ENVELOPING ALGEBRAS OF SEMI-SIMPLE LIE ALGEBRAS

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In a recent paper we studied systems of equations of the form

$$[[x_i,x_j],x_k] = \delta_{ki}x_j - \delta_{kj}x_i, \qquad i, j, k = 1, 2, \ldots, n. \tag{1}$$

$$\phi(x_1) = 0 \tag{2}$$

where as usual $[a,b] = ab - ba$ and $\phi(\lambda)$ is a polynomial.[1] Equations of this type have arisen in quantum mechanics. In our paper we gave a method of determining the matrix solutions of such equations. The starting point of our discussion was the observation that if the elements x_i satisfy (1) then the elements x_i, $[x_j,x_k]$ satisfy the multiplication table of a certain basis of the Lie algebra $\mathfrak{S}_{n+1}$ of skew symmetric $(n + 1) \times (n + 1)$ matrices. We proved that if (2) is imposed as an added condition, then the algebra generated by the x's has a finite basis, and we obtained the structure of the most general associative algebra that is generated in this way.

In this paper we shall generalize a portion of these results to arbitrary simple and semi-simple Lie algebras. Our main results are stated in §**1**. In §**2** we reduce the considerations to the case of an algebraically closed base field. In §**3** we give a summary of known definitions and structural results and in §**4** we prove some basic lemmas that are needed to complete the proof. The algebraically closed case is then treated in §**5**. In §**6** we give some applications of our results to representation theory. One of these is a generalization of a result due to Harish-Chandra that we have been privileged to see prior to publication.[2]

1. Let $\mathfrak{L}$ be a Lie algebra over a field Φ. Thus $\mathfrak{L}$ is a vector space over Φ in which there is defined a bilinear composition $[a,b]$ satisfying the identities

$$[a,a]=0, \quad [[a,b],c] + [[b,c],a] + [[c,a],b] = 0. \tag{3}$$

As is well known any associative algebra $\mathfrak{A}$ determines a Lie algebra $\mathfrak{A}_l$ relative to the composition $[a,b] = ab - ba$. We define an *imbedding* of a Lie algebra $\mathfrak{L}$ in an associative algebra $\mathfrak{A}$ to be a homomorphism $a \rightarrow a^S$ of $\mathfrak{L}$ into $\mathfrak{A}_l$. This means that

$$(a + b)^S = a^S + b^S, \qquad (\alpha a)^S = \alpha a^S$$
$$[a,b]^S = [a^S,b^S] \equiv a^S b^S - b^S a^S. \tag{4}$$

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[1] [5] of the bibliography. [2] [2].

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The subalgebra of $\mathfrak{A}$ generated by the elements a^S is called the *enveloping algebra* of the imbedding S.

We shall say that the imbedding S is a *cover* of the imbedding $T(S \geqslant T)$ if there exists a homomorphism of the enveloping associative algebra $\mathfrak{E}_S$ of S onto the enveloping associative algebra $\mathfrak{E}_T$ of T. This means that $T = SG$ where G is a homomorphism of $\mathfrak{E}_S$ onto $\mathfrak{E}_T$. Evidently G is unique. If $S \geqslant T$ and $T \geqslant S$, then we say that S and T are *equivalent*. This is the case if and only if $T = SG$ where G is an isomorphism of the enveloping algebras.

There exists a *universal imbedding* U that is a cover of every imbedding. We can obtain such an imbedding as follows. Let $x_1, x_2, \ldots$ be a basis for $\mathfrak{L}$ over Φ and let $[x_i, x_j] = \sum \gamma_{ijk} x_k$ be the multiplication table. Let $\mathfrak{F}$ be the free associative algebra generated by the x_i, $\mathfrak{B}$ the two-sided ideal generated by the elements $y_{ij} = x_i x_j - x_j x_i - \sum \gamma_{ijk} x_k$ and let $\mathfrak{U} = \mathfrak{F}/\mathfrak{B}$. If $\bar{x}_i$ is the coset $x_i + \mathfrak{B}$, $[\bar{x}_i, \bar{x}_j] = \bar{x}_i \bar{x}_j - \bar{x}_j \bar{x}_i = \sum \gamma_{ijk} \bar{x}_k$. Hence the mapping $a = \sum \alpha_i x_i \to \sum \alpha_i \bar{x}_i \equiv \bar{a}$ is an imbedding of $\mathfrak{L}$. The enveloping algebra of this imbedding is $\mathfrak{U}$. It is easy to see that this imbedding is universal. Also it is known that the distinct monomials $\bar{x}_1^{k_1} \bar{x}_2^{k_2}, \ldots, \bar{x}_r^{k_r}$ form a basis for $\mathfrak{U}$.[3] In particular the $\bar{x}_i$ are linearly independent so that the universal imbedding is $1-1$. It is therefore convenient to identify $\mathfrak{L}$ with its image $\bar{\mathfrak{L}}$ in this imbedding and to write a for $\bar{a}$, $\mathfrak{L}$ for $\bar{\mathfrak{L}}$. The universal imbedding thus becomes the identity mapping. We call $\mathfrak{U}$ the *universal* (*associative*) *algebra* of the Lie algebra $\mathfrak{L}$.

