathfrak{a} = \mathfrak{i} < \mathfrak{u}$.

Moreover, using the preservation results of the current article, together with the preservation results of [25], as well as the fact that $\mathbb{Q}_{\mathcal{I}}$ has the Sacks property, we obtain:

Corollary 5.15. *It is relatively consistent that* $\operatorname{cof}(\mathcal{N}) = \mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{a}_T = \mathfrak{u} = \omega_2$.

Proof. Work over a model of CH. Let $\mathcal{A}_0$ be Shelah's selective independent family and let $\mathcal{A}_1$ be a tight mad family. Using an appropriate bookkeeping device define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets such that for even α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}(\mathcal{C})$ for some uncountable partition $\mathcal{C}$ of 2^ω into compact sets, for odd α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}_{\mathcal{I}}$ for some maximal ideal $\mathcal{I}$ on ω, and such that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{a}_T = \mathfrak{u} = \omega_2$. The iteration $\mathbb{P}_{\omega_2}$ has the Sacks property and therefore $\operatorname{cof}(\mathcal{N}) = \omega_1$. By the indestructibility of selective independence the family $\mathcal{A}_0$ remains maximal independent in $V^{\mathbb{P}_{\omega_2}}$ and so a witness to $\mathfrak{i} = \omega_1$. Moreover, by the preservation properties of tight MAD families (see [25]), and the above preservation theorems, $\mathcal{A}_1$ is a witness to $\mathfrak{a} = \omega_1$ in the final model. ⊣

§6. Appendix: The problem of Vaughan.

We conclude the paper with an overview of the problem of Vaughan and point many of the difficulties surrounding a possible solution of it, in particular the fact that the most common forcing methods do not seem to help with the problem:

(1) *Finite support iteration of ccc forcings of length a regular cardinal over a model of CH.* This approach cannot work since in the models obtained in this way, the size of the continuum is equal to *cov* $(\mathcal{M})$ and it is known that *cov* $(\mathcal{M}) \leq \mathfrak{i}$.[5]

(2) *Countable support iteration of definable proper forcings of length ω_2 over a model of CH.* It follows by the results of Džamonja, Hrušák, and Moore in [43] that in all of these models the equality $\mathfrak{b} = \mathfrak{a}$ will hold, so in particular we will have that $\mathfrak{a} \leq \mathfrak{i}$.

(3) *Countable support iteration of non-definable proper forcings of length ω_2 over a model of CH.* This approach could work; however a model of $\mathfrak{i} < \mathfrak{a}$ obtained by this method will also be a model of $\omega_1 = \mathfrak{d} < \mathfrak{a}$ (since $\mathfrak{d} \leq \mathfrak{i}$), thus solving the problem of Roitman, which is considered to be one of the hardest problems in the theory of cardinal invariants.

[5] Shelah proved that $\mathfrak{d} \leq \mathfrak{i}$ (see the Appendix of [50]). This result was improved by Balcar, Hernandez-Hernández, and Hrušák in [2] where they proved that $cof\ (\mathcal{M}) \leq \mathfrak{i}$.

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(4) *Forcing with ultrapowers and iterating along a template.* The method of forcing with ultrapowers and iterating along a template was introduced by Shelah in [48] to build models of $\mathfrak{d} < \mathfrak{a}$ and $\mathfrak{u} < \mathfrak{a}$. This is a very powerful method that has been very useful and has been successfully applied to this day. Unfortunately, it seems that all forcings obtained using this method tend to increase $\mathfrak{i}$ for the same reason they increase $\mathfrak{a}$. To learn more about this powerful method, see [8–10, 16, 20, 22, 39].

(5) *Short finite support iterations over models of MA.* Performing a finite support iteration of length ω_1 over a model of MA (for example) is a powerful method to add "small witnesses" of some cardinal invariants while keeping others large. Models obtained in this way are often called "dual models" (see [12] for several interesting results and applications of these methods). In [2] a dual model was constructed to add a small maximal independent family in order to build a model of $\mathfrak{i} < non(\mathcal{N})$. Unfortunately, it is not clear how one could avoid adding a small MAD family with this method. Moreover, it seems likely that the principle $\diamondsuit_{\mathfrak{d}}$ of Hrušák will hold in these models[6] (see [28]).

In principle, it could be possible to construct a model of $\mathfrak{i} < \mathfrak{a}$ using matrix iterations (see [7, 11, 38] to learn more about this method), but one would need to be very careful in order to avoid problems like in the points 1 and 5 above.

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[6] $\diamondsuit_{\mathfrak{d}}$ is the following principle: There is a family $\{d_\alpha \mid \alpha \in \omega_1\}$ such that $d_\alpha : \alpha \longrightarrow \omega$ and for every $F : \omega_1 \longrightarrow \omega_1$ there is $\alpha \geq \omega$ such that $F \restriction \alpha \leq^* d_\alpha$. In [28] it was proved that $\diamondsuit_{\mathfrak{d}}$ implies both $\mathfrak{d} = \omega_1$ and $\mathfrak{a} = \omega_1$.

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CENTRO DE CIENCIAS MATEMÁTICAS
UNAM, MEXICO CITY, MEXICO
E-mail: chapi@matmor.unam.mx

INSTITUTE OF MATHEMATICS, UNIVERSITY OF VIENNA
AUGASSE 2-6, 1090 VIENNA, AUSTRIA
E-mail: vera.fischer@univie.ac.at

CENTRO DE CIENCIAS MATEMÁTICAS
UNAM, MEXICO CITY, MEXICO
E-mail: oguzman@matmor.unam.mx

INSTITUTE OF MATHEMATICS
PAVOL JOZEF ŠAFÁRIK UNIVERSITY IN KOŠICE
JESENNÁ 5, 040 01 KOŠICE, SLOVAKIA
E-mail: jaroslav.supina@upjs.sk