If S is any imbedding of $\mathfrak{L}$ the mapping $a \to a^S$ can be extended in one and only one way to a homomorphism of $\mathfrak{U}$ onto the enveloping algebra $\mathfrak{E}_S$. We denote this extension by S also. It is clear that if $\mathfrak{K}_S$ is the kernel of S, then S is also equivalent to the natural imbedding in $\mathfrak{U}/\mathfrak{K}_S$. Also it is easy to see that $S \geqslant T$ if and only if $\mathfrak{K}_S \subseteq \mathfrak{K}_T$.

In this paper we assume throughout that $\mathfrak{L}$ has a finite basis but we consider imbeddings of $\mathfrak{L}$ in associative algebras that need not be finite dimensional. Our main results show, however, that under certain simple conditions we can conclude that the enveloping algebras have finite bases. To state the results we need to introduce several definitions.

Let $\mathfrak{L}$ be a semi-simple Lie algebra over a field of characteristic 0.[4] As is well known, $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \ldots \oplus \mathfrak{L}_s$ where the $\mathfrak{L}_i$ are simple and uniquely determined. If a is any element of $\mathfrak{L}$, $a = a_1 + a_2 + \ldots + a_s$, a_i in $\mathfrak{L}_i$. We call a_i the ith *component* of a. We call a subset $\Gamma = \{a, b, \ldots\}$ *total* if for any $i = 1, 2, \ldots, s$ there is a c in the set whose ith component $c_i \neq 0$. It is easy to see that Γ is total if and only if the ideal generated by Γ is $\mathfrak{L}$ itself.

We can now state the main results of this note as the following two theorems.

THEOREM 1. *Let $\mathfrak{L}$ be a semi-simple Lie algebra with a finite basis over a field of characteristic 0 and let Γ be a total subset of $\mathfrak{L}$. Let S be an imbedding*

[3]The results in the universal algebra quoted here are due independently to Birkhoff [1] and to Witt [9].

[4]Definitions of semi-simplicity and simplicity are given in the next section.

of $\mathfrak{L}$ such that for each $c \in \Gamma$, c^S is algebraic. Then the enveloping algebra of S has a finite basis.

THEOREM 2. *Let $\mathfrak{L}$ and Γ be as in Theorem 1 and let $\{S\}$ be the collection of imbeddings of $\mathfrak{L}$ such that for every c in Γ, c^S is algebraic of degree $\leqslant$ a fixed t. Then their exists an imbedding T such that the enveloping algebra of T is finite dimensional and such that $T \geqslant S$ for every $S \in \{S\}$.*

2. We shall now show that it suffices to prove these theorems for algebraically closed base fields. Assume that Φ is arbitrary and let Ω be its algebraic closure. Then it is well known that if $\mathfrak{L}$ is a semi-simple so is $\mathfrak{L}_\Omega$. It is necessary for our purposes to obtain the decomposition of $\mathfrak{L}_\Omega$ into simple algebras from that of $\mathfrak{L}$.

For this purpose we consider first the structure of $\mathfrak{M}_\Omega$ for any simple non-associative algebra $\mathfrak{M}$. Let P be the multiplication centralizer (extended centre of Φ).[5] Then we know that P is a field containing Φ and that $\mathfrak{M}$ can be regarded as an algebra over P. Let $x_1, x_2, \ldots, x_m$ be a basis for $\mathfrak{M}$ over P and accordingly write $\mathfrak{M} = \mathrm{P}x_1 + \mathrm{P}x_2 + \ldots + \mathrm{P}x_m$. We assume now that P is separable over Φ. (This will certainly be the case if Φ has characteristic 0.) We form P_Ω. Then it is well known that $\mathrm{P}_\Omega = \Omega^{(1)} \oplus \Omega^{(2)} \oplus \ldots \oplus \Omega^{(r)}$ where the $\Omega^{(i)}$ are one dimensional algebras over Ω isomorphic to Ω and r is the dimensionality $(\mathrm{P}:\Phi)$.[6] If $\rho \in \mathrm{P}$ we have $\rho = \rho^{(1)} + \rho^{(2)} + \ldots + \rho^{(r)}$, $\rho^{(i)} \in \Omega^{(i)}$, and the correspondences $\rho \rightarrow \rho^{(i)}$ are isomorphisms of P into $\Omega^{(i)}$. Now P_Ω acts as a set of endomorphisms in $\mathfrak{M}_\Omega$ and

$$\mathfrak{M}_\Omega = \mathrm{P}_\Omega x_1 + \mathrm{P}_\Omega x_2 + \ldots + \mathrm{P}_\Omega x_m.$$

Since the elements of P_Ω commute with the right and the left multiplications, $\mathfrak{M}^{(i)} = \Omega^{(i)}\mathfrak{M}_\Omega$ is an ideal in $\mathfrak{M}_\Omega$. We have the decomposition $\mathfrak{M}_\Omega = \mathfrak{M}^{(1)} \oplus \mathfrak{M}^{(2)} \oplus \ldots \oplus \mathfrak{M}^{(r)}$. Also $\mathfrak{M}^{(i)} = \Omega^{(i)}x_1 + \Omega^{(i)}x_2 + \ldots + \Omega^{(i)}x_m$ and it is easily seen that $\mathfrak{M}^{(i)} \cong (\mathfrak{M} \text{ over } \mathrm{P})_\Omega$.[7] Since $\mathfrak{M}$ is central simple over P, $(\mathfrak{M}$ over $\mathrm{P})_\Omega$ is simple. Hence, $\mathfrak{M}_\Omega$ is a direct sum of r isomorphic simple algebras over Ω.

Now let $a \in \mathfrak{M}$ and write $a = \sum a_j x_j$, a_j in P. Then $a = a^{(1)} + a^{(2)} + \ldots + a^{(r)}$ where $a^{(i)} = \sum a_j^{(i)} x_j \in \mathfrak{M}^{(i)}$. Thus it is clear that if $a \neq 0$, then each $a^{(i)} \neq 0$.

We return now to the consideration of Lie algebras. Let $\mathfrak{L}$ be semi-simple over a field of characteristic 0 and let $\Gamma = \{a, b, \ldots\}$ be a total subset of $\mathfrak{L}$. It is clear from the above remarks that $\mathfrak{L}_\Omega$ is semi-simple and that Γ is total for $\mathfrak{L}_\Omega$. Now let S be an imbedding of $\mathfrak{L}$ in an associative algebra $\mathfrak{A}$. Then S can be extended in one and only one way to an imbedding of $\mathfrak{L}_\Omega$ in $\mathfrak{A}_\Omega$ and the enveloping algebra of this extension is $\mathfrak{E}_\Omega$, $\mathfrak{E}$ the enveloping algebra of S. If the elements $a^S, b^S, \ldots$ are algebraic in $\mathfrak{A}$ they are algebraic of the same degree in $\mathfrak{A}_\Omega$. Hence, if Theorem 1 holds for algebraically closed fields, then we can

[5] [4] p. 546.
[6] [6] p. 97.
[7] Cf. [6] p. 115.

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conclude that the dimensionality $(\mathfrak{E}_\Omega:\Omega)$ is finite. Since $(\mathfrak{E}:\Phi) = (\mathfrak{E}_\Omega:\Omega)$ this proves the result for $\mathfrak{L}$.

Now let $\{S\}$ be the collection of imbeddings of $\mathfrak{L}$ such that c^S is algebraic of degree $\leqslant t$ for every c in Γ. Let $\mathfrak{U}$ be the universal associative algebra of $\mathfrak{L}$ and let $\mathfrak{K}_S$ be the kernel of the homomorphism of $\mathfrak{U}$ onto the enveloping algebra $\mathfrak{E}_S$. We form the intersection $\mathfrak{D}$ of the ideals $\mathfrak{K}_S$, S in $\{S\}$ and we let T be the natural imbedding determined by $\mathfrak{U}/\mathfrak{D}$. It is clear that $T \geqslant S$ for every S. Hence, it suffices to show that $\mathfrak{U}/\mathfrak{D}$ is finite dimensional. We require now the following.

Lemma. Let $\mathfrak{R}$ be a vector space over a division ring Δ and let P be a division ring extension of Δ. Then if $\{\mathfrak{S}_\alpha\}$ is a collection of subspaces of $\mathfrak{R}$, $(\bigcap \mathfrak{S}_\alpha)_\mathrm{P} = \bigcap \mathfrak{S}_{\alpha\mathrm{P}}$ holds in $\mathfrak{R}_\mathrm{P}$.

Proof. Assume first that our collection consists of two subspaces $\mathfrak{S}_1, \mathfrak{S}_2$. Let (u_i, g_j) be a basis for $\mathfrak{S}_1$ such that (g_j) is a basis for $\mathfrak{D} = \mathfrak{S}_1 \cap \mathfrak{S}_2$ and let (v_k, g_j) be a basis for $\mathfrak{S}_2$. The set (u_i, v_k, g_j) is linearly independent over Δ and hence also over P (in $\mathfrak{R}_\mathrm{P}$). It is obvious that $\mathfrak{S}_{1\mathrm{P}} \cap \mathfrak{S}_{2\mathrm{P}} = \mathfrak{D}_\mathrm{P}$. Thus the result holds for two spaces. By induction it holds for a finite number. Hence, by the descending chain condition it holds also for any number of subspaces in a finite dimensional space. Now consider the general case. Clearly $\bigcap \mathfrak{S}_{\alpha\mathrm{P}} \supseteq \mathfrak{D}_\mathrm{P}$, $\mathfrak{D} = \bigcap \mathfrak{S}_\alpha$. Let $y' = \sum_1^m \rho_j e_{i_j} \in \bigcap \mathfrak{S}_{\alpha\mathrm{P}}$ where $(e_1, e_2, \ldots)$ is a basis for $\mathfrak{R}$ over Δ. Let $\mathfrak{H} = [e_{i_1}, e_{i_2}, \ldots, e_{i_m}]$ be the space spanned by the e_{i_j} and set $\mathfrak{U}_\alpha = \mathfrak{H} \cap \mathfrak{S}_\alpha$. Then $y' \in (\mathfrak{H}_\mathrm{P} \cap \mathfrak{S}_{\alpha\mathrm{P}}) = \mathfrak{U}_{\alpha\mathrm{P}}$. Hence $y' \in (\bigcap \mathfrak{U}_{\alpha\mathrm{P}})$. Since the $\mathfrak{U}_\alpha$ are subspaces of the finite dimensional space $\mathfrak{H}$, $\bigcap \mathfrak{U}_{\alpha\mathrm{P}} = (\bigcap \mathfrak{U}_\alpha)_\mathrm{P}$. Hence $y' \in (\bigcap \mathfrak{U}_\alpha)_\mathrm{P} \subseteq (\bigcap \mathfrak{S}_\alpha)_\mathrm{P} = \mathfrak{D}_\mathrm{P}$. This proves that $\bigcap \mathfrak{S}_{\alpha\mathrm{P}} \subseteq \mathfrak{D}_\mathrm{P}$. Hence $\bigcap \mathfrak{S}_{\alpha\mathrm{P}} = \mathfrak{D}_\mathrm{P}$.

In the above notation we now have $\mathfrak{D}_\Omega = \bigcap \mathfrak{K}_{S\Omega}$ where Ω is the algebraic closure of the base field Φ. On the other hand, $\mathfrak{U}_\Omega$ is the universal algebra of $\mathfrak{L}_\Omega$ and $(\mathfrak{U}/\mathfrak{D})_\Omega = \mathfrak{U}_\Omega/\mathfrak{D}_\Omega = \mathfrak{U}_\Omega/\bigcap \mathfrak{K}_{S\Omega}$. Moreover, each $\mathfrak{U}_\Omega/\mathfrak{K}_{S\Omega}$ determines an imbedding $\tilde{S}$ of $\mathfrak{L}_\Omega$ in which the elements c^S, c in Γ, are algebraic of degree $\leqslant t$. If Theorem 2 holds in the algebraically closed case, then there exists a $\tilde{T}$ such that $\tilde{T} \geqslant \tilde{S}$ for every $\tilde{S}$ and $\mathfrak{E}_{\tilde{T}}$ is finite dimensional. It follows that $\mathfrak{U}_\Omega/\bigcap \mathfrak{K}_{S\Omega}$ is finite dimensional. Hence $\mathfrak{U}/\mathfrak{D}$ is finite dimensional. This will prove Theorem 2 in the general case.

3. We recall at this point some of the standard definitions and results of the theory of Lie algebras of characteristic 0.[8] If $a \in \mathfrak{L}$ the mapping A: $x \to [x,a]$ is called the *adjoint* mapping determined by a. This mapping is a linear transformation in the vector space $\mathfrak{L}$ and the correspondence $a \to A$ is an imbedding of $\mathfrak{L}$ in the associative algebra of linear transformations. This imbedding is called the *adjoint representation*. The elements of $\mathfrak{L}$ can be classified according to the nature of their adjoint mappings. Thus a is said to be *nilpotent*, if A is nilpotent. Also a is called *regular* if A has the minimum

[8]See [8] for the results of this section.

number of 0 characteristic roots for the adjoint mappings of $\mathfrak{L}$. If a is regular the subspace $\mathfrak{L}_0$ belonging to the characteristic root 0 of A is a nilpotent subalgebra of $\mathfrak{L}$. An algebra is said to be *nilpotent* if there exists an integer N such that the Lie product of any N of its elements is 0. A nilpotent subalgebra $\mathfrak{H}$ is called a *Cartan subalgebra* if $\mathfrak{H}$ is a maximal in the sense that the only elements z such that $zA^m = 0$ for every $a \in \mathfrak{H}$ and a suitable integer m are the elements of $\mathfrak{H}$. It is known that the nilpotent algebra $\mathfrak{L}_0$ determined as above by a regular element is a Cartan subalgebra.

A somewhat weaker condition than nilpotency is solvability. This states that the derived series $\mathfrak{L} \supseteq \mathfrak{L}' = [\mathfrak{L},\mathfrak{L}] \supseteq \mathfrak{L}'' = [\mathfrak{L}',\mathfrak{L}'] \ldots$ leads to 0. An algebra is *semi-simple* if it has no solvable ideals, *simple* if it has no proper ideals. It is a fundamental theorem that any semi-simple Lie algebra is a direct sum of simple Lie algebras.

It is known that any Cartan subalgebra $\mathfrak{H}$ of a semi-simple Lie algebra is commutative. If the base field Φ is algebraically closed, we can use any Cartan algebra $\mathfrak{H}$ to obtain a certain canonical basis for $\mathfrak{L}$. This consists of a basis $(h_1, h_2, \ldots, h_l)$ for $\mathfrak{H}$ and elements $e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}, \ldots$ such that the subscripts $\alpha, \beta, \ldots$ are linear functions on the vector space $\mathfrak{H}$. These functions are called *the roots* of $\mathfrak{H}$ and their significance is given in the first line of the following multiplication table:

$$
(5) \qquad
\begin{aligned}
[e_\alpha, h] &= \alpha(h)e,\\
[e_{-\alpha}, e_\alpha] &= h_\alpha \in \mathfrak{H},\\
[e_\alpha, e_\beta] &= \begin{cases} 0 & \text{if } \alpha + \beta \text{ is not a root,}\\ N_{\alpha\beta}\, e_{\alpha+\beta} \neq 0 & \text{if } \alpha + \beta \text{ is a root.}\end{cases}
\end{aligned}
$$

It is known that there are l linear independent roots and that the h_α generate the whole of $\mathfrak{H}$. It is known that the e_α can be normalized so that if α, β are any two roots such that $\alpha + \beta$ is also a root, then $h_{\alpha+\beta} = h_\alpha + h_\beta$. If α and β are any two roots, the roots of the form $\alpha + \nu\beta$, ν an integer, form an unbroken *α-string*

$$\beta - k\alpha,\ \beta - (k-1)\alpha, \ldots, \beta, \ldots, \beta + k'\alpha.$$

The value $\alpha(h_\alpha) \neq 0$ and $2\beta(h_\alpha)/\alpha(h_\alpha) = k - k'$. Thus $\beta(h_\alpha) = 0$ if and only if β is the centre term of its α-string.

We shall say that the root ρ is *connected to* α if there exists a sequence of roots $\alpha, \beta, \ldots, \rho$ such that for any two consecutive terms β, γ, $\gamma(h_\beta) \neq 0$. If β is not the centre term of its α-string, then as we have seen, β is connected with α. The same conclusion holds also if β is the centre term provided that the α-string, containing β, contains more than one term. For if these conditions hold either $\beta + \alpha$ or $\beta - \alpha$ is a root. In the former case $(\beta + \alpha)(h_\alpha) = \alpha(h_\alpha) \neq 0$ and $\beta(h_{\alpha+\beta}) = \beta(h_\alpha + h_\beta) = \beta(h_\alpha) + \beta(h_\beta) = \beta(h_\beta) \neq 0$. Hence the sequence α, $\alpha + \beta$, β shows that β is connected with α. A similar argument can be used if $\beta - \alpha$ is a root. Thus we see that if σ is not connected with α, then the α-string containing σ contains this term only. Then $[e_\sigma e_\alpha] = [e_\sigma, e_{-\alpha}] = [e_\sigma, h_\alpha] = 0$.

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It follows easily that if α is any root the space spanned by e_α, h_α, e_ρ, h_ρ for all the ρ that are connected to α is an ideal. Hence, if $\mathfrak{L}$ is simple any root ρ is connected with any other root α.

It is clear from the multiplication table (5) that any element $h \in \mathfrak{H}$ is *semi-regular* in the sense that its adjoint mapping H has simple elementary divisors. We shall now prove the converse that any semi-regular element h can be imbedded in a Cartan subalgebra. First let $\mathfrak{L}_0$ be the subalgebra corresponding to the characteristic root zero of H. Since 0 is a simple root, $\mathfrak{L}_0$ is just the set of elements z such that $[z,h] = 0$. Let h' be an element of $\mathfrak{L}_0$ that is regular in $\mathfrak{L}_0$ and let $\mathfrak{H}$ be the Cartan subalgebra of $\mathfrak{L}_0$ determined by this element. Thus, $\mathfrak{H}$ is the totality of elements $z \in \mathfrak{L}_0$ such that $z(H')^m = 0$ for some m. Here, $h' \rightarrow H'$ in the adjoint representation. Now it is clear that $\mathfrak{H}$ is the intersection of the space belonging to the characteristic root 0 of H with the space belonging to the characteristic root 0 of H'. Since $\mathfrak{H}$ contains h and h' it follows from the definition that $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{L}$. This implies that $\mathfrak{H}$ is commutative.

4. The proof of the main theorems for algebraically closed base fields depends on some lemmas which we shall now derive.

LEMMA 1. *Let $\mathfrak{A}$ be an associative algebra over a field of characteristic 0 and let e and h be elements of $\mathfrak{A}$ such that $[e,h] = eh - he = e$. Then if h is algebraic of degree m, e is nilpotent of degree $\leqslant m$.*

Proof. From $[e,h] = e$ we obtain $eh = (h+1)e$. Hence for any polynomial $\phi(\lambda)$

$$e\phi(h) = \phi(h+1)e. \tag{6}$$

Hence, also

$$e^k\phi(h) = \phi(h+k)e^k. \tag{7}$$

Now let $\phi(\lambda)$ be the minimum polynomial of h (of degree m). Then by (7) $\phi(h+k)e^k = 0$. We multiply this equation for $k < m$ on the right by e^{m-k} and obtain $\phi(h+k)e^m = 0$, $k = 1,2,\ldots,m$. Also $\phi(h)e^m = 0$. Since Φ is of characteristic 0, these relations imply that $e^m = 0$.

LEMMA 2. *If e and h are elements of an associative algebra of characteristic 0 such that $[[e,h],e] = 0$ and e is algebraic of degree m, then $[e,h]$ is nilpotent of index $\leqslant 2^m - 1$.*

This is essentially Lemma 2 of [3]. The proof given there needs to be corrected by the replacement of $2h-1$ by $2^h - 1$ and $2k-1$ by $2^k - 1$.

LEMMA 3. *Let h, e, f be elements of an associative algebra over a field of characteristic 0 such that*

$$[e,h] = 2e, \quad [f,h] = -2f, \quad [e,f] = h \tag{8}$$

and suppose that e is nilpotent of index m. Then h is algebraic and its minimum polynomial is a factor of

$$\mu(\lambda) = \prod_{j=1}^{2m-1} (\lambda + m - j). \tag{9}$$

Proof. We use the argument that gives (7) to prove that

(10) $$e^k\phi(h) = \phi(h+2k)e^k, \quad \phi(h)f^k = f^k\phi(h+2k).$$

Also,

(11) $$[e^n,f] = n(h+n-1)e^{n-1}$$

and by induction on r we can prove for $r \leqslant n$

(12) $$[^r\, e^n,f], f], \ldots, f] = \sum_{k=1}^{[r/2]} c_{nrk} f^k \prod_{j=1}^{r-2k} (h+n-j)e^{n-r+k}$$

where

$$c_{nrk} = \binom{n}{k}\binom{n-k}{r-2k} r!.$$

Assume now that $e^m = 0$. Then, we can prove that

(13) $$\prod_{j=1}^{2r-1} (h+m-j)e^{m-r} = 0.$$

For this is true for $r = 0$. If we assume it true for $0, 1, 2, \ldots, r-1$ and we multiply (12) for $n = m$ on the left by $\prod_{r+1}^{2r-1} (h+m-j)$ we obtain (13). For $r = m$ this gives the lemma.

5. Now let $\mathfrak{L}$ be semi-simple over an algebraically closed field of characteristic 0 and let $\Gamma = \{a,b,\ldots\}$ be a total subset of $\mathfrak{L}$. Let S be an imbedding of $\mathfrak{L}$ in $\mathfrak{A}$ such that every c^S, c in Γ, is algebraic of degree $\leqslant t$. To prove the first theorem it suffices to show that $\mathfrak{L}$ has a basis $y_1,y_2,\ldots, y_n$ such that every y_i^S is algebraic. For if $[y_i y_j] = \sum \gamma_{ijk} y_k$ then

$$y_i^S y_j^S = y_j^S y_i^S + \sum \gamma_{ijk} y_k^S.$$

It follows that the monomials $(y_i^S)^{m_1}(y_2^S)^{m_2}\ldots(y_n^S)^{m_n}$, $m_i < T$ the maximum degree of the y_i^S generate the space $\mathfrak{E}$. Hence $(\mathfrak{E}:\Phi) \leqslant T^n$.

We suppose first that $\mathfrak{L}$ is simple and $\Gamma = \{a\}$. We note first that we can suppose that a is a nilpotent element of $\mathfrak{L}$. For if the adjoint mapping A is not nilpotent, then there is an $e \neq 0$ and a $\rho \neq 0$ such that $[e,a] = eA = \rho e$. If we replace a by $a' = \rho^{-1}a$, then $[e,a'] = e$. Hence by Lemma 1, e^S and the adjoint transformation E corresponding to e are nilpotent. Also degree $e^S \leqslant t$. Thus, we can assume that $\Gamma = \{e\}$ where E is nilpotent.

We apply next a result due to Morosov that asserts that if e is a non-zero nilpotent element of a semi-simple Lie algebra over an algebraically closed field of characteristic 0, then e can be imbedded in a three-dimensional simple subalgebra.[9] In fact, we can find elements f, h such that

(14) $$[e,h] = 2e, \quad [f,h] = -2f, \quad [e,f] = h.$$

The first of these equations shows that $[[e,h],e] = 0$. Since e^S is algebraic, $[e,h]^S$ is nilpotent. Hence $2e^S$ and e^S are nilpotent of index $\leqslant t$.

We observe next that the adjoint transformation H of h and the element h^S

[9][7]. A simple and complete proof of this result will be given by the present author in Trans. Amer. Math. Soc.

are roots of polynomials of the form (9). Hence the minimum polynomials of these elements have distinct roots in Ω. It follows that H has simple elementary divisors and that h is semi-regular.

We now imbed h in a Cartan subalgebra $\mathfrak{H}$ and we choose a canonical basis $(h_1,h_2,\ldots,h_l,e_\alpha,e_{-\alpha},\ldots)$ for $\mathfrak{L}$. Since there are l linearly independent roots and $h \neq 0$ we can find an α such that $\alpha(h) \neq 0$. Then $[e_\alpha,h] = \alpha(h)e_\alpha$ and $[e_{-\alpha},h] = -\alpha(h)e_{-\alpha}$. By Lemma 1, $e_\alpha{}^S$ and $e_{-\alpha}{}^S$ are nilpotent of index $\leqslant 2t-1$. The elements $e_\alpha,e_{-\alpha}$ and h_α span a three-dimensional simple Lie algebra with multiplication table

$$[e_\alpha,h_\alpha] = \alpha(h_\alpha)e_\alpha,\ [e_{-\alpha},h_\alpha] = -\alpha(h_\alpha)e_{-\alpha},\quad [e_{-\alpha},e_\alpha] = h_\alpha.$$

If we set $e'_\alpha = 2\alpha(h_\alpha)^{-1}e_\alpha$, $e'_{-\alpha} = e_{-\alpha}, h'_\alpha = 2\alpha(h_\alpha)^{-1}h_\alpha$, then we obtain

$$[e'_\alpha,h'_\alpha] = 2e'_\alpha,\ [e'_{-\alpha},h'_\alpha] = -2e'_{-\alpha},\quad [e'_{-\alpha},e'_\alpha] = h'_\alpha.$$

Hence, by Lemma 3, h'^S_α satisfies an equation of the form (9).

Now let ρ be any root. Since $\mathfrak{L}$ is simple, we can find a sequence of roots $\alpha,\beta,\ldots,\rho$ beginning with α and ending with ρ such that consecutive terms, β,γ, have the property $\gamma(h_\beta) \neq 0$. Now $[e_\beta,h_\alpha] = \beta(h_\alpha)e_\beta \neq 0$ and $[e_{-\beta},h_\alpha] = -\beta(h_\alpha)e_{-\beta} \neq 0$. Since h^S is algebraic, $e_\beta{}^S$ and $e_{-\beta}{}^S$ are nilpotent. If, as before, we introduce $e'_\beta = 2\beta(h_\beta)^{-1}e_\beta$, $e'_{-\beta} = e_{-\beta}$, $h'_\beta = 2\beta(h_\beta)^{-1}h_\beta$, $h_\beta = [e_{-\beta},e_\beta]$ then we see that $e'_\beta{}^S,e'_{-\beta}{}^S$ are nilpotent and that $h'_\beta{}^S$ satisfies an equation of the form (9). Continuing in this way we obtain e'_ρ, $e'_{-\rho}$, h'_ρ such that $e'_\rho{}^S$, $e'_{-\rho}{}^S$ are nilpotent and $h'_\rho{}^S$ satisfies an equation of the form (9). We obtain in this way a basis consisting of certain of the h'_α and all of the e'_α, $e'_{-\alpha}$ and we have

$$(e'_\alpha{}^S)^u = 0,\ (e'_{-\alpha}{}^S)^u = 0 \qquad (15)$$
$$\prod_{i=1}^{2u-1} (h'_\alpha{}^S + u - i) = 0.$$

It is now clear that the enveloping algebra $\mathfrak{E}_S$ has a finite basis. Hence, Theorem 1 is proved in the present case.

Now let $\mathfrak{U}$ be the universal algebra of $\mathfrak{L}$ and let $\mathfrak{K}$ be the two-sided ideal generated by the elements

$$(e'_\alpha{}^S)^u,\ (e'_{-\alpha}{}^S)^u,\ \prod_{i=1}^{2u-1} (h'_\alpha{}^S + u - i).$$

Since (15) holds, it is clear that the natural imbedding in $\mathfrak{U}/\mathfrak{K}$ is a cover of S. Also we can choose a u for which (15) holds that depends only on t and on the basis $(h'_\alpha,e'_\alpha,\ldots)$ for $\mathfrak{L}$. Hence, the imbedding determined by $\mathfrak{U}/\mathfrak{K}$ is a cover of every imbedding S that has the property that a^S is algebraic of degree $\leqslant t$. Hence, we have also established Theorem 2 in the special case.

Finally let $\mathfrak{L}$ be semi-simple, $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \ldots \oplus \mathfrak{L}_s$ where the $\mathfrak{L}_i$ are simple.

For each i we choose a c such that $c_i \neq 0$. If c_i is nilpotent in $\mathfrak{L}_i$ by Morosov's theorem, there is an element h_i in $\mathfrak{L}_i$ such that $[c_i,h_i] = c_i$. Hence, $[c,h_i] = c_i$ and $[[c,h_i],c] = [c_i,c] = 0$. Hence, by Lemma 2, c_i^S is nilpotent of index $\leqslant 2^t-1$. If c_i is not nilpotent in $\mathfrak{L}_i$, there exists an element e_i in $\mathfrak{L}_i$ such that $[e_i,c_i] = [e_i,c] = \rho e_i \neq 0$. It follows that e_i^S is nilpotent of index $\leqslant t$. As in the above discussion, we can use the element c_i or e_i to prove that $\mathfrak{L}_i$ has a basis of the form $(h'_\alpha,e'_\alpha, \ldots)$ satisfying (15). The set theoretic sum of the bases obtained in this way for the $\mathfrak{L}_i$ is a basis for $\mathfrak{L}$. This basis can be used as in the simple case to complete the proofs of Theorems 1 and 2.

6. By a *representation* of a Lie algebra we mean as usual an imbedding in the associative algebra of linear transformations of some finite dimensional vector space. Irreducibility is defined as usual. We consider now the set of irreducible representations S such that the minimum polynomial of c^S is of degree $\leqslant t$ for every c in Γ. Then S determines a representation of the finite dimensional algebra $\mathfrak{E}_T$ given in Theorem 2. If we recall that a finite dimensional associative algebra has only a finite number of inequivalent irreducible representations we obtain the following

THEOREM 3. *Let $\mathfrak{L}$ and Γ be as in Theorem 1. Then there exists only a finite number of inequivalent irreducible representations of $\mathfrak{L}$ such that the degree of the minimum polynomial of every c^S, c in Γ, does not exceed a fixed integer t.*

Since the minimum polynomial of a linear transformation has degree $\leqslant$ the dimensionality of the space we have the

COROLLARY (Harish-Chandra). If $\mathfrak{L}$ is a semi-simple Lie algebra of characteristic 0, $\mathfrak{L}$ has only a finite number of inequivalent irreducible representations of a given degree.[10]

We consider next a more special application, namely, we study a system of equations of the form (1) to which is added the equation $\phi(x) = 0$ where $x = \sum_1^n \xi_i x_i \neq 0$ and $\phi(\lambda)$ is a polynomial. We seek linear transformations (or matrices) X_i that satisfy such a system. It is known that the correspondence $x_i \rightarrow X_i$ defines a representation of the Lie algebra $\mathfrak{S}_{n+1}$ of $(n+1) \times (n+1)$ skew symmetric matrices.[11] If $X = \sum \xi_i X_i$ then $\phi(X) = 0$. Hence, by Theorem 3 there exist only a finite number of inequivalent irreducible sets of linear transformations that satisfy our system. This generalizes our earlier result noted in the introduction.

[10][2]. It should be noted that Harish-Chandra has proved that there are only a finite number of inequivalent representations of given degree. This result follows readily from the present corollary and the theorem that any representation is completely reducible.

[11][5] p. 156.

